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Optimal search strategy of moving object in multichannel system

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Проведено аналіз задачі пошуку сигналу в багатоканальній системі зв'язку з одним пошуковим пристроєм. Побудована оптимальна стратегія переміщення пошукового пристрою в багатоканальній системі і отримано відповідну оцінку ефективності пошуку.

Проведён анализ задачи поиска сигнала в многоканальной системе связи с одним поисковым устройством. Построена оптимальная стратегия перемещения поискового устройства в многоканальной системе и получено соответствующая оценка эффективности поиска.

Introduction

Mathematical apparatus used in the search theory is diverse [1, 2, 3, 10] although even for search tasks with simple parameters, finding the optimal strategy in an analytical view is usually not possible. Thus finding the solution that leads to the "simplest" problems of mathematical programming, effectively solvable using quantitative methods is relevant.

Consider the following search problem. A given system consisting of N channels with one dynamic signal and one search device (SD), where SD is moving only through the first M channels while the signal is moving on all N channels, $N, M \leq N$, in discrete time period $t = 0, 1, 2, \ldots$ the signal and the SD are transitioning from channel to channel independently from each other according the the Markov model. Assume that probabilities $a_{ij}, i, j = \overline{1, N}$ of signal transition from i-channel to j-channel and the corresponding probabilities $B_{ij}, i, j = \overline{1, M}$ for the SD are time independent. At the start moment $t = 0$ the signal is located in the channel j_o ($j_0 \leq N$) wile the SD is located in channel j_1 ($j_1 \leq M$).

The following rule is used to indicate the discovery of the impulse by the SD: the signal located in the i-channel at the t_1 time period and leaving it at the t_2 time period is considered disovered in the t $(t_1 \leq t \leq t_2)$ time period, if the SD is also located in the i channel, the discovery is considered valid only if it occurred for the first time since the t_1 time period.

Probability of signal discovery in the t time period is denoted as $K_{j_o,j_1}(t)$. The probability of the signal emergence in any of the connection channels (transition in another channel) in the t time frame is denoted as $L_{j_0}(t)$, $C_{j_o,j_1}(t) = K_{j_0,j_1}(t)/L_{j_0}(t)$ denotes the relationship between. $C_{j_o,j_1}(t)$ value where t_0 is the given discrete time period is referred to as search effectiveness coefficient, while $C = \lim_{t\to\infty} C_{j_0,j_1}(t)$ value, if such limit exists it is referred to as a stationary effectiveness coefficient.

Worth noting that these effectiveness coefficients have practical applications.

$$\sum_{t=0}^{t_o} K_{j_o,j_1}(t)\,,\ \sum_{t=0}^{t_o} K_{j_o,j_1}(t)$$

The following values need to be determined: effectiveness coefficient $C_{j_o,j_1}(t)$, stationary effectiveness coefficient C, probabilities $B_{ij}, i, j = \overline{1, M}$ of SD transition between channels where stationary effectiveness coefficient is maximum.

1 Main relationships for a non-inflated system

The signal movement thorough the channel is described by a uniform Markov chain $\xi(t)$ with discrete time and multitude of states $\{1, 2, \ldots, N\}$ and a probability matrix of transitioning by 1 step $A = \left\{a_{ij} : i, j = \overline{1, N}\right\}$. The initial state $\xi(0) = j_0$. $\xi(t) = i$ if at time period t the signal is located in the i channel.

The SD movement thorough the channel is described by a uniform Markov chain $\eta(t)$ with multitude of states $\{1, 2, \ldots, M\}$ and a probability matrix of transitioning by 1 step $B = \left\{b_{ij} : i, j = \overline{1, M}\right\}$. The initial state $\eta(0) = j_1$. $\eta(t) = i$ if at time period t the SD is located in the i channel.

