Збірник праць Ін-ту математики НАН України (2023) т. 20, №1, 843–872

The theory of dynamical systems of conflict in the framework of functional analysis

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Abstract. In this article, we give an introduction to the mathematical setting of problems related to the phenomenon of conflict in terms of constructions in Hilbert spaces. The struggle (conflict, game) between opponents (adversaries, players) will be represented by operator transformations of vectors in Hilbert spaces and probabilistic distributions on the territory of life resources. The phenomenon of conflict as a contradiction between opponents appears in mathematical terms as an intersection the domains of definition for operators and overlapping of corresponding measures.

Conflict interaction between opponents in the physical sense is described by the specific transformation of states in a Hilbert space. In turn, this is a mapping that changes the spectral measurements. Thus, a complex dynamical system arises, which we call a dynamical system of conflict. Then the following main problems arise as fundamental questions. What reasonable law of engagement (game or war) should be adopted to resolve the initial intersections? What is a fair limiting distribution of the resource territory?

In a more general formulation, solving conflict problems means the detailed describing of all possible outcomes on opponents states of the type: victories, defeats, states of equilibrium, compromises as fixed points together with their basins of attraction.

Анотація. Стаття присвячена введенню математичної постановки задач, пов'язаних із феномен конфлікту, в термінах конструкцій у гільбертових просторах. Боротьба (конфлікт, гра) між опонентами (супротивниками, гравцями) буде представлена операторними перетвореннями векторів у гільбертових просторах та ймовірнісними розподілами на просторі життєвих ресурсів. Феномен конфлікту, як суперечність між опонентами, в математичних термінах означає перетин областей визначення для операторів та перекриття носіїв відповідних мір.

Конфліктна, у фізичному сенсі, взаємодія між опонентами описується специфічним перетворенням станів у гільбертовому просторі. У свою чергу, ці відображення змінюють спектральні вимірювання. Таким чином, виникає складна динамічна система, яку ми називаємо динамічною

The author was partly supported in framework of the project "Mathematical modelling of complex dynamical systems and processes caused by the state security" (No. 0123U100853)

Keywords: dynamical system of conflict; opponent; Hahn-Jordan decomposition; rigged Hilbert space; equilibrium

DOI: http://dx.doi.org/10.3842/trim.v20n1.530

системою конфлікту. Як основні проблеми постають наступні фундаментальні питання. Який розумний закон конфліктної взаємодії (гри чи війни) треба прийняти для розв'язання початкових перетинів? Яким має бути справедливий граничний розподіл території життєвого ресурсу?

У більш загальному формулюванні, розв'язання конфліктних проблем означає детальний опис усіх можливих результатів про стани опонентів типу: перемоги, поразки, рівноваги, компроміси, як нерухомі точки, разом з басейнами їх притягання.

Having created the world,
God has appointed for everyone
a place at paradise, but
Devil invented a conflict

1. Introduction

1.1. **A bit of history.** The classical approaches to the conflict phenomenon and its applications have been discussed in many publications (see, for example, [5–8], [11], [14–17], [20], [10, 21–23], [27]). Here we shortly recall only some of well-known relating equations, models and versions:

– the Malthus-Verhulst population equation describing the dynamics of internal competition,

$$\frac{dP}{dt} = (b-d)P - cP^2,$$

– the logistic equation

$$x_{n+1} = rx_n\,(1-x_n)$$

which has been used to explain many population phenomena and existence of different evolution cycles,

– the Lotka-Volterra equations

$$\dot{N} = aN - bNP, \qquad \dot{P} = -cP + dNP$$

with wide spectrum of applications to behavior of hostile essences (for example, the predator-prey model).

Besides, there are many problems with collision situations reflected in the game theory (see, for example, [39] "Theory of Games and Economic Behavior", by John von Neumann and Oskar Morgenstern).

1.2. **Transition to a new vision.** However, it is impossible to understand the global picture of conflict phenomena on a universal scale, without moving from classical approaches to posing problems in terms of the modern theory of dynamic systems, using of Hilbert or Banach spaces, theory of self-adjoint operators and methods of functional analysis.

Today, it is obvious that a wide range of phenomena in the natural environment, such as, for example, biological populations or the dramatic evolution of society, can be understood at least partially only with the help of complex systems theory.

So, in world there are several powerful scientific centers to study of complex systems (of type Santa Fe Institute in Mexico). In this direction the main mathematical instruments are the non-linear analysis, the theory of dynamical systems, and the computer modelling. According to the theory developing in [19, 40] well constructed non-linear dynamical system has always enough equilibrium states, both repulsive and attractive which separate the phase space into zones with different regimes of behavior for trajectories. The equilibrium states here play the same role as eigenvectors for operator in the linear analysis.

Moreover, we have to use all powerful instruments already developed in mathematical physics and functional analysis. This transition to using of contemporary methods is similar to going from classical physics to quantum mechanics and quantum field theory.

One of the important feature of a new approach to description of results (instead of construction of deterministic trajectories) is the going to statistical (probabilistic) interpretation of results. It means that in general that it is impossible to predict, who will be a winner and how many he obtains in each single case of the game. Thus, all prediction results of conflict interactions will be presented in a form of probabilistic distributions on infinitely dimensional space. So, we need to use the methods of Hilbert or Banach spaces.

Further, a notion of the conflict transformation as a some mapping in the states space may be adequately represented by a specific linear operator in a Hilbert space. It should correspond to the physical process of conflict interaction between large (in a real, infinity) amount of alternative sides (opponents, players, agents).

We start with obvious remark that only two forms of rough interaction are observed in our living environment, namely, repulsive and attractive. It follows that the construction of a universal conflict transformation in the dynamical equations can be based on two operations corresponding to simple mathematical signs: *minus* and *plus*. This is similar to the creation and annihilation operators in quantum field theory. This is why we have to look for construction of equations with two basic transformations that represent the attraction and repulsion in each dynamical conflict model and that are analogous to the creation and annihilation operators in quantum field theory. It is important that these equations are necessarily non-linear, since the value of the terms corresponding to each opponent changes non-additively at each moment of the conflict game.

Finally, we note that despite the important impact of abstract outcomes on our understanding of various processes, especially in multi-component and multi-agent models represented by mean-field games with total payoffs, from the point of view of concrete applications, the results of an abstract theory are of little use. In fact, we need to develop a more advanced universal conflict theory of the type of axiomatic approach in quantum field theory.

In the next sections, we will consider only the simplest versions of conflict transformations without taking into account external influences. For more discussion and results see the constructions in [2], [4], [24]-[25], [28]-[37], [35].

2. The universal law of conflict interaction

Who should be here,
me or my enemy, that is the question

Here we discuss the mathematical writing of a heuristic version of the universal law of interaction between alternative opponents. In other words, we are trying to build a simple mapping that describes the elementary act of physical collision between abstract adversaries. In the following constructions of complex conflict systems, this map will be transformed into a more convenient and perfect form.

Consider a standard situation. Let A and B be two alternative opponents living in the common resource space Ω. Alternativeness means that any interaction between A and B occurs according to the law of mutual repulsion in the sense probable presence. Therefore, we will use the probabilistic approach.

Let $\mathbf{P}^A = \mathbf{P}^A(\Delta)$ ($\mathbf{P}^B = \mathbf{P}^B(\Delta)$) denote probability of presence A (B) in some disputed region $\Delta \subset \Omega$ at the initial moment of discrete time. It is then natural to expect that any single act of conflict between these opponents will change these initial probabilities to new ones determined by simple formulas:

$$\mathbf{P}^A_{\text{new}}(\Delta) = \mathbf{P}^A(1 - \mathbf{P}^B), \qquad \mathbf{P}^B_{\text{new}}(\Delta) = \mathbf{P}^B(1 - \mathbf{P}^A). \tag{2.1}$$

The right-hand sides of these equalities contain the products of two values: the probability of being in the Δ region for one of the rivals and the probability of not being in the same region for the second.

We call the law (2.1) a generalized formula of William Shakespeare, considering $\mathbf{P}^C_{\text{new}}(\Delta) = \mathbf{P}^C(1 - \mathbf{P}^C)$, $C = A, B$ as its usual variant, which was implemented in the logistic equation.

We take this universal law as the basic part of all formulas describing the conflict struggle. So, in specific models of dynamical systems of conflict, this law of struggle is consistently repeated until the moment of victory or defeat, or to a certain kind of balance (compromise), or, finally, to the achievement of cyclic orbits.

