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Advances in theory of evolution equations of many colliding particles

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Abstract. The review presents rigorous results of the theory of fundamental equations of evolution of many-particle systems with collisions and also considers their connection with nonlinear kinetic equations describing the collective behavior of particles in scaling approximations.

Анотація. В огляді подано строгі результати теорії фундаментальних еволюційних рівнянь систем багатьох частинок із зіткненнями, а також розглянуто їх зв'язок із нелінійними кінетичними рівняннями, які описують колективну поведінку частинок у скейлінгових наближеннях.

Contents

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1. Preface

This review presents the modern theory of evolution equations for systems of many particles with collisions. The traditional approach to describing the evolution of both finitely and infinitely many classical particles is based on the description of the evolution of all possible states by means of the reduced distribution functions governed by the BBGKY hierarchy (Bogolyubov–Born–Green–Kirkwood–Yvon), which for finitely many particles is an equivalent to the Liouville equation for the probability distribution function [14, 82, 83].

As is known, in the basis of the description of many-particle systems, there are notions of the state and the observable. The functional of the mean value of observables defines a duality between observables and states, and as a consequence, there exist two approaches to the description of evolution. Thus, an equivalent approach to the description of evolution is to describe the evolution by means of observables governed by the so-called dual BBGKY hierarchy for reduced observables [10, 51].

In certain situations [14], the collective behavior of many-particle systems can be adequately described by the kinetic equations. The conventional philosophy of the description of kinetic evolution consists of the following: if the initial state is specified by a one-particle reduced distribution function, then the evolution of the state can be effectively described by means of a one-particle reduced distribution function governed by the nonlinear kinetic equation in a suitable scaling limit. A well-known historical example of a kinetic equation is the Boltzmann equation, which describes the process of particle collisions in rarefied gases [13].

Nowadays, a number of papers have appeared that discuss possible approaches to the rigorous description of the evolution of many colliding particles [25, 29, 52, 88, 89, 92, 94]. In particular, this is related to the problem of the rigorous derivation of the Boltzmann kinetic equation from the underlying hierarchies of fundamental evolution equations. The conventional method of deriving the Boltzmann equation consists of constructing the Boltzmann–Grad scaling asymptotics of the BBGKY hierarchy solution represented by series expansions within the framework of perturbation theory. The most advanced and rigorous results to date have been obtained for systems of colliding particles, which is why the motivation for writing this work is to discuss the theory of evolution equations for such many-particle systems [4, 52, 98].

In what follows, we mainly consider three challenges are left open until recently [52].

One of them is related to the construction of solutions to the Cauchy problem for hierarchies of fundamental evolution equations for systems of many particles with collisions, using the example of hard spheres with elastic collisions. It is established that the cluster expansions of groups of operators for the Liouville equations for observables and a state of many hard spheres underlie the classification of possible non-perturbative solution representations of the Cauchy problem for the dual BBGKY hierarchy and the BBGKY hierarchy, respectively. As a consequence, these solutions are represented in the form of series expansions whose generating operators are the cumulants of the groups of operators for the Liouville equations. In a particular case, the non-perturbative solutions of these hierarchies are represented in the form of the perturbation (iteration) series as a result of applying analogs of the Duhamel equation to their generating operators. The paper also formulated the Liouville hierarchy of evolution equations for correlation functions of the state and established that the dynamics of correlations underlie the description of the evolution of an infinitely many hard spheres that are governed by the BBGKY hierarchy for the reduced distribution functions or the hierarchy of nonlinear evolution equations for the reduced correlation functions [53].

Another challenge considered below is an approach to the description of the kinetic evolution within the framework of the evolution of the observables of many colliding particles [51]. The problem of a rigorous description of the kinetic evolution of hard sphere observables is considered by giving the example of the Boltzmann–Grad asymptotics of the non-perturbative solution of the dual BBGKY hierarchy. One of the advances of this approach is the opportunity to construct kinetic equations, taking into account the correlations of particles in the initial state and also the description of the process of propagation of initial correlations in scaling approximations.

In addition, this paper discusses the approach to describing the evolution of a state by means of the state of a typical particle of a system of many hard spheres, or, in other words, we consider the origin of the description of the evolution of the state of hard spheres by the Enskog-type kinetic equation [47]. One of the applications of the method is related to the challenge of the rigorous derivation of kinetic equations of the non-Markovian type based on the dynamics of correlations, which allow us to describe the memory effects in complex systems.

Thus, this review presents rigorous results in the theory of fundamental evolution equations for many-particle systems with collisions, as well as nonlinear kinetic equations describing their collective behavior in scaling approximations.