Then

$$K_{j_0 j_1}(t) = \mathcal{P}\left\{ \begin{array}{c} \xi(t) = i, \eta(t) = i, \quad \eta(\tau) \neq i,\ t - u_\xi(t) < \tau < t, \\ i = \frac{\overline{1,M}}{\xi(0)} = j_0, \quad \eta(0) = j_1 \end{array} \right\},$$

Where $u_\xi(t)$ is the underjump of the Markov chain

$$u_\xi(t) = t - \sup\{\tau \le t : \xi(\tau) \ne \xi(t)\}.$$

Let us introduce the notation:

$P_\xi(t,j,i) = \mathcal{P}\{\xi(t) = i/\xi(0) = j\}$ - probability of signal being located in the i-th channel at time period t under condition that initially the signal was located in the j-th channel.

$P_\eta(t,j,i) = \mathcal{P}\{\eta(t) = i/\eta(0) = j\}$ - probability of SD being located in the i-th channel at time period t under condition that initially it was located in the j-th channel.

$Q_\eta(t,j,i) = \mathcal{P}\{\eta(t) = i,\ \ \eta(\tau) \ne i,\ 0 < \tau < t\,|\eta(0) = j\}$ - probability of SD being first located in the i-th channel at time t under condition that initially it was located in the j-th channel, $i \ne j$, $t \ge 1$.

$h_\xi(t,j,i) = \mathcal{P}\{\xi(t) = i,\ \xi(t-1) \ne i\,|\xi(0) = j\}$ – probability of signal transitioning in the i-th channel at time period t from any other channel other than i, under condition that it was initially located in the j-th channel, $t > 0$.

$\overline{F_\xi}(n,i) = \mathcal{P}\{\xi(t+1) = i, \xi(t+2) = i, \ldots, \xi(t+n) = i/\xi(t) = i\}$ - probability that the signal stays in the i-th channel during the time period $[t; t+n]$, under condition that it was located in the i-th channel at time period t.

The following relations are fair

$$P_\xi(t,j,i) = \sum_{k=1}^{N} a_{jk} P_\xi(t-1,k,i),\ t \ge 1,\ P_\xi(o,j,i) = \delta_{ji},$$

$$P_\eta(t,j,i) = \sum_{k=1}^{M} \beta_{jk} P_\eta(t-1,k,i),\ t \ge 1,\ P_\eta(o,j,i) = \delta_{ji},$$

$$Q_\eta(t,j,i) = \sum_{k \ne i} \beta_{jk} Q_\eta(t-1,k,i),\ t \ge 2,\ Q_\eta(1,j,i) = \beta_{ji},\ j \ne i,$$

$$h_\xi(t,j,i) = \sum_{k \ne i} a_{ki} P_\xi(t-1,j,k),\ t \ge 1,$$

$$L_{j_o(t)} = \sum_{i=1}^{N} h_\xi(t,j_0,i),$$

where

$$\delta_{ji} = \begin{cases} 1,\ when\ j = i; \\ 0,\ when\ j \ne i. \end{cases}$$

Theorem 1.2. *For probability of signal discovery in time frame t the following relation is applicable*

$$K_{j_0 j_1}(t) = \sum_{i=1}^{M} \Big\{ h_\xi(t, j_o, i)\, P_\eta(t, j_1, i) + \delta_{j_0 i} \overline{F_\xi}(t, i)\, Q_\eta(t, j_1, i) +$$

$$\left. \sum_{n=1}^{t-1} h_\xi(t-n,\ j_0, i)\, \overline{F_\xi}(n, i) \sum_{k \neq i} P_\eta(t-n, j, k)\, Q_\eta(n, k, i) \right\} ,\ t > 0 \quad (1)$$

Proof. The proof is based on the total probability formula, according to which, the probability of event A out of total pool of events $E_1, \ \ldots, \ E_n$, $(E_i \ \neq \emptyset, \ i = \overline{1, \ n})$ is

$$\mathcal{P}(A) = \sum_{i=1}^{n} \mathcal{P}(A \mid E_i) \cdot \mathcal{P}(E_i)$$

While constructing the relationships, it must be taken into account that the signal and SD transitions are independent.

Consider the initial fixes conditions $\xi(0) = \ j_0, \eta(0) = J_1$.