In particular, if continuous time is used, then the above heuristic law of conflict transformation can be written in the form of a system of nonlinear differential equations:

$$\frac{d}{dt}\mathbf{P}^A = \mathbf{P}^A(1 - \mathbf{P}^B), \qquad \frac{d}{dt}\mathbf{P}^B = \mathbf{P}^B(1 - \mathbf{P}^A).$$

To clarify this law, we are taking another important step: regionalization of the resource space of the conflict. Formally, this means the decomposition Ω to a set of partitioned regions:

$$\Omega = \bigcup_{i=1}^{m} \Omega_i, \quad 2 \leqslant m \leqslant \infty. \tag{2.2}$$

In fact, all real conflict processes take place in some space or territory that is always there is naturally divided into separate ones parts (we call them regions) that are separated from each other and in each of which there are purely local relations between the presence of conflicting parties (opponents). The structure of such partitions can be significantly different in the mathematical sense, from Ω_i as ideal subsets to manifolds with complex even fractal supports. In what follows we assume that some decomposition (2.2) are fixed. However, it may change in the course of conflict resolution similar as it happened when one state intervenes on another. Next, we assume that some separation (2.2) is fixed. However, this can change during conflict resolution, such as when one state intervenes in another. Moreover, in models describing the infinite repetition of biological populations that compete with each other, it is necessary to carry out a self-similar division of each region Ω_i at all moments of time $t = 1, 2, \ldots$:

$$\Omega_i = \Omega_{i_1} = \bigcup_{i_2=1}^{m} \Omega_{i_1 i_2}, \qquad \ldots, \qquad \Omega_{i_1 \cdots i_{k-1}} = \bigcup_{i_k=1}^{m} \Omega_{i_1 \cdots i_k}, \ k = t.$$

Here we will make only one step. Let $\mathbf{P}_i^A \equiv \mathbf{P}^A(\Omega_i, t)$ and $\mathbf{P}_i^B \equiv \mathbf{P}^B(\Omega_i, t)$ denote the independent probabilities of capturing region Ω_i by opponents A and B, respectively, at time t. Then, if we assume the uniformity of such distributions into regions Ω_i, then the above law takes the form of a system of $2m$ differential equations:

$$\frac{d}{dt}\mathbf{P}_i^A = \lambda \mathbf{P}_i^A(1 - \mathbf{P}_i^B), \qquad \frac{d}{dt}\mathbf{P}_i^B = \lambda \mathbf{P}_i^B(1 - \mathbf{P}_i^A), \quad i \in \overline{1, m},$$

where λ is a normalization factor.

In what follows we use discrete time, so instead of the above differential equations let's move on to the differences:

$$p_i^{t+1} = \lambda p_i^t(1 - r_i^t), \qquad r_i^{t+1} = \lambda r_i^t(1 - p_i^t), \quad t = 0, 1, \dots, \tag{2.3}$$

where the notation is entered, $p_i^t := \mathbf{P}^A(\Omega_i, t)$, $r_i^t = \mathbf{P}^B(\Omega_i, t)$. Since we use a statistical approach, vectors $p^t = (p_1^t, \dots, p_m^t)$, $r^t = (r_1^t, \dots, r_m^t)$ are stochastic for each t:

$$\sum_{i=1}^m p_i^t = 1 = \sum_{i=1}^m r_i^t.$$

It is not difficult to notice that the normalization coefficient λ in (2.3) depends on time and has the form

$$\lambda = 1/z^t, \qquad z^t = 1 - \theta^t, \qquad \theta^t := (p^t, r^t) = \sum_{i=1}^m p_i^t r_i^t.$$

Difference equations (2.3) establishes the simplest probabilistic law for conflict transformation which we will denote as $\star$. Models of dynamical conflict systems

$$\{p^0, r^0\} \xrightarrow{\star,\ t} \{p^t, r^t\},\ t = 1, 2, \dots$$

generated by the difference system equations of the type (2.3), has already been studied in a number of publications (see [28–34, 36, 37]). One of our main ones results confirm the convergence of all trajectories of the dynamic systems described above to equilibrium states. We call this result the conflict theorem. It can be formulated as follows (see [32] and references therein).

Theorem 2.1. *Each trajectory $\{p^t, r^t\}$ of CDS generated by the system of equations (2.3) starting with of an arbitrary point $\{p^0, r^0\}$ given by a pair of stochastic vectors p^0, r^0 which are different, $(p^0, r^0) \neq 1$, converges to the limit state (fixed point),*

$$\{p^t, r^t\} \longrightarrow \{p^\infty, r^\infty\},\ t \longrightarrow \infty,$$

which consists with two orthogonal vectors, $p^\infty \perp r^\infty$. That is,

– *if at the initial moment of time the inequality $p_i^0 > r_i^0$ was fulfilled for some coordinates, then*

$$p_i^\infty > 0, \qquad r_i^\infty = 0,$$

– *if $p_k^0 < r_k^0$, then*

$$p_k^\infty = 0, \qquad r_k^\infty > 0,$$

– *and if $p_j^0 = r_j^0$, then*

$$p_j^\infty = r_j^\infty = 0.$$

Values of non-zero boundary coordinates $p_i^\infty > 0$, $r_k^\infty > 0$ *are proportional to the initial differences* $d_i = p_i^0 - r_i^0$, $d_k = r_k^0 - p_k^0$, *i.e.*

$$p_i^\infty = d_i/D, \qquad r_k^\infty = d_k/D,$$

where the proportionality coefficient D *is independent of indices of non-zero coordinates.*

Thus, the struggle of opponents for possession of regions in space (2.2) is only able to redistribute the initial values probabilities of their presence in different regions in accordance with the law alternative conflict of the form (2.3). That is, the dominance of one of the players in some Ω_i leads to the disappearance of another in the same region. Presence is redistributed in such a way that opponents are located in different regions that do not overlap. Therefore, all trajectories of the dynamic system (2.3) converge to the equilibrium states given by orthogonal vectors in the sense of $\mathbb{R}^m$ if $m < \infty$ or in the Hilbert space l_2 if $m = \infty$. Any compromise states in pure alternative dynamics are impossible.

3. Connection with perturbation theory

Perturbation theory in mathematical physics provides a powerful tool for its use in conflict theory. First, let's briefly recall the background.

3.1. **Perturbed operators and their scattering.** Let a Hilbert space $\mathcal{H}$ be the state space of some physical system and H be its free Hamiltonian. For example, $\mathcal{H} = L_2(\mathbb{R}^3)$, and $H = -\Delta$ – the Laplace operator. Then, according to quantum mechanics, the dynamics is described by the Schrödinger equation. Its solutions are written by vector functions of the form $\Psi(t) = \exp^{-itH} \Psi(0)$, which describe the time evolution of the trajectories, starting from states $\Psi(0) \in \mathcal{H}$.

Let us now consider two perturbations V_1 and V_2 of H, which can be considered as the influence of alternative opponents on free evolution. Suppose both sums, $H_k = H + V_k$, $k = 1, 2$, are well-defined self-adjoint operators of $\mathcal{H}$. Then, again according to the abstract Schrödinger equation

$$i\frac{d}{dt}\Psi(t) = H_k\Psi(t),$$

there appear two different time evolutions:

$$\Psi_1(t) = e^{-itH_1}\Psi(0), \qquad \Psi_2(t) = e^{-itH_2}\Psi(0).$$

The physical collision between such different behaviors is described in perturbation theory by the so-called scattering operator

$$S = W^+(W^-)^*,$$

where

$$W^{\pm} = \lim_{t \to \pm\infty} e^{itH_2} e^{-itH_1}$$

is defined as the wave operators [26]. The existence problem of the wave and scattering operators are non-trivial and has long history. Especially in the case of singular perturbations (see, for example, [33]).

Here we want to look at H_k as operator strategies that correspond to the alternative behavior of a certain entity (population, political ideologies, opinions, etc.). Then $\Psi_k(t)$ represent its independent time evolutions with the initial state $\Psi(0)$. But instead of the collision in the above form of the scattering operator, we propose to find a new description of the confrontation between opposite sides in the form of a discrete-time sequence of acts of redistribution for the initial positions in the life space according to the universal nonlinear rule.

3.2. **Infinite-dimensional space of living resources.** Thus, we need to define the conflict composition $\star$ in terms of vector-states of Hilbert space,

$$\Psi_1(t) \star \Psi_2(t) = ?, \ t = 0, 1, 2, \ldots$$

The transformation $\star$ will correspond to interaction between alternative sides for presence in the living (resources) space $\mathcal{H}$.

One of the way is to engage an observation the position operator Q. Its spectrum, $\Omega := \sigma(Q)$, we will treat as the living (or resources) space for alternative opponents. Then the independent time evolutions in such space may be described in terms of probability measures

$$\mu_k^t(\Delta) := (\mathbb{E}_Q(\Delta)\Psi_k^t, \Psi_k^t), \qquad \Delta \in \mathcal{B},$$

where $\mathcal{B}$ is the Borel algebra of subsets on Ω, and $\mathbb{E}_Q(\Delta)$ denotes the operator spectral measure (the identity resolution of operator Q).

It is worth recalling the physical interpretation. According to the principles of quantum physics (see, for example, J. von Neumann "Mathematische Fundamentals of Quantum Mechanics") the values $\mu_1^t(\Delta)$, $\mu_2^t(\Delta)$ can be considered as the independent probabilities of finding opposite sides represented by states Ψ_i in the subset $\Delta \subseteq \Omega$. In other words, its are the mathematical expectations to observe (to measure) the presence of opponents with strategies H_1, H_2 in states Ψ_1^t, Ψ_2^t restricted to subspace $\mathbb{E}_Q(\Delta)\mathcal{H}$.

The conflict interaction causes a certain deformation of these independent expectations.

Now we will explain the idea for construction of the mathematical transformation $\star$ corresponding to a conflict interaction in a Hilbert space.

We suppose that each conflict dynamics has its own time, continuous or discrete, in general different from the independent (free) time evolutions of physical systems. Here we develop an abstract approach to definition of the conflict transformation. Let μ_i^t, ν_i^t denote a couple of probability measures of above type on the measurable space $(\Omega = \sigma(Q), \mathcal{B})$. Putting

$$\mathbf{P}^t_{\mu_1}(\Delta) := \mu_1^t(\Delta), \qquad \mathbf{P}^t_{\mu_2}(\Delta) := \mu_2^t(\Delta),\ \Delta \in \mathcal{B},$$

we define conflict transformation in terms of these measures:

$$\begin{Bmatrix} \mu_1^t \\ \mu_2^t \end{Bmatrix} \xrightarrow{\star,\ t} \begin{Bmatrix} \mu_1^{t+1} \\ \mu_2^{t+1} \end{Bmatrix} \qquad (\mu_1^0 = \mu_1,\ \mu_2^0 = \mu_2),$$

where we use discrete time $t = 0, 1, \ldots$. At each step, the measures are modified using a generated William Shakespeare type formula (2.1):

$$\mu_1^{t+1}(\Delta) = \mathbf{P}^{t+1}_{\mu_1}(\Delta) \simeq \mathbf{P}^t_{\mu_1}(\Delta) \cdot (1 - \mathbf{P}^t_{\mu_2}(\Delta)),$$
$$\mu_2^{t+1}(\Delta) = \mathbf{P}^{t+1}_{\mu_2}(\Delta) \simeq \mathbf{P}^t_{\mu_2}(\Delta) \cdot (1 - \mathbf{P}^t_{\mu_1}(\Delta)).$$

The inverse problem, i.e., the reconstruction of vectors Ψ_1^t, Ψ_2^t on the measures μ_1^t, μ_2^t will be considered in further sections.