1.1. **A chronological overview of the theory of evolution equations for many colliding particles.** The theory of kinetic equations begins with the work of L. Boltzmann [9], where an evolution equation for collision dynamics was formulated based on phenomenological models of kinetic phenomena. Later, to generalize the Boltzmann equation for the case of dense gases or fluids, D. Enskog [26] formulated a kinetic equation for a system of many hard spheres, now known as the Enskog equation.

The idea that equations formulated on the basis of phenomenological models of phenomena, such as hydrodynamics equations or kinetic equations, should be derived from fundamental evolutionary equations for systems of many particles, namely the Liouville equations, apparently goes back to the works of D. Hilbert [76] and H. Poincaré [85]. At the Second International Congress of Mathematicians, held in Paris at the beginning of the 20th century, D. Hilbert formulated this idea in his list of open questions as follows: "Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes that lead from the atomistic view to the laws of motion of continua".

The approach to describe the evolution of the state of many-particle systems in a way equivalent to the Liouville equation for the probability distribution function based on the hierarchy of evolution equations for reduced distribution functions, known in our time as the BBGKY hierarchy, was most consistently formulated in the work of M. M. Bogolyubov [6], and independently by M. Born and H. S. Green [11], J. G. Kirkwood [77], J. Yvon [100].

In his famous monograph "Problems of the Dynamical Theory in Statistical Physics" [6], which was actually the manuscript of a 1945 report at the Institute of Mathematics in Kyiv, M. M. Bogolyubov also formulated a consistent approach to the problem of deriving kinetic equations from the dynamics of many particles. Using the methods of perturbation theory, an approach was developed to construct a generalization of the Boltzmann equation, known as the Bogolyubov kinetic equation, as well as justify the Vlasov and Landau kinetic equations for the first time. Thanks to this work, the irreversibility mechanism of the evolution of systems of many particles, whose dynamics are described as reversible in time by the fundamental equations of motion, became clear. A little later, in the Proceedings of the Institute of Mathematics, M. M. Bogolyubov published a paper on the derivation of the equations of hydrodynamics from the BBGKY hierarchy [5]. These works became widely known as a result of G. E. Uhlenbeck's lectures [16]. Bogolyubov's ideas became the cradle of modern kinetic theory, as M. H. Ernst noted in his review [27]. The Bogolyubov method of deriving the Boltzmann kinetic equation directly from the BBGKY hierarchy is presented in modern terminology in the book [14].

Since these lines are written in the work dedicated to the 160th anniversary of the birth of Dmytro Oleksandrovych Grave, the first academician of the Ukraine Academy of Sciences in mathematics and the founder of the Institute of Mathematics in 1920, it should be reminded that M. M. Bogolyubov was one of the students of D. O. Grave, thanks to whom he became an outstanding scientist in the field of mathematical physics. It is known that in 1922 at the age of thirteen, M. Bogolyubov became a participant in Grave's famous mathematician seminar. In 1925, at the request of professor D. O. Grave, the Small Presidium of Ukrgolovnauka made a decision: "In view of his phenomenal abilities in mathematics, to consider M. Bogolyubov as a post-graduate student of the research department of mathematics in Kyiv from June, 1925", and already in 1928 he defended his doctoral thesis. By the way, this historical precedent convincingly illustrates the significance of a scientific school for the development of mathematics.

Rigorous methods for the description of the equilibrium state by the Gibbs distribution functions [71], i.e., by solutions of the steady BBGKY hierarchy, originate from the works of M. M. Bogolyubov and D. Ya. Petrina within the framework of a canonical ensemble [7] and D. Ruelle within the framework of a grand canonical ensemble [90] and were investigated in numerous works as a new direction of the progress of modern mathematical physics in the 70-80s. In our time, mention above work of M. M. Bogolyubov, D. Ya. Petrina and B. I. Khatset was included in the special issue of the Ukrainian Journal of Physics dedicated to the 90th Academy of Sciences of Ukraine, which was was republished the most significant works of Ukrainian physicists over the entire period of the Academy's existence, in other words, works that contributed to the golden fund of world physical science (Golden Pages of Ukrainian Physics [8]).

Note that the mathematical description of the Gibbs equilibrium states for infinitely many particles forms the principal part of modern statistical mechanics. The main rigorous results about the equilibrium Gibbs states were presented in the book [64].

The mathematical theory of the BBGKY hierarchy originates from the works of D. Ya. Petrina and V. I. Gerasimenko [56–58, 81, 82] in the early 80s. The dual BBGKY hierarchy for reduced functions of observables was introduced by V. I. Gerasimenko in the middle of the 1980s, and the theory of these evolution equations began to develop in the last two decades (see [52] and references therein).