Discovery of the signal in the I channel in the t time frame can occur:

1. In case if the signal transitioned into the i-th channel in the t time period from any other channel other than i while the SD was located in the i-th channel at the t time period (probability of the event $h_\xi(t, j_o, i) P_\eta(t, j_1, i)$);
2. In case the signal transitioned into the i-th channel in time period $t - n$ $(1 \leq n \leq t)$ and stayed there until time period t, inclusive, while the SD, for the first time starting from the $t - n$ time period transitioned into the I channel in the t time period.

If $n = t$ the probability of the second event is

$$\delta_{j_0 j_1} F_\xi(t, i) Q_\eta(t, i_1, i)$$

If $1 \leq n \leq t-1$, then the second event can occur with the probability $Q_\eta(n, k, i) h_\xi(t-n, \ j_0, i) F_\xi(n, i)$ under the condition that at the moment $t - n$ the SD was located in the k channel. The distribution of the channel number, in which the SD was located at the $t - n$ time period, $P_\eta(t - n, \ j_1, k)$ using the full probability formula for a fixed n, $1 \leq n \leq$

$t-1$ and then adding upp the probabilities for all n, $1 \leq n \leq t$ will yield the probability of discovering the signal in the i channel in the second scenario.

Summing the probabilities of non interfering events, corresponding to the two cases for all numbers of i channels which are used by the transitioning SD, $1 \leq i \leq m$, will yield the the relationship (1) □

Theorem 1.3. *If $\xi(t)$, $\eta(t)$ are irreducible Markov chains with standard distribution $\left\{P_\xi(i), i=\overline{1,M}\right\}$, $\left\{P_\eta(i), i=\overline{1,N}\right\}$ respectively, then a limit exists*

$$K = \lim_{t \longrightarrow \infty} K_{j_0 j_1}(t) =$$

$$= \sum_{i=1}^{M} h_\xi(i) \left[P_\eta(i) + \sum_{n=1}^{\infty} F_\xi(n,i) \sum_{k \neq i} P_\eta(k) \cdot Q_\eta(n,k,i) \right] \quad (2)$$

independent of initial states of j_0, j_1, where

$$P_\xi(i) = \lim_{t \to \infty} P_\xi(t, j_0, i),$$

$$P_\eta(i) = \lim_{t \to \infty} P_\eta(t, j_1, i),$$

$$h_\xi(i) = \lim_{t \to \infty} h(t, j_0, i).$$

Proof. The proof is derived from theorem 1 and ergodic quality of standard distribution. While at the same time relationships [4, p, 8.30] are fair

$$\begin{aligned} P_\xi(i) &= \sum_{j=1}^{N} \alpha_{ji} P_\xi(j), \\ P_\eta(i) &= \sum_{j=1}^{M} \beta_{ji} P_\eta(j), \\ h_\xi(i) &= \sum_{j \neq 1} \alpha_{ji} P_\xi(j). \end{aligned} \quad (3)$$

□

Corollary 1.1. *If theorem 2 condition is fulfilled, the static coefficient of effectiveness equals*

$$C = \frac{K}{L}, \quad (4)$$

where

$$L = \lim_{t\to\infty} L_{j_0}(t) = \sum_{i=1}^{N} h_\xi(i).$$

1.1 Construction of enlarged systems

For the purpose of simplification of relationship (2) for large M a phase enlargement method is used [5, 6]. As en example, a random channel is fixed, for example c with number i, where the SD can be located. Markov chain $\eta(t)$ multiple states $\{1,\dots,M\}$ are broken into two classes $\{i\}$ and $\bar{i} = \{1,2,\dots,i-1,i+1,M\}$.In this case the first class consists only from I states, and the second one consists from other than I states.

Lemma 1.1. *If the following relationships are fulfilled*

$$\max_{j\neq i} \beta_{ji} / \min_{j\neq i} (1-\beta_{ji}) \ll 1, \quad \beta_{nk} \neq 0, n, k = \overline{1, M}, \tag{5}$$

then the time of Markov chain $\eta(t)$ in multidute of states $\bar{i}$ is closely describes by a geometric law of distribution with parameter $1 - y_i$, where

$$y_i = \sum_{j\neq i} P_\eta(j)\,\beta_{ji} / \sum_{j\neq i} P_\eta(j). \tag{6}$$

Proof. If conditions (5) are fulfilled, then transition into the i state is a rare condition, the process of transitioning amongst a multitude of $\bar{i}$ states until departure from it can be considered established, which in turn, on empirical level, justifies the use the phase enlargement method [6].