3.3. **Singular perturbation as a cause of conflict.**

Conflict is caused
by the singular structure of matter

Here we show that alternative behavior strategies corresponding to the above operators H_k arise naturally under a singular perturbation of the free Hamiltonian.

Briefly, the splitting of unity (free Hamiltonian) into contradiction sides (H_k operators) can be described within the framework of singular perturbation theory.

In our approach, the phenomenon of conflict between alternative parties looks like a kind of explosion of free evolution, ”exit from paradise”. Mathematically, this means splitting the free Hamiltonian H on two or more branches of conflicting evolution. We associate this path with the singular perturbation H. But first we mention the usual approach again.

Let us consider a free energy operator H with domain $\mathcal{D}(H)$ in a Hilbert space $\mathcal{H}$. Let

$$\mathcal{D} = \{\Psi \in \mathcal{D}(H), \|\Psi\| = 1\}$$

denote some initial set of states for the physical system associated with operator H. We note that in general in real situation, each concrete physical system involves in the “life” not all vectors from $\mathcal{D}(H)$. So, $\mathcal{D}$ is only a part of $\mathcal{D}(H)$, but it is assumed that $\mathcal{D}$ is a dense subset in $\mathcal{H}$. Each vector $\Psi \in S$ has a pure deterministic free time evolution described by the Schrödinger equation, $\Psi(t) = \exp^{-itH} \Psi$.

If the physical influence on the system is described as a "rough" perturbation written in the form $H_1 = H + V_1$, where V_1 is a fairly good operator that has an interpretation of the external field, then the problem is to describe the perturbed picture of the new evolution, including the spectral analysis of H_1. Moving on to another perturbation V_2, we get some new evolution picture, $\Psi(t) = \exp^{-itH_2}\Psi$ evolution. There is no conflicting phenomenon on this way.

Further we need in the following

Definition 3.3.1 ([33]). A self-adjoint operator $\widetilde{H}$ is called *(purely) singularly perturbed with respect to* H if the linear set

$$\mathcal{D}_\Gamma = \{f \in \mathrm{Dom}(H) \cap \mathrm{Dom}(\widetilde{H}) \mid Hf = \widetilde{H}f\}$$

is dense in the Hilbert space $\mathcal{H}$.

Here, Γ is associated with some extremely small set in physical space that is responsible for the singular perturbation.

For more detailed facts connected with definition of singular perturbation see [33]

Formally every singular perturbed operator may appear in the following way. At first one consider a restriction of H into some dense linear subset $\mathcal{D}_\Gamma = \mathcal{D} \subseteq \mathcal{D}(H)$ with consequent extension to any a new self-adjoint operators H_i such that:

$$H_i \restriction \mathcal{D}_\Gamma = H \restriction \mathcal{D}_\Gamma, \quad i \geqslant 2.$$

We will always assume that $\mathcal{D}_i = \mathcal{D}_\Gamma \cap \mathcal{D}(H_i)$ are dense in $\mathcal{H}$.

Now, the evolution of the physical system associated with different operators H_i will have a certain kind of uncertainty because these operators are quite close (they are identical on a dense set) but still different (due to a singular perturbation on a very small set Γ). Hence, opposite and conflicting paths may arise for the evolutions started from the vectors $\Psi \in \cap_i \mathcal{D}_i$. The theory of the dynamic system of conflict is designed to give a description of this kind of evolution and to solve the problem of "fair" redistribution of the conflict territory $\Omega = \sigma(Q)$ which the spectrum of Q.

To this aim, we need to select and use an explicit law of conflict interactions that will govern the time dependence of the trajectories of the conflict dynamical system.

3.4. **The law of conflict interaction in terms of states.** Here we will describe one of the possible variants of conflict dynamics in terms pairs of non-orthogonal states $\Psi \in \mathcal{D}_1$, $\Phi \in \mathcal{D}_2$ which correspond to two conditional opponents. It will be performed by a composition marked $\star$ and operating in the Hilbert space $\mathcal{H}$.

Let Q denote the self-adjoint operator in $\mathcal{H}$ corresponding to the observation, which is called a position. Its spectrum represents the resource area for all other observations.

Put in correspondence to a couple Ψ, Φ their spectral representations with respect the operator Q:

$$\begin{aligned}\Psi &\to \mu(\Delta) := (\mathbb{E}_Q(\Delta)\Psi, \Psi),\\ \Phi &\to \nu(\Delta) := (\mathbb{E}_Q(\Delta)\Phi, \Phi),\ \ \Delta \in \mathcal{B},\end{aligned}$$

where $\mathbb{E}_Q(\Delta)$ stands for the operator spectral measure of Q and $\mathcal{B}$ denotes the Borel σ-algebra. In what follows we denote this map by symbol K.

These probability measures μ, ν we interpret as two starting independent distributions for a couple of some opponents (individuals) along the territory $\Omega = \sigma(Q)$ where $\sigma(Q)$ denotes the spectrum of Q. If the states of the opponents are orthogonal, $\Psi \perp \Phi$, then the measure carriers μ, ν have a zero intersection and there is no conflict between the opponents. But if

$$\mathrm{supp}(\mu) \cap \mathrm{supp}(\nu) \neq \varnothing,$$

then the opponents start to struggle (conflict interaction) with the aim of displacing each other from the territory of joint coexistence.

The simplest version for the evolution law $\star$ of opponent sides,

$$\begin{Bmatrix}\mu^0\\ \nu^0\end{Bmatrix} \xrightarrow{\star,\ t} \begin{Bmatrix}\mu^t\\ \nu^t\end{Bmatrix}, \qquad (\mu^0 = \mu,\ \nu^0 = \nu),\ t \geqslant 0,$$

may be described by the nonlinear equations of a view:

$$\frac{d}{dt}\mu^t \simeq \Theta^t\mu^t - \eta^t, \qquad \frac{d}{dt}\nu^t \simeq \Theta^t\nu^t - \eta^t, \tag{3.1}$$

where $\Theta^t = \Theta(\mu^t, \nu^t) := (H\Phi^t, \Psi^t)$ stands for the multiplicative Hamiltonian of the system and $\eta^t = \eta^t(\mu, \nu)$ denotes the so-called *conflict occupation measure*. Here we used an isometric correspondence

$$K : \mathcal{H} \ni \Psi^t \longleftrightarrow \mu^t_\Psi \longleftrightarrow \psi(t) \in L_2(\Omega, d\mathbb{E}_Q(x)),\ x \in \Omega \tag{3.2}$$

($\psi(t)$ is the density of μ^t_Ψ with respect to the spectral measure of Q) which is constructed on the basis of the spectral theorem for the operator Q (see [9] for details). In fact formulas (3.1) are analogies of the products $\mathbf{P}^t_\mu(\Delta) \cdot (1 - \mathbf{P}^t_\nu(\Delta))$ from the above Shakespeare's formula.

Now we need to give some explanation about the concept of the conflict occupation measure in the abstract situation. Let μ, ν be a pair of positive measures on $(\Omega, \mathcal{B})$. Of course, we can assume that as above

$$\mu(\Delta) = (\mathbb{E}_Q(\Delta)\Psi, \Psi)_\mathcal{H}, \qquad \nu(\Delta) = (\mathbb{E}_Q(\Delta)\Phi, \Phi)_\mathcal{H}$$

with the assumption that $\mathrm{supp}(\mu) \cap \mathrm{supp}(\nu) \neq \varnothing$.

Suppose that the conflict territory Ω is separated into a finite amount of regions: $\Omega = \bigcup \Omega_i$. Then we define the conflict occupation (intervention) measure for the starting couple $\{\mu, \nu\}$ as $\eta := \eta_\nu + \eta_\mu$, where

$$\eta_\nu(\Delta) := \mathrm{Var}_\mu(\nu) = \sup_{\Delta = \cup \Delta_i} \sum_i \chi_\omega(\Delta_i)\nu(\Delta_i), \quad \Delta, \Delta_i \in \mathcal{B},$$

$$\chi_\omega(\Delta_i) = \begin{cases} 1, & \text{if } \omega(\Delta_i) \geqslant 0, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\eta_\mu(\Delta) := \mathrm{Var}_\nu(\mu) = \sup_{\Delta_i \subseteq \Delta \bigcap \Omega_i} \sum_i \chi_{-\omega}(\Delta_i)\mu(\Delta_i),$$

$$\chi_{-\omega}(\Delta_i) = \begin{cases} 1, & \text{if } -\omega(\Delta_i) \geqslant 0, \\ 0, & \text{otherwise} \end{cases}$$

where $\omega := \mu - \nu$ is the signed measure associated with μ, ν. In particular, if Δ is a set of *absolute domination* for μ:

$$\mu(\Delta') \geqslant \nu(\Delta'), \ \forall \Delta' \subseteq \Delta,$$

then $\eta_\nu(\Delta) = \nu(\Delta)$. And similarly for η_μ, if ν absolute dominates on some Δ, i.e.,

$$\nu(\Delta') \geqslant \mu(\Delta'), \ \forall \Delta' \subseteq \Delta,$$

then $\eta_\mu(\Delta) = \mu(\Delta)$. Thus, the value $\eta_\nu(\Delta)$ estimates the "intervention strength" of opponent for μ on a set where it has absolute dominance, and vice versa, $\eta_\mu(\Delta)$ has a similar but opposite meaning, it evaluates the strength of occupation of another opponent which is represented by μ on a set Δ, where now ν has absolute dominance.