Mathematical methods for deriving nonlinear kinetic equations from the BBGKY hierarchy began to develop intensively in the early 1980s [12, 79]. One of the achievements of this period was the formal derivation of the Boltzmann equation from the dynamics of an infinite number of hard spheres in the Boltzmann–Grad limit.

In the approach to the problem of deriving kinetic equations from particle dynamics, which was formulated by H. Grad [73] and has now become generally accepted, the philosophy of the description of kinetic evolution looks like this: if the initial state is specified by a one-particle distribution function, then the evolution of the state of many particles can be effectively described by means of a one-particle distribution function governed by a nonlinear kinetic equation in a certain scaling approximation.

The Boltzmann–Grad asymptotics of a solution of the BBGKY hierarchy, represented as an iteration series for infinitely many hard spheres, was first constructed by C. Cercignani [12] and O. E. Lanford [79] and rigorously justified in a series of papers [59–61, 83] by D. Ya. Petrina and V. I. Gerasimenko (some details are given in sections 2.2 and 3.1). Incidentally, it should be noted that the results of the papers [57, 59] were discussed with Academician M. M. Bogolyubov at that time and were submitted by him for publication.

Rigorous results of the theory of evolution equations for hard spheres and the derivation of the Boltzmann equation from the BBGKY hierarchy in the Boltzmann–Grad limit were summarized in monographs C. Cercignani, V. I. Gerasimenko and D. Ya. Petrina [14], C. Cercignani, R. Illner and M. Pulvirenti [15], H. Spohn [95] at the end of the 90th.

The last two decades of progress in solving the problem of rigorous derivation of kinetic equations from the collisional dynamics of particles are represented in numerous recent works [4, 20, 25, 28, 29, 87–89, 92, 94]. The challenges of this area of contemporary mathematical physics are also discussed in the latest review [52]. With respect to the modern progress in the theory of evolution equations of quantum many-particle systems, we refer to the overview [38].

In these notes, recent advances in the theory of evolutionary equations for many colliding particles will be considered; more precisely, we focus on the dynamics of many hard spheres with elastic collisions.

1.2. **Evolution equations of finitely many hard spheres.** The description of many-particle systems is based on the concepts of an observable and a state. The mean value functional (expectation values) of observables defines the duality between observables and a state, and as a result, there are two approaches to describing evolution. The evolution of the system of finitely many colliding particles considered below is governed by such fundamental evolution equations as the Liouville equation for observables or its dual equation for a state.

Within the framework of a non-fixed, i.e., arbitrary but finite average number of identical particles (non-equilibrium grand canonical ensemble), the observables and the state of a hard sphere system are described by the sequences of functions

$$A(t) = (A_0, A_1(t, x_1), \ldots, A_n(t, x_1, \ldots, x_n), \ldots)$$

at instant $t \in \mathbb{R}$ and by the sequence

$$D(0) = (D_0, D_1^0(x_1), \ldots, D_n^0(x_1, \ldots, x_n), \ldots)$$

of the probability distribution functions at the initial moment, respectively. These functions are defined on the phase spaces of the corresponding number of particles, i.e., $x_i \equiv (q_i, p_i) \in \mathbb{R}^3 \times \mathbb{R}^3$ is phase coordinates that characterize a center of the i hard sphere with a diameter of $\sigma > 0$ in the space $\mathbb{R}^3$ and its momentum and are symmetrical with respect to arbitrary permutations of their arguments. For configurations of a system of identical particles of a unit mass interacting as hard spheres the following inequalities are satisfied: $|q_i - q_j| \geqslant \sigma$, $i \neq j \geqslant 1$, i.e., the set

$$\mathbb{W}_n \equiv \big\{(q_1, \ldots, q_n) \in \mathbb{R}^{3n} \mid |q_i - q_j| < \sigma$$

for at least one pair $(i, j) : i \neq j \in (1, \ldots, n)\big\}$, $n > 1$, is the set of forbidden configurations.

A mean value functional of the observable of a hard sphere system is represented by the series expansion [83]:

$$\langle A \rangle(t) = (I, D(0))^{-1}(A(t), D(0)), \tag{1.1}$$

where the following abbreviated notation

$$(A(t), D(0)) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} A_n(t, x_1, \ldots, x_n) D_n^0(x_1, \ldots, x_n) dx_1 \cdots dx_n$$

was used and the coefficient $(I, D(0))$ is a normalizing factor (grand canonical partition function).