Class $\bar{i}$ is enlarged in a single state, which is also called $\bar{i}$. The functioning of the enlarged system according to the enlargement theorem is closely described by the stationary Markov chain $\eta^{(i)}$ with two possible states i, $\bar{i}$ and probability matrix of one step transitioning

$$\Lambda^{(i)} = \begin{pmatrix} \beta_{ii} & 1-\beta_{ii} \\ y_i & 1-y_i \end{pmatrix},$$

Where

$$y_i = \sum_{j\neq i} P_\eta(j)\,\beta_{ji} / \sum_{j\neq i} P_\eta(j).$$

Hence, the lemma statement is proved. □

At the same time, the definition of “more less” and “closely described” are discussed in [7]. The general limit theorems, on which the method of phase enlargement in relation to Markov and half-Markov processes is based upon, is being proved in [5].

1.2 Defining the task mathematical programming

Theorem 1.4. *Under condition if geometric distribution of time spent by SD in state* $\bar{i}$, $i = \overline{1,\ M}$, *the problem of optimizing the stationary coefficient of search effectiveness boils down to a non-linear programming problem in relation to variables* $x_i, y_i, i = \overline{1, M}$,

$$C = \frac{1}{L} \sum_{i=1}^{M} h_{\xi}(i)\, R(i) \to \max \tag{7}$$

$$\sum_{i=1}^{M} \frac{y_i}{x_i + y_i} = 1, \tag{8}$$

$$0 \le x_i \le 1, \quad 0 \le y_i \le 1, \quad i = \overline{1, M}, \tag{9}$$

$$\frac{x_i y_i}{x_i + y_i} \le \sum_{j \ne i} \frac{x_j y_j}{x_j + y_j}, \tag{10}$$

where

$$R(i) = \begin{cases} \frac{y_i}{x_i+y_i} + \frac{x_i y_i}{x_i+y_i} \frac{1}{d_i+x_i}, \quad if\ a_{ii} \ne 0; \\ \frac{y_i}{x_i+y_i}, \quad if\ a_{ii} = 0,\ y_i \ne 0; \\ 0, \quad if\ a_{ii} \ne 0,\ y_i = 0; \end{cases}$$

$$d_{i=} \frac{1}{a_{ii}} - 1,$$

for probabilities β_{ij} *transitions of SD from channel to channel for optimal search strategy is calculated:*

$$\beta_{ii} = x_i^*, i = \overline{1,\ M}$$

when $i \ne j$, β_{ij} *can always be found as a solution to the system of linear equations*

$$\begin{cases} c\dfrac{y_i^* x_i^*}{y_i^* + x_i^*} = \sum\limits_{j \ne i} \dfrac{y_j^*}{y_j^* + x_j^*} V \cdot \beta_{ji} \\ x_i^* = \sum\limits_{j \ne i} \beta_{ij}, \end{cases} \tag{11}$$

where $x_i^*, y_i^*, i = \overline{1,\ M}$ *in turn are the solution to the problem of mathermatical programming* (7)-(10).

Proof. Similar to proof of theorem 2, using Markov chain $\eta^{(i)}(t)$ instead of Markov chain $\eta(t)$for every fixed $i, i = \overline{1,\ M}$ yields

$$K = \sum_{i=1}^{M} h_\xi\left(i\right)\left[P_{\eta^{(i)}}\left(i\right) + P_{\eta^{(i)}}\left(\bar{i}\right)\sum_{n=1}^{\infty} F_\xi(n,i)Q_{\eta^{(i)}}(n,\bar{i},i)\right], \qquad (12)$$

for reaching the probability of first encounter by Markov chain $\eta^{(i)}\left(t\right)$ of i state the following relation