It should be noted that in general the conflict occupation measure is not probabilistic, i.e., $\eta(\Omega) < 1$. In addition, we can say that conflict confrontation is very weak whenever $\eta(\Omega) \sim 0$ and extremely strong if $\eta(\Omega)$ is close to 1.

We propose some illustrative example: $\eta_\nu(\Delta)$ may estimate how many English-speaking persons lives in Ukraine, while $\eta_\mu(\Delta)$ gives values of Ukrainian-speaking persons there are in some fixed region Δ in the USA. These values refer to the initial time $t = 0$ and will change according to the conflict dynamics in a form (3.1).

Theorem 3.5 (Theorem of conflict in terms of states in a Hilbert space)**.** *Let states of a couple opponents at a time moment $t = 0$ are given by two unite non-orthogonal vectors $\Psi, \Phi \in \mathcal{H}$. Using the mapping* (3.2) *put in* *correspondence to* Ψ, Φ *measures* μ_Ψ, ν_Φ*. Assume that the set*

$$\operatorname{supp}(\mu_\Psi)\bigcap \operatorname{supp}(\nu_\Phi) \neq \varnothing.$$

Then the trajectory of the conflict dynamical system

$$\begin{Bmatrix}\Psi^0\\ \Phi^0\end{Bmatrix} \xrightarrow{\star,\ t} \begin{Bmatrix}\Psi^t\\ \Phi^t\end{Bmatrix}, \qquad (\Psi^0 = \Psi,\ \Phi^0 = \Phi), \quad t \geqslant 0,$$

with the conflict composition $\star$ *generated by equations of type* (3.1) *in terms of measures* $\mu_\Psi(\cdot) = (\mathbb{E}_Q(\cdot)\Psi, \Psi)$*,* $\nu_\Phi(\cdot) = (\mathbb{E}_Q(\cdot)\Phi, \Phi)$*, converges to a fixed point* $\{\Psi^\infty, \Phi^\infty\}$*. Thus, there exist the limits in the strong sense in* $\mathcal{H}$

$$\Psi^\infty = \lim_{t\to\infty} \Psi^t, \qquad \Phi^\infty = \lim_{t\to\infty} \Phi^t.$$

That is

$$\Psi^\infty \perp \Phi^\infty$$

and

$$\operatorname{supp}(\mu_{\Psi^\infty})\bigcap \operatorname{supp}(\nu_{\Phi^\infty}) = 0.$$

Proof. Here we give only some sketch of our arguments.

At first, we come from equations (3.1) to its difference variants, i.e., to the conflict dynamics at the discrete time

$$\begin{Bmatrix}\Psi^0\\ \Phi^0\end{Bmatrix} \xrightarrow{\star,\ t} \begin{Bmatrix}\Psi^t\\ \Phi^t\end{Bmatrix}, \quad t = 0, 1, \ldots$$

where each couple Ψ^t, Φ^t is defined by the iteration procedure in accordance with dynamics of the associated measures:

$$\begin{aligned} \mu_\Psi^{t+1}(\Delta) &= \frac{\mu_\Psi^t(\Delta)(1+\Theta^t) - \eta^t(\Delta)}{1+\Theta^t+W^t}, \\ \nu_\Phi^{t+1}(\Delta) &= \frac{\nu_\Phi^t(\Delta)(1+\Theta^t) - \eta^t(\Delta)}{1+\Theta^t+W^t}, \end{aligned} \tag{3.3}$$

where $\Delta \in \mathcal{B}$, $\Theta^t = (H\Psi^t, \Phi^t)$, and $W^t = \eta^t(\Omega)$.

Further, for proving of the limiting measures

$$\mu_\Psi^\infty = \omega^+/z_\mu, \qquad \nu_\Phi^\infty = \omega^-/z_\nu,$$

we use a suitable version of arguments (see, for instance [32]), where ω^+/z_μ, ω^-/z_ν denote the normalized components of the Hahn-Jordan decomposition for signed measure $\omega = \mu_\Psi - \nu_\Phi = \omega^+ + \omega^-$. And finally, we come back from $L_2(\Omega, d\mathbb{E}_Q(x))$ to the Hilbert space using the inverse transformation K^{-1}:

$$\Psi^\infty = K^{-1}\mu_\Psi^\infty, \qquad \Phi^\infty = K^{-1}\nu_\Phi^\infty. \qquad \square$$

A more general theorem is also true if we modify the conflict interaction law (3.3). Namely, instead of $\Theta^t = (H\Psi^t, \Phi^t)$, we take $\Theta_1^t = (H_1\Psi^t, \Phi^t)$ and $\Theta_2^t = (H_2\Psi^t, \Phi^t)$, where H_1, H_2 is a pair of singularly perturbed operators. They arise as a pair of self-adjoint extensions of the symmetric restriction H to the domain $\mathcal{D}_\Gamma$, where $\Gamma \subset \Omega$ is the zero set with respect to the spectral measure of the operator Q:

$$\int_\Gamma \|\phi(x)\|^2 d\mathbb{E}_Q(x) = 0, \quad \forall \phi \in L_2(\Omega, d\mathbb{E}_Q(x)).$$

3.6. **The method of rigged spaces.** Quite often, the conflict struggle arises between rather close, almost identical living conceptions. This kind of proximity can be accurately described in terms of singular perturbations of the general free Hamiltonian. Then, as in the real situation, each adversary will have its own Hamiltonian, which determines the strategy of its evolution in time.

Let us describe this approach in more details.

Let H be a strongly positive self-adjoint operator in a Hilbert space $\mathcal{H}$ and

$$\mathcal{H}_- \sqsupset \mathcal{H} \sqsupset \mathcal{H}_+ = \mathrm{Dom} H$$

denotes the associated rigged (equipped) space (for details see [9]). Here $\sqsupset$ stands for the dense inclusion. This rigged space is only some part of the so-called H-scale of Hilbert spaces,

$$\mathcal{H}_{-k} \sqsupset \mathcal{H} = \mathcal{H}_0 \sqsupset \mathcal{H}_{+k} = \mathrm{Dom} H^{k/2}, \quad k > 0.$$

The concept of a singular perturbation first appears in physical considerations (see [3]) related to the problem of the expression

$$-\Delta + \lambda\delta, \ \lambda \in \mathbf{R}^1,$$

where $-\Delta$ is the Laplace operator, and δ stands for the Dirac delta function treated as a singular one-point potential. The corresponding linear functional

$$l_\delta(\varphi) = \langle \varphi, \delta \rangle := \int_{\mathbb{R}^3} \delta_{x_0}(x)\varphi(x)dx = \varphi(x_0), \quad \varphi \in C(\mathbb{R}^3)$$

is singular in $L_2(\mathbb{R}^3, dx)$ since its null set $\ker(l_\delta)$ creates a dense domain in L_2. In an abstract approach, the singularity property was extended to quadratic forms in the rigged spaces.

Definition 3.6.1 ([33])**.** A positive quadratic form $\gamma(\cdot,\cdot)$ on $\mathcal{H}_+$ is called *singular* in $\mathcal{H}$ if for each $\Psi \in \mathcal{H}$ there exists a sequence $\varphi_n \in \mathcal{H}_+$ such that $\varphi_n \to \Psi$ and $\gamma[\varphi_n] = \gamma(\varphi_n, \varphi_n) \to 0$ as $n \to \infty$.

It is clear that a quadratic form γ is singular if its null set

$$\ker(\gamma) = \{\varphi \in \mathcal{H}_+ | \;\; \gamma[\varphi] = 0\} =: \mathcal{D}_0$$

creates a dense domain in $\mathcal{H}$, i.e. $\mathcal{D}_0^{\text{cl}} = \mathcal{H}$ (cl = closure). Thus, γ contains practically unobservable physical information.

Despite this "almost zero information", singular quadratic forms can exert a strong physical influence on the H Hamiltonian. In particular, singular quadratic forms can carry singular perturbations H, which significantly change the behavior of evolutions in time.

Let γ be fixed and its null set $\ker(\gamma) = \mathcal{D}_0$ be dense in $\mathcal{H}$. Then the restrictions

$$\mathbf{H} := H|\mathcal{D}_0$$

defines some symmetric operator in $\mathcal{H}$. We assume that its deficiency indices are equal and nonzero:

$$n^+(\mathbf{H}) = n^-(\mathbf{H}) \neq 0.$$

The family of its self-adjoint extensions

$$\{\widetilde{H} = \widetilde{H}^* \mid \widetilde{H}|_{\mathcal{D}_0} = \mathbf{H}_{\mathcal{D}_0}\}$$

represents all possible candidates for a singular perturbed operator. The resolvent of each operator $\widetilde{H}$ allows an explicit construction according to the so-called Krein's formula. We will give only the most famous sample of Krein's formula:

$$\widetilde{H}^{-1} = H_F^{-1} + \widetilde{B} P_{\mathcal{N}_0},$$

where $P_{\mathcal{N}_0}$ denotes the orthogonal projector onto the defect subspace $\mathcal{N}_0$ of $\mathbf{H}$, and $\widetilde{B}$ represents the extension parameter. It is some self-adjoint operator in $\mathcal{N}_0$. Above H_F denotes the Friedrichs extension of $\mathbf{H}$ which often coincides with $\mathbf{H}$. Depending on the operator $\widetilde{B}$, the corresponding singular perturbed extension $\widetilde{H}$ can have many new spectral properties in comparison with the original H.