$$Q_{\eta^{(i)}}\left(n,\bar{i},i\right) = y_i(1-y_i)^{n-1} \qquad (13)$$

Implemented denomination

$$x_i = 1 - \beta_{ii}$$

System of equations for stationary distribution $P_{\eta^{(i)}}$ for Markov chain $\eta^{(i)}(t)$ is the following

$$\begin{cases} cP_{\eta^{(i)}}\left(i\right) = P_{\eta^{(i)}}\left(i\right)(1-x_i) + P_{\eta^{(i)}}(\bar{i})y_i \\ P_{\eta^{(i)}}\left(i\right) = P_{\eta^{(i)}}\left(i\right)x_i + P_{\eta^{(i)}}(\bar{i})(1-y_i) \\ P_{\eta^{(i)}}\left(i\right) + P_{\eta^{(i)}}\left(\bar{i}\right) = 1 \end{cases} \qquad (14)$$

Solving system (14) yields

$$P_{\eta^{(i)}}\left(i\right) = y_i/(x_i + y_i) \qquad (15)$$

From relations (4), (12), (15) taking into account denotations (13),(6) it is possible to obtain the expression for effectiveness coefficient C, shown previously in the left part of (7)

As SD is located in one of the M first channels

$$\sum_{i=1}^{M} P_{\eta^{(i)}}\left(i\right) = 1$$

From this follows (15) and the limit (8)

Limit (9) – is the consequence of standard demands towards the probability of transition.

System of linear equations (11) is the consequence of relations (6) with (15), (8) taken into account and is denoted (13). It serves to determine the probabilities $\beta_{ij}, i \neq j$, of SD transitions from channel to channel using the given parameters $x_i^*, y_i^*,\ i = \overline{1, M}$. But sustem (11) not always yields a non negative solution.

Limitation (10) – is a needed and valuable condition of existence of non-negative solution to the $\beta_{ij}, i \neq j$ system of linear equations (11). Such solution is not singular [8, p.4] and must be chosen with conditions from (5) taken into account for $i = \overline{1, M}$. □

Note, if the solution to the problems of mathematical programming (7)-(10) some of the parameters y_i^* do not exist, then the probability of the SD to be located in these channels is zero and as follow, for optimal search strategy, the SD foes not transition into channels with these numbers.

1.3 Case of multiple channels with same probability characteristics in the same system of channels.

Consider a real important case, of a multi channels system with channel groups of the same type, meaning that the Markov chain $\xi(t)$has the same probability component types related to same type channels. In this case if channels I and j are same type channels then $a_{ik} = a_{jk}$,

$$a_{ki} = a_{kj}, k = \overline{1, M}.$$

Theorem 1.5. *If the first k channels of* $1 < k \leq M$ *are of the same type, then*

1. *for* $a_{11} \neq 0$ *the solution to the problem of mathematical programming* (7)-(10) *is showm t=by the relation*

$$y_i^* = y_j^*,\ i, j = \overline{1, k};\ x_i^* = 1,\ \text{if } y_i^* \neq 0,\ i = \overline{1, k}; \tag{16}$$

   or for optimal search strategy the stationary probabilities of SD being located is identical channels are equal, $P_{\eta^{(i)}}(i) = P_{\eta^{(i)}}(j)$, $i,\ j = \overline{1, k}$ *(if* $P_\eta(i) \neq 0$);

2. *for* $a_{11} = 0$ *the optimal search strategy may be chosen in such a manner, as to satisfy the relation* (16).

Proof. Let $x_i^*, y_i^*, i = \overline{1, M}$ be the solution to the problem of mathematical programing (7)-(10).

Consider the case $a_{11} \neq 0$

Study the auxiliary problem of mathematical programming related to variables $x_i,\ P_i,$ where $P_i = y_i/(x_i + y_i)$, $i = \overline{1, k}$, while fixing the remaining M-k variables in (7)-(10) in the following way: $x_i = x_i^*,\ P_i = y_i^*/(x_i^* + y_i^*),\ i = \overline{k+1, M}$,

$$C' = \sum_{i=1}^{k} \frac{x_i P_i (1 - P_i)}{d_i (1 - P_i) + x_i P_i} \to max \tag{17}$$

$$\sum_{i=1}^{k} P_i = D, \tag{18}$$

$$0 \leq x_i \leq 1,\ P_i \geq 0,\ x_i \leq \frac{1 - P_i}{P_i} \tag{19}$$

$$x_i P_i \leq \sum_{i \neq j} x_j P_j + S, \tag{20}$$

Where

$$D = 1 - \sum_{i=k+1}^{M} \frac{y_i^*}{x_i^* + y_i^*},\ S = \sum_{i=k+1}^{M} \frac{x_i^* y_i^*}{x_i^* + y_i^*}$$

Expression $f(x_i, P_i) = \frac{x_i P_i (1-P_i)}{d_i(1-P_i)+x_i P_i}$ under the sum sign in (17) with fixed $P_i > 0$ is an increasing fraction-linear function for x_i, and $f(x_i,\ 0) = 0$. Hence, taking into account the limitation $0 \leq x_i \leq 1$, with fixed P_1, $i = \overline{1,\ k}$, C' reahces maximum with $x_i = 1,\ i = \overline{1, k}$.

On the other hand, its not hard to verify that $\frac{\partial^2 (1,P)}{\partial P^2} < 0,\ P > 0$. Therefore $f(1, P)$ is strictly a convex upwardly function from P and therefore the following relation from [9] holds true

$$f\left(1, \sum_{i=1}^{k} P_i/k\right) \geq \frac{1}{k} \sum_{i=1}^{k} f(1, P_i),\ P_i > 0,\ i = \overline{1,\ k}$$

the equality is achieved only in case $P_i = P_j, i, j = \overline{1, k}$ i.e. tanking into consideration the limitation (18), for $P_i = D/k, i = \overline{1, k}$.

Note that as k>1, then the discovered $P_i < 1/2,\ i = \overline{1, k}$ and inequalities $1 \leq \frac{1-P_i}{P_i}$ and (20) are solved automatically. Hence, solution to (17)-(20) and therefore (7)-(10) in this case satisfy (16)

Consider the case $a_{11} = 0,\ D > 0$. Given:

$$K = h_\xi(1)\sum_{i=1}^{k}\frac{y_i}{x_i + y_i} + \sum_{i=k+1}^{M} h_\xi\,(i)\,R\,(i)\,.$$

Then the values $x_i^{**},\ y_i^{**},\ i = \overline{1, M}$are related in the following way

$x_i^{**} = 1,\ i = \overline{1, k};\ x^{**} = x^*,\ i = \overline{k+1, M};\ y_i^{**} = \frac{D/k}{1-D/k},\ i = \overline{1, k};\ y_i^{**} = y_i^*,\ i = \overline{k+1, M}$ and also are the solution to the problems of mathematical programming (7)-(10). □

Model example. Consider a case $M = N = 2,\ a_{11} \neq 0,\ a_{22} = 0$. Then $h_\xi\,(1) = h_\xi\,(2)\,,\ x_1 = y_2,\ y_1 = x_2$. Denote $y = y_1,\ P = \frac{y_2}{x_2+y_2}$, after transformation a simple problem os mathematical prograining is obtaines, equivalent to problems (7)-(10).

$$\begin{cases} \frac{P_y}{d_1+y} \to max \\ 0 \le y \le 1, 0 \le P \le \frac{1}{1+y} \end{cases}$$

which has the solution

$$y^* = \begin{cases} \sqrt{d_1},\ if\ \ 0 < d_1 \le 1, \\ 1, \qquad if\ \ d_1 > 1, \end{cases} ,P^* = \frac{1}{1+y^*}$$

Therefore for optimal search strategy:

$$\beta_{22} = \begin{cases} 1 - \sqrt{\frac{1}{a_{11}} - 1},\ when\ a_{11} \ge \frac{1}{2}; \\ 0,\ when\ a_{11} < \frac{1}{2}. \end{cases}$$

There exists no analytical solution for the problem for $N > 2$. For quantitative solution in case the quantity of channels is no less ten two an algorithm was developed and a corresponding program.

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