We note that each operator $\widetilde{B}$ is closely connected with singular quadratic form γ. Since it is usually assumed that γ is continuous on $\mathcal{H}_+$ there exists the associated bounded operator $B_\gamma : \mathcal{H}_+ \to \mathcal{H}_-$. And all $\widetilde{B}$ in fact is also obtained using some singular quadratic form γ_B in the rigged Hilbert space.

It is important that all operators $\widetilde{H}$ are the same on the dense in $\mathcal{H}$ domain $\mathcal{D}_0$. So, one able to consider the conflict dynamical system with many, $m \geqslant 2$, opponents. Every of them will have its own strategy of behavior in a form of the operator H_i. Thus, for any family of self-adjoint extensions $\{H_i\}_{i=1}^m$ defined by the above formula,

$$H_i^{-1} = H_F^{-1} + \widetilde{B}_i P_{\mathcal{N}_0}, \tag{3.4}$$

we have

$$H_i\Psi = \mathbf{H}\Psi, \quad \Psi \in \mathcal{D}_0.$$

This fact causes a certain uncertainty of the evolution in time of arbitrary vectors $\Psi \in \mathcal{H}$, if they are interpreted as the states of some biological system. Since the singular perturbation is located on a very small, in the physical sense, set, we denote it Γ (for example, at a single point, as a delta potential, or on a fractal that has zero Lebesgue measure), it is difficult to take into account, it is not visible. From the point of view of biological essence, this kind of violation does not exist. But nevertheless, different elements of the biological population under the influence of a singular influence choose different strategies of behavior corresponding to Hamiltonian among the set $\{\widetilde{H}\}$. Thus, any singular perturbation can be interpreted as the cause of the splitting of the behavior of the biological population into different branches of evolution over time with subsequent confrontational struggle between them.

So, if we fix some family of self-adjoint extensions $\{H_i\}_{i=1}^m$, $m \geqslant 2$ and associate to each H_i some kind of "society" (as a linear set of vectors $\mathcal{D}_i$ from $\mathcal{H}$), then its evolution admits description similar to the previous ones for a couple of players. Thus, we have the Theorem of conflict for a complex system with many opponents fighting each one against all others.

Theorem 3.7. *For $\Psi_i \in \mathcal{H}$, $i \in \overline{1,m}$, put $\mu_i(\cdot) = (\mathbb{E}_Q(\cdot)\Psi_i, \Psi_i)$ and define the "mean field" measures $\nu_i(\cdot) = \frac{1}{m-1}\sum_{k\neq i}\mu_k$. Consider the conflict dynamical system with trajectories*

$$\{\mu_1^0, \dots, \mu_m^0\} \xrightarrow{\star,\ t} \{\mu_1^t, \dots, \mu_m^t\},\ t \geqslant 0$$

produced by formulas

$$\frac{d}{dt}\mu_i^t = \Theta_i^t\mu_i^t - \eta_i^t, \qquad \mu_i^0 = \mu_i, \tag{3.5}$$

where $\Theta_i^t := (H_i^{-1}\Psi_i^t, 1/(m-1)\sum_{k\neq i}\Psi_k)$ with H_i defined by the Krein formula (3.4) and $\eta_i = \eta_{\mu_i} + \eta_{\tilde{\nu}_i}$ is the occupation measure for a couple μ_i, ν_i. Assume that all differences

$$D_{ii'} = \widetilde{B}_i - \widetilde{B}_{i'}, \quad i, i' \in \overline{1,m}$$

are compact operators. Then, under some pure technical additional assumptions on the family μ_i, for each $\Delta \in \mathcal{B}$ there exist limits

$$\mu_i^\infty(\Delta) = \lim_{t\to\infty}\mu_i^t(\Delta), \qquad \mu_i^\infty \perp \mu_{i'}^\infty, \quad i \neq i'.$$

That is,

$$\mu_i^\infty = \omega_i^+/z_{\omega_i},$$

where ω_i^+/z_{ω_i} *denotes the normalized positive component of the Hahn-Jordan decomposition for family of signed measures* $\omega_i = \mu_i - \nu_i$*.*

In particular, for the vectors $\Psi_i^t := K^{-1}\mu_i^t$ *we have convergence in the strong sense:*

$$\lim_{t\to\infty} \Psi_i^t = \Psi_i^\infty,$$

where all vectors $\Psi_i^\infty := K^{-1}\mu_i^\infty$ *are orthogonal in* $\mathcal{H}$*.*

One may consider the family of the limiting vectors $\{\Psi_i^\infty\}_{i=1}^m$ as the equilibrium state for the starting conflict system.

The proof of this theorem faces new difficulties due to the fact that in (3.5) the functional Θ_i^t is not the same for all components, but depends on H_i. Besides, it is necessary to coordinate the procedure of the Hahn-Jordan decomposition for family of signed measures ω_i. In application this decomposition means the separation of the common living territory between alternative players.

We recall that according to the well-known Hahn-Jordan theorem [13], there exist two kinds of decomposition connected with a signed measure of a view $\omega_i = \mu_i - \nu_i$. Namely, for the set, $\Omega_{\omega_i} = \mathrm{supp}(\omega_i)$ and for ω_i, $\omega_i = \omega_i^+ - \omega_i^-$, where measures ω_i^+ and ω_i^- are orthogonal. The first decomposition has a form

$$\Omega_{\omega_i} = \Omega_{\omega_i^+} \bigcup \Omega_{\omega_i^-},$$

where $\Omega_{\omega_i^\pm} = \mathrm{supp}(\omega_i^\pm)$. That is

$$\Omega_{\omega_i^+} \bigcap \Omega_{\omega_i^-} = \varnothing,$$

and where the positive and negative components ω_i^+, ω_i^- are uniquely definition as follows,

$$\omega_i^+(\Delta) = Var_+(\omega_i, \Delta) := \sup_{\Delta'\subseteq\Delta} \omega_i(\Delta'),$$
$$\omega_i^-(\Delta) = Var_-(\omega_i, \Delta) := -\inf_{\Delta'\subseteq\Delta} \omega_i(\Delta'), \quad \Delta', \Delta \in \mathcal{B}.$$

This theorem states that

$$\mu_i^t(\Delta) \to \mu_i^\infty(\Delta) \geqslant 0, \qquad \text{if } \Delta \subseteq \Omega_{\omega_i^+},$$
$$\mu_i^t(\Delta) \to 0, \qquad \text{if } \Delta \subseteq \Omega_{\omega_i^-}.$$

The set

$$\Omega_{\Psi_1,\ldots,\Psi_m} = \bigcup_i \mathrm{supp}(\omega_{i=1}^m)$$

naturally to treat as the common living territory for m players associated with vectors $\Psi_1, \dots, \Psi_m$. Due to conflict transformation,

$$\Omega_{\omega_i^+} = \text{supp}(\omega_i^+) = \text{supp}(\mu_i^\infty),$$

and therefore the set $\Omega_{\Psi_1,\dots,\Psi_m}$ was separated into new family of regions without intersections:

$$\Omega_{\Psi_1,\dots,\Psi_m} = \bigcup_{i=1}^{m} \Omega_{\omega_i^+} = \bigcup_i \text{supp}(\mu_i^\infty).$$

Thus, the conflict is resolved since there appears the equilibrium state $\Psi_1^\infty, \dots, \Psi_m^\infty$ described in terms of the associate limiting measures:

$$\left\{ \begin{matrix} \mu_1^t \\ \vdots \\ \mu_m^t \end{matrix} \right\} \xrightarrow{\star,\ t} \left\{ \begin{matrix} \mu_1^\infty = \omega_1^+/z_{\mu_1} \\ \vdots \\ \mu_m^\infty = \omega_m^+/z_{\mu_m} \end{matrix} \right\}, \qquad \mu_i^\infty \perp \mu_{i'}^\infty,\ i \neq i'$$

where

$$\text{supp}(\mu_i^\infty) = \Omega_{\omega_i^+}.$$

The last equality shows that every measure ${\mu_i}^\infty$ is concentrated in region of the initial absolute domination of μ_i over all $\mu_{i'}$, $i' \neq i$.

On this way we may consider a model of an abstract society represented by two conflict clusters of vectors $S = S_1 \bigcup S_2$. Each subsystem S_1, S_2 contains a finite number, m_1, m_2, of players represented by unite vectors

$$\Psi_i \in S_1 \subset \mathcal{D}_1 \subset \mathcal{D}(H), \qquad \Phi_k \in S_2 \subset \mathcal{D}_2 \subset \mathcal{D}(H).$$

Then there appear two clusters of the associated probability measures:

$$\mu_i(\cdot) = (\mathbb{E}_Q(\cdot)\Psi_i, \Psi_i), \Psi_i \in \mathcal{D}_1, \qquad \nu_k(\cdot) = (\mathbb{E}_Q(\cdot)\Phi_k, \Phi_k), \Phi_k \in \mathcal{D}_2$$

and their "mean field" alternative variants

$$\tilde{\mu}_i = 1/m_2 \sum_k \nu_k, \qquad \tilde{\nu}_k = 1/m_1 \sum_i \mu_i.$$

Theorem 3.8 (Conflict between two clusters of opponents)**.** *All trajectories of the conflict dynamical system*

$$\{\mu_1^0, \dots, \mu_{m_1}^0, \quad \nu_1^0, \dots, \nu_{m_2}^0\} \xrightarrow{\star,\ t} \{\mu_1^t, \dots, \mu_{m_1}^t, \quad \nu_1^t, \dots, \nu_{m_2}^t\},\ t \geqslant 0,$$

generated by formulas of type

$$\frac{d}{dt}\mu_i^t = \Theta^t \mu_i^t - \eta_\nu^t, \qquad \frac{d}{dt}\nu_k^t = \Theta^t \nu_k^t - \eta_\mu^t,$$

converges to a fixed point (an equilibrium state):

$$\{\mu_1^\infty, \dots, \mu_{m_1}^\infty, \quad \nu_1^\infty, \dots, \nu_{m_2}^\infty\}, \qquad \mu_i^\infty \perp \nu_k^\infty,\ \forall i, k.$$

3.9. **Towards physical examples.** Let H denote the Hamiltonian of some physical system of some elements (particles) with $+/-$ charge or $+1/-1$ sign of spines.

Then subspaces $\mathcal{D}_i \subset \mathcal{D}(H),\ i = 1, 2$ differ due to chosen priority with respect to positive-negative charges or right-left sign of spins.

At starting time moment there are many vectors Ψ_i, Φ_k with the mixed distribution along an above physical property. The conflict interaction between possessing to one of them produces the separation all mixed states into two orthogonal subspaces:

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2.$$

The close pictures appear in the Ising model describing the behavior of nuts in a lattice of magnetic dipoles with $+1$ or -1 spins, as well as in potential theory which deals with the existence problem of minimizing signed measures supported on a condenser $A = A_+ \bigcup A_-$.

Else one example gives a model of quantum harmonic oscillator. Let

$$\mathcal{H} = L_2(\mathbb{R}^1, dx), \qquad H = 1/2m(Q^2 + P^2), \qquad Q \simeq x, \qquad P \simeq id/dx.$$

The operator H has the discrete spectrum:

$$H\mathbf{e}_i = \lambda_i \mathbf{e}_i, \qquad \lambda_i = (i + 1/2)\overline{\mathbf{h}}\omega, \qquad \omega \text{ is the oscillation phase,}$$

where

$$\mathbf{e}_i(x) \simeq \exp(-x^2/2)\mathbf{H}_i(x), \qquad \mathbf{H}_i(x) \text{ are Hermite polynomials.}$$

In the simplest case we consider the system $\{\Psi, \Phi\}$ with two unit vectors, $\Psi, \Phi \in \mathcal{D}(H)$:

$$\Psi = \sum_{i=0}^{\infty} a_i \mathbf{e}_i, \qquad \Phi = \sum_{i=0}^{\infty} b_i \mathbf{e}_i, \quad a_i, b_i \in \mathbb{C},$$

$$\|\Psi\| = \sum_i p_i = 1 = \sum_i r_i = \|\Phi\|, \qquad p_i = |a_i|^2, \qquad r_i = |b_i|^2$$

The conflict interaction between Ψ, Φ deforms their free dynamics given by the Schrödinger equation. The problem of "right" redistribution of starting priorities along the spectrum we write in a symbolic form as $\Psi^t \star \Phi^t$? In other words we regard Ψ, Φ as opponents whose projection weights (amplitudes) p_i, r_i on $\mathbf{e}_i$ show their priority relations with respect to eigenvalues λ_i.

Let us separate all basic oscillators $\mathbf{e}_i$ into two subsets $\mathcal{D}_\Psi, \mathcal{D}_\Phi$ using the priority domination produced by Ψ and Φ:

$$\mathbf{e}_i \in \mathcal{D}_\Psi, \text{ if } i \in N_\Psi = \{i \mid p_i \geqslant r_i\},$$

$$\mathbf{e}_k \in \mathcal{D}_\Phi, \text{ if } k \in N_\Phi = \{k \mid r_k > p_k\},$$

where recall, p_i (r_k) is a probability to find Ψ (Φ) in a pure state $\mathbf{e}_i$, ($\mathbf{e}_k$). Now we are able to define the occupation discrete measure $\eta(\Psi, \Phi) = \eta$:

$$\eta = (\eta_i)_{i=0}^{\infty}, \qquad \eta_i = \begin{cases} p_i, & \text{if } p_i \leqslant r_i \\ r_i, & \text{if } r_i < p_i \end{cases}$$

We assert the existence of an equilibrium limiting state for such conflict dynamics at discrete time. Namely, by the Theorem of conflict there exist the limits

$$\lim_{t\to\infty} p_i^t = p_i^{\infty}, \qquad \lim_{t\to\infty} r_k^t = r_k^{\infty},$$

such that $p_i^{\infty} = 0$, $i \in N_{\Phi}$, $r_k^{\infty} = 0$, $k \in N_{\Psi}$. The system $\{\Psi^{\infty}, \Phi^{\infty}\}$ with the limit vectors

$$\Psi^{\infty} = \sum_{i \in N_{\Psi}} a_i^{\infty} \mathbf{e}_i, \qquad \Phi^{\infty} = \sum_{k \in N_{\Phi}} b_k^{\infty} \mathbf{e}_k$$

creates a fixed point, $\Psi^{\infty} \perp \Phi^{\infty}$. It is easy to see that this equilibrium state is extremely unstable.

4. Connections with potential theory

In this section, we are going to substantiate our idea about the connection between the well-known Gaussian minimization problem in potential theory and the existence of a limit state of equilibrium in conflict theory. In other words, we want to show that the minimizing measure in the potential theory as the equilibrium charge on the capacitor essentially coincides with state of compromise redistribution in dynamical system of conflict (DSC). To formulate our goal in more detail, we need additional preparations.

But at first we recall shortly some facts from the classical theory of capacities of compact sets [12,38] following the papers [41–45]. This theory was initiated by Wiener and developed by many scientists, we remind only Frostman, Riesz, de la Vallee-Poussin. The modern notion of inner and outer capacities was originated by Cartan. He observed that the cone of all positive measures on $\mathbb{R}^3$ with finite Newtonian energy is complete in the energy norm. The using of the strong and the so-called vague topologies enabled Fuglede to extend a theory of capacities for measures on a locally compact space X for positive definite kernels κ on X. Ohtsuka developed this approach for vector-valued Radon measures $\mu_i, i \in I$ on X, where $\dim I \leqslant \infty$.

The last fact is important for application in theory of dynamical systems of conflict when we want to study the models with an arbitrary amount of opponents associated with vectors $\Psi_i, i \in I$ in a Hilbert space $\mathcal{H}$ of type $L_2(\Omega, d\mathbb{E}_Q(x))$ with $\Omega = X \subseteq \mathbb{R}^n$ where Q is so-called the position operator.

In [45] was developed and investigated the theory of inner capacities and inner capacitary measures and proved a series of theorem on convergence of above measures and their potentials for monotone families of sets. We used some results from [45] to prove Theorem 4.1 (see below).

Let $\mathcal{M}, \mathcal{M}^+$ denote the sets of all signed and positive measures on X, and $\mathcal{M}_1^+(X)$ denotes the set of all probability measures on X.

Let $G(x, y), x, y \in X$ be a positive definite kernel on a locally compact Hausdorff space X. For a given kernel G the function

$$U_G^\mu(x) \equiv G(x, \mu) := \int G(x, y) d\mu(y), \qquad \mu \in \mathcal{M}$$

is called the potential of a measure μ and the value

$$E_G[\mu] \equiv G(\mu, \mu) := \iint G(x, y) d\mu(x) d\mu(y)$$

is its energy.

Consider a compact set $K \subset X$. Given G and a compact set $K \subset X$, the value

$$c(K) \equiv \mathrm{cap}_G(K) := \left[\inf \{E_G[\mu] : \ \mu \in \mathcal{M}_1^+(X), \ \mu(K) = 1\}\right]^{-1}$$

is called [1, 12, 38] the *interior capacity of a set K with respect to G.*

The capacity of the set can be defined in another way [1] in terms of smooth functions,

$$c(K) := \inf\left\{\|\varphi\|^2_{\mathcal{H}_+} : \ \varphi \in C_0^\infty, \ \varphi(x) \geqslant 1 \text{ on } K\right\},$$

where $\mathcal{H}_+$ means a positive Hilbert space built on a given kernel G (in fact, it is some Sobolev space).

One of main results of the potential theory asserts that under rather wide assumptions on a set K there exists the so-called *equilibrium measure* $\gamma \in \mathcal{M}^+$ supported on K such that

$$c(K) = E_G[\gamma] = \|\gamma\|^2_{\mathcal{H}_+}$$

and

$$U_G^\gamma(x) = 1, \quad \forall x \in K,$$

where $\|\cdot\|_{\mathcal{H}_+}$ denotes the norm in the Hilbert space associated with G.

Further we assume that $X \subseteq \mathbb{R}^n$ and that a kernel G is perfect in the B. Fuglede's sense [18]. In particular, $G(x, y)$ is a symmetric and lower semi-continuous. So, using the notation from [45], we may put $G(x, y) = \kappa(x, y)$, where $\kappa(x, y)$ denotes one of the perfect kernels of type:

– Newton

$$\kappa(x, y) = |x - y|^{2-n}, \ n \geqslant 3,$$

– Riesz

$$\kappa(x, y) = |x - y|^{\alpha - n}, \ 0 < \alpha < n,$$

– or general Green ones

$$\kappa(x,y) = g_\Gamma(x,y), \quad \Gamma \subset \mathbb{R}^n.$$

For given strictly positive definite kernel κ on X we define the Hilbert space $\mathcal{H}_\kappa \equiv \mathcal{H}_+$ by the standard procedure of compactification and factorization of the linear space from signed measures $\mathcal{M}(X)$ with the inner product

$$(\omega_1, \omega_2)_{\mathcal{H}_\kappa} := \kappa(\omega_1, \omega_2)$$

and the norm $\|\omega\| = \sqrt{\kappa[\omega]}$.

According to [45] there are various ways for solution for "the problem of minimizing energy integrals over various unite charge distributions" on a set K with a presence of an extreme field. This problem is frequently referred to as the Gauss variational ones.

In particular, for arbitrary $K \subset \mathbb{R}^n$ and the kernel κ denote by $\mathcal{M}^+(K)$ the cone of all positive measures μ concentrated on K and which have finite energy $E_\kappa[\mu]$. It is known that this cone is strongly complete in the norm of $\mathcal{H}_\kappa$. Then for any $K \subset \mathbb{R}^n$ with finite inner capacity $c(K)$, there exists the equilibrium measure γ_K which is uniquely determined by the two relations

$$\kappa[\gamma_K] = \|\gamma_K\|^2_{\mathcal{H}_\kappa} = c(K),$$
$$\kappa(x, \gamma_K) = 1 \text{ on } K.$$

It is remarkable that similar kind of the result is true for a set K which is replaced by the condenser of a view $\Omega = \Omega_- \bigcup \Omega_+$ and instead of positive measures μ need to take signed measures ω. In the simplest reading, the solution of the minimizing problem meant that for wide kind of sets of type a condenser with finite κ-capacity there exists the signed measure

$$\gamma = \gamma_+ - \gamma_- \in \mathcal{M}(\Omega), \tag{4.1}$$

$\gamma_+ \in \mathcal{M}^+(\Omega_+)$, $\gamma_- \in \mathcal{M}^+(\Omega_-)$, such that

$$E_\kappa[\gamma] = \kappa(\gamma,\gamma) = c_\kappa(\Omega), \qquad \kappa(x,\gamma) = 1 \quad \gamma - \text{almost everywhere.}$$

Here $c_\kappa(\Omega)$ denotes a capacity of the condenser Ω with respect a given kernel $\kappa(x,y)$.

Our hypothesis is that the minimizing measure γ may be constructed as the limiting distribution in the DSC. In a symbol we are going to write (see below Theorem 4.1)

$$\mu_+^\infty = \gamma_+, \qquad\qquad \mu_-^\infty = \gamma_-,$$

with γ_+ and γ_- defined as follows:

$$\lim_{t\to\infty} \{\mu_A^t \star \mu_B^t\} = \{\mu_+^\infty, \mu_-^\infty\},$$

where $\star$ denotes the conflict interaction in the space of probability measures on Ω. In other worlds it means that the measures μ_A^t, μ_B^t corresponds to two alternative opponents A, B after the conflict interaction at the time moment t creates a sequence of signed measures ω^t which converges to the minimizing measure γ under a fixed condenser Ω. That is, the conflict interaction mapping $\star$ have to be constructed with using the kernel κ, and we extend our considerations to construction of the DSC in the Hilbert space $\mathcal{H}_\kappa$ with the inner product

$$(\omega_1, \omega_2)_{\mathcal{H}_\kappa} := \int \kappa(x,y) d(\omega_1 \otimes \omega_2)(x,y)$$

and the norm

$$\|\omega\|_{\mathcal{H}_\kappa} = \sqrt{\kappa(\omega, \omega)}.$$

So we formulate our intentions in the form of a theorem, although we do not have a complete proof of it today.

Theorem 4.1. *Let H, Q be a self-adjoint operators in Hilbert space $\mathcal{H}$. Assume the spectrum Q is absolutely continuous, $\sigma(Q) = \sigma_{ac} = X \subseteq \mathbb{R}^n$, $n \geqslant 1$, H is strongly positive, and its inverse has an integral representation in $L_2(\sigma(Q), d\mathbb{E}_Q(x))$ given by a kernel:*

$$(H_{-1}\Psi, \Phi) = (\mu_\Psi, \mu_\Phi)_{L_2(\sigma(Q), d\mathbb{E}_Q(x))} = \int_{\sigma(Q)} \kappa(x,y) d\mu_\Psi(x) d\mu_\Phi(y),$$

where

$$\mu_\Psi(\cdot) = (\mathbb{E}_Q(\cdot)\Psi, \Psi), \;\; \mu_\Phi(\cdot) = (\mathbb{E}_Q(\cdot)\Phi, \Phi)$$

and the associated with H^{-1} integral kernel $\kappa(x,y)$ is perfect in sense Fuglede (see [18]).

Let H_A, H_B denote a couple self-adjoint extension of the symmetric operator

$$\mathbf{H} := H_{\mathcal{D}_0}, \quad \mathcal{D}_0 \subset \mathcal{D}(H)$$

which has a nontrivial deficiency indices $n^+(\mathbf{H}) = n^-(\mathbf{H}) \neq 0$. These operators, H_A, H_B, are corresponded to the Hamiltonians of two opponent sides, A and B. They admit interpretations as the strategies of the dynamical behavior.

Let the DSC associated with H_A, H_B is fixed by a number of components $m_A + m_B = m < \infty$ and a decomposition of $\sigma(Q) \equiv \Omega$ into some kind of condenser:

$$\Omega = \Omega_- \bigcup \Omega_+, \qquad \Omega_- = \bigcup_{j \in N_-} \Omega_j, \qquad \Omega_+ = \bigcup_{i \in N_+} \Omega_i,$$
$$N_- = \{1, \ldots, m_B\}, \qquad N_+ = \{1, \ldots, m_A\}, \tag{4.2}$$

such that

$$\Omega_i^{\rm cl} \bigcap \Delta_j^{\rm cl} = \varnothing, \ i \neq j.$$

The states of this DSC are consisted from a set of m vectors

$$\{\Psi_i^t \in \mathcal{D}(H_A)\}_{i \in N_+}, \qquad \{\Phi_j^t \in \mathcal{D}(H_B)\}_{j \in N_-}$$

which change

$$\begin{Bmatrix} \Psi_i^t \\ \Phi_j^t \end{Bmatrix} \xrightarrow{\star} \begin{Bmatrix} \Psi_i^{t+1} \\ \Phi_j^{t+1} \end{Bmatrix},$$

in the discrete time $t = 0, 1, \dots$ in accordance with the law of conflict interaction $\star$ given in the terms of associated measures:

$$\begin{aligned} \mu_{\Psi_i}^{t+1}(\Delta) &= \frac{\mu_{\Psi_i}^t(\Delta)(1+\Theta^t) - \eta_i^t(\Delta))}{1+\Theta^t+W_A^t}, \\ \mu_{\Phi_j}^{t+1}(\Delta) &= \frac{\mu_{\Phi_j}^t(\Delta)(1+\Theta^t) - \eta_j^t(\Delta))}{1+\Theta^t+W_B^t}, \end{aligned} \tag{4.3}$$

where $\Delta \in \mathcal{B}$,

$$\Theta^t = \sum_{i \in I_A, j \in I_B} (H\Psi_i^t, \Phi_j^t), \quad W_A^t = \sum_i \eta_i^t(X), \quad W_B^t = \sum_j \eta_j^t(X), \tag{4.4}$$

and where the occupation measures $\eta_i^t(\cdot)$ and $\eta_j^t(\cdot)$ are defined under assumption that every Ω_i is a set of absolute domination for μ_{Ψ_i} in the sense that:

$$\mu_{\Psi_i}(\Delta') \geqslant \nu_i(\Delta'), \quad \forall \Delta' \subseteq \Omega_i, \tag{4.5}$$

and similarly every Ω_j, $j \in I_B$ is a set of absolute domination for μ_{Φ_j}:

$$\mu_{\Phi_j}(\Delta') \geqslant \nu_j(\Delta'), \quad \forall \Delta' \subseteq \Omega_j, \tag{4.6}$$

where measures ν_i, ν_j are defined as the "mean fields" created by the opponent sides:

$$\nu_i(\Delta) = 1/m_B \sum_{j \in I_B} \mu_{\Phi_j}(\Delta) + 1/(m_A - 1) \sum_{k \neq i} \mu_{\Psi_k}(\Delta),$$

$$\nu_j(\Delta) = 1/m_A \sum_{i \in I_A} \mu_{\Psi_i}(\Delta) + 1/(m_B - 1) \sum_{k \neq j} \mu_{\Phi_k}(\Delta).$$

That is, $\eta_i^t(\cdot)$, $\eta_j^t(\cdot)$ are defined as it was described in Subsection 3.4 starting with signed measures $\omega_i := \mu_{\Psi_i} - \nu_i$ and $\omega_j := \mu_{\Phi_j} - \nu_j$.

Then the minimizing problem for above fixed condenser $\Omega = \sigma(Q)$ of view (4.2) with above assumptions (4.5) and (4.6), admits unique solution γ_Ω in the following space of probability measures

$$\mathcal{M}_+^1(\Omega) \times \mathcal{M}_+^1(\Omega) :=$$

$$:= \{\mu_{\Psi_i},\ \mu_{\Phi_j}, \Psi_i \in \mathcal{D}(H_A), \Phi_j \in \mathcal{D}(H_B),\ \|\Psi_i\| = \|\Phi_j\| = 1\}.$$

In particular, it means that there exists the trajectory of DSC that converges to the equilibrium signed measure $\gamma_\Omega = \gamma_{\Omega_-} + \gamma_{\Omega_+}$ where

$$\gamma_{\Omega_+} = \lim_{t\to\infty} \sum_{i\in I_A} \mu^t_{\Psi_i}, \qquad \gamma_{\Omega_+} = \lim_{t\to\infty} \sum_{j\in I_B} \mu^t_{\Phi_j}.$$

Here we represent only some arguments for proving of the above theorem. At first, we remark that in according Theorems 3.5 and 3.7, there exist limiting measures

$$\mu^\infty_{\Psi_i} = \lim_{t\to\infty} \mu^t_{\Psi_i}, \qquad \mu^\infty_{\Phi_j} = \lim_{t\to\infty} \mu^t_{\Phi_j}, \qquad i \in I_A, j \in I_B,$$

which are supported on the corresponded subsets Ω_i or Ω_j, respectively. The last fact follows from assumption that for every starting measure μ_{Ψ_i} or μ_{Φ_j} the set Ω_i or respectively Ω_j is a set of absolute domination:

$$\mu_{\Psi_i}(\Delta') \geqslant \nu_i(\Delta'), \qquad \forall \Delta' \subset \Omega,$$
$$\mu_{\Phi_j}(\Delta') \geqslant \nu_j(\Delta'), \qquad \forall \Delta' \subset \Omega.$$

By this, the values of occupation measures $\eta^t_i(\Delta')$, $\eta^t_j(\Delta')$, which, we recall, estimate the "intervention strength" of opponents and the "strength of competition" with $\mu^t_{\Psi_k}$, $k \neq i$ and $\mu^t_{\Phi_k}$, $k \neq j$, respectively, on sets Ω_i, Ω_j converges to zero with $t \to \infty$ for any $\Delta' \subseteq \Omega_i$ and $\Delta' \subseteq \Omega_j$. And vice versa, by the same reason of absolute dominance, values of all measures $\mu^t_{\Psi_i}(\Delta')$ and $\mu^t_{\Phi_j}(\Delta')$ goes to zero when Δ' did not belong to Ω_i and Ω_j, respectively.

Note that $\eta_\mu(\Delta)$ has a similar but opposite meaning, it evaluates the strength of occupation of another opponent which is represented by μ on a set Δ, where now ν has absolute dominance.

Here it is worth recalling the property of absolute dominance in the abstract case. Consider a couple of probability measures $\mu, \nu \in \mathcal{M}^+(X)$, $\mu \neq \nu$. Assume that for some Δ one of the measures μ or ν has a local priority with respect other. It means that for any Borel $\Delta' \in \Delta$ following inequality is fulfilled:

$$\mu(\Delta') \geqslant \nu(\Delta') \qquad \text{or} \qquad \mu(\Delta') \leqslant \nu(\Delta').$$

Using this tool we introduce on of the main characteristic of an opponent, its dominant territory as the maximal subset where one of above inequalities are fulfilled. In the case of two measures, this problem has a solution in terms of the classical Hahn decomposition for the charge $\omega = \mu - \nu$.

In the theory for conflict dynamical systems with many alternative opponents associated with an arbitrary family of probability measures $\{\mu_i\}$ on the measurable space $(\Omega, \mathcal{B})$ there appears a similar task as a part of the equilibrium states problem. Importantly that supports of limiting measures corresponding to the equilibrium state just coincide with subsets from decomposition of Ω into the maximal regions of dominance for each measure μ_i over all others in the sense, $\mu_i(F) \geqslant \mu_k(F)$, $\forall k \neq i$.

We assert that under fixed integral kernel κ constructed by a suitable operator H, the minimizing measures γ_i for each subset Ω_i admits construction among a family of the limiting measures μ_i^∞ which appear in DSC with arbitrary sets of starting opponent components $m = m_A + m_B < \infty$.

It should be noted that above the decomposition (4.2) was originally fixed, while in theory the complex systems composed with a family of $m \geqslant 2$ advisories, say $A, B, C, \dots$ which are associated with probabilities measures μ_i, $i \geqslant 3$ on a space $(\Omega, \mathcal{B})$ the analogous decomposition

$$\Omega = \bigcup_{i=1}^{m} \mathbf{\Omega}_i^+$$

appears as result of long fighting. Here $\mathbf{\Omega}_i^+$ is not unique and dependents on the starting relations between measures μ_i. Therefore, it coincides with the union of all subsets of absolute dominance for one of the opponents. Equivalently, this means that every μ_i exceeds all other measures in the sense that

$$\mu_i(F) \geqslant \mu_k(F), \quad \forall k \neq i, \quad \forall F \subseteq \mathbf{\Omega}_i^+ \bigcap \mathcal{B}.$$

Although the same set also appears as support of the limit measure μ_i^∞.

In fact, it is a nontrivial problem to generalize the above kind of decomposition for a case of several opponents. From the one hand side it may be performed using the classic Hahn-Jordan decomposition for each couple of measures μ_i, ν_i which define a signed measure $\omega_i = \mu_i - \nu_i$, where

$$\nu_i := 1/(m-1) \sum_{k \neq i} \mu_k.$$

The difficulties connected with a fact that $\mu_i(F) \geqslant \nu_i(F)$ does not imply that $\mu_i(F) \geqslant \mu_k(F)$ for $k \neq i$ and therefore the set of absolute domination for μ_i in general is less than a positive subset of ω_i. So we need to extend the classic Hahn-Jordan decomposition into positive and negative components, $\omega = \omega_- + \omega_+$, on the case of multipolar expansions for a family of signed measures: $\omega_{i,k} = \mu_i - \mu_k,\ i, k \in \overline{[1, m]}, m \geqslant 3$. On the other hand in the theory of DSC with many opponents, this problem is closely related to the problem of finding equilibrium states and the limiting redistribution of resource space as a result of the conflict interaction.

4.2. **A case of conflict interaction between a finite number of abstract societies.** Let A_i, $i \in \overline{1, m}$ denote $m > 1$ abstract societies living in the resource territory $\Omega \subset X$, which is a compact set in $\mathbb{R}^n$. Match each

A_i with a set of some individuals. Mathematically, this means a family of vectors Ψ_{α_i}, $\alpha_i \in I_i < \infty$ in the Hilbert space $\mathcal{H}$ or probability measures $\mu_{\alpha_i} \in \mathcal{M}_1^+(\Omega)$ associated with the vectors Ψ_{α_i} in the spectral representation of the position operator Q with $o(Q) = \Omega$. We assume that each A_i has priority in some region $\Omega_i \subset \Omega$ of the resource area. In other words, Ω_i denotes the domain of absolute dominance for the family of measures μ_{α_i}:

$$\Omega_i = \cup_F \{\mu \in \mathcal{M}_1^+(X) \mid \mu_{\alpha_i}(E) \geqslant \mu_{\alpha_k}(E),\ \forall E \subseteq F,\ \forall k \neq i\}.$$

Let $\{A_i, \Omega = \bigcup_i \Omega_i,\ \star\}$ denote the dynamical system of conflict between abstract societies A_i. Here $\star$ and corresponds to the law of conflict interaction and controls the behavior of trajectories in terms of vectors $\Psi^t_{\alpha_i}$ or measures $\mu^t_{\alpha_i}$, $t = 0, 1, \ldots$ The mathematical definition of $\star$ depends on the specific real model. In general, in what form this mapping is implemented, the question is open. It can be constructed similarly to formulas (4.3) with additional terms corresponding to the division of the entire system into clusters. The question about the form of the energy functional Θ^t, see (4.4), is especially important since it is defined by some Hamiltonian H whose inverse H^{-1} and its quadratic form $(H^{-1}\cdot, \cdot)_{\mathcal{H}}$ is used for construction of the integral kernel $\kappa_H(\cdot, \cdot) \equiv \kappa(\cdot, \cdot)$. The question of the form of the energy functional Θ^t, see (4.4), is particularly important, since it is determined by some Hamiltonian H, the inverse of which H^{-1} and its quadratic form $(H^{-1}\cdot, \cdot)_{\mathcal{H}}$ is used to construct the integral kernel $\kappa_H(\cdot, \cdot) \equiv \kappa(\cdot, \cdot)$.

Thus, under all technical assumptions about the structure of the resource space $\Omega = \bigcup_i \Omega_i$, as a generalized condenser, the kernel κ_H, and the properties of the potential function $\kappa(x, \mu)$, where the measures of μ we take from $\mathcal{M}_1^+(\Omega)$ (for details see [41–45]), we can formulate our hypothesis in the form of a theorem as follows.

Theorem 4.3. *The minimizing measure $\gamma_\Omega = \sum_i \gamma_i$ for which*

$$\kappa(\gamma_\Omega, \gamma_\Omega) = \|\gamma_\Omega\|^2 \quad \text{and} \quad \kappa(\gamma_i, \gamma_i) = c_\kappa(\Omega_i),$$

where $c_\kappa(\cdot)$ is the capacity of the set Ω_i can be defined among the family of limit measures μ_i^∞ that arise in the above-described dynamical system of conflict with arbitrary sets of starting combinations $\mu_i \in \mathcal{M}_i(\Omega)$.

It is important that the minimizing measure γ_Ω as an equilibrium state of the conflict system allows approximate construction using iterative sequences according to formulas of the type (4.3).

Finally, we note that an essential technical trick in the proof of the above theorem is based on the notion of inner balayage for measures in the sense of the following definition [45].

For the set $\Omega_i \subset \Omega$, the inner balayage μ^{Ω_i} $\mu \in \mathcal{M}^+(\Omega)$ is defined as the measure with minimum energy $\kappa(\nu, \nu)$ in the class

$$\Gamma^+_{\Omega_i,\mu} := \{\nu \in \mathcal{M}^+(\Omega) \mid \kappa(\cdot, \nu) \geqslant \kappa(\cdot, \mu) \text{ on } \Omega_i\},$$

i.e.

$$\|\mu^{\Omega_i}\|^2 = \min_{\nu \in \Gamma^+_{\Omega_i,\mu}} \|\nu\|^2.$$

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