We remark that in the particular case of a system of $N < \infty$ hard spheres the observables and a state are described by the one-component sequences:

$$A^{(N)}(t) = (0, \ldots, 0, A_N(t), 0, \ldots)$$

and

$$D^{(N)}(0) = (0, \ldots, 0, D_N^0, 0, \ldots),$$

respectively, and therefore, functional (1.1) has the following representation

$$\langle A^{(N)}\rangle(t) = (I, D^{(N)}(0))^{-1}(A^{(N)}(t), D^{(N)}(0)) \equiv$$
$$\equiv (I, D^{(N)}(0))^{-1}\frac{1}{N!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^N} A_N(t, x_1, \ldots, x_N)D^0_N(x_1, \ldots, x_N)dx_1 \cdots dx_N,$$

where

$$(I, D^{(N)}(0)) = \frac{1}{N!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^N} D^0_N(x_1, \ldots, x_N)dx_1 \cdots dx_N$$

is the normalizing factor (canonical partition function), and it is usually assumed that the normalization condition $(I, D^{(N)}(0)) = 1$ holds.

Let $\mathcal{C}_\gamma$ be the space of sequences $b = (b_0, b_1, \ldots, b_n, \ldots)$ of bounded continuous functions $b_n = b_n(x_1, \ldots, x_n)$ that are symmetric with respect to permutations of the arguments $x_1, \ldots, x_n$, equal to zero on the set of forbidden configurations $\mathbb{W}_n$ and equipped with the norm:

$$\|b\|_{\mathcal{C}_\gamma} = \max_{n\geqslant 0}\frac{\gamma^n}{n!}\|b\|_{\mathcal{C}_n} = \max_{n\geqslant 0}\frac{\gamma^n}{n!}\sup_{x_1,\ldots,x_n}|b_n(x_1, \ldots, x_n)|,$$

where $0 < \gamma < 1$. We also introduce the space $L^1_\alpha = \oplus_{n=0}^\infty \alpha^n L^1_n$ of sequences $f = (f_0, f_1, \ldots, f_n, \ldots)$ of integrable functions $f_n = f_n(x_1, \ldots, x_n)$ that are symmetric with respect to permutations of the arguments $x_1, \ldots, x_n$, equal to zero on the set $\mathbb{W}_n$ and equipped with the norm:

$$\|f\|_{L^1_\alpha} = \sum_{n=0}^\infty \alpha^n \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} |f_n(x_1, \ldots, x_n)|dx_1 \cdots dx_n,$$

where $\alpha > 1$ is a real number. If $A(t) \in \mathcal{C}_\gamma$ and $D(0) \in L^1_\alpha$ mean value functional (1.1) exists and determines the duality between observables and states.

The evolution of the observables

$$A(t) = (A_0, A_1(t, x_1), \ldots, A_n(t, x_1, \ldots, x_n), \ldots)$$

is described by the Cauchy problem for the sequence of the weak formulation of the Liouville equations for hard spheres with elastic collisions [14]. On the space $\mathcal{C}_\gamma$ a non-perturbative solution $A(t) = S(t)A(0)$ of the Liouville equation of many hard spheres is determined by the following group of operators [83]:

$$
\begin{aligned}
&(S(t)b)_n(x_1,\dots,x_n) = S_n(t,1,\dots,n)b_n(x_1,\dots,x_n) \doteq \\
&\doteq \begin{cases} b_n(X_1(t,x_1,\dots,x_n),\dots,X_n(t,x_1,\dots,x_n)), & \\ \qquad\qquad\qquad\qquad \text{if}(x_1,\dots,x_n) \in (\mathbb{R}^{3n}\times(\mathbb{R}^{3n}\backslash\mathbb{W}_n)), & \\ 0, \qquad\qquad\qquad\quad \text{if}(q_1,\dots,q_n)\in\mathbb{W}_n, & \end{cases}
\end{aligned}
\tag{1.2}
$$

where for arbitrary $t \in \mathbb{R}$ the function $X_i(t)$ is a phase trajectory of *ith* particle constructed in book [14] and the set $\mathbb{M}_n^0$ of the zero Lebesgue measure, which consists of the phase space points that are specified such initial data that during the evolution generate multiple collisions, i.e., collisions of more than two particles, more than one two-particle collision at the same instant, and an infinite number of collisions within a finite time interval.

On the space $\mathcal{C}_\gamma$ one-parameter mapping (1.2) is an isometric $*$-weak continuous group of operators, i.e., it is a C_0^*-group [19]. The infinitesimal generator $\mathcal{L} = \oplus_{n=0}^{\infty}\mathcal{L}_n$ of the group of operators (1.2) has the structure:

$$
\mathcal{L}_n = \sum_{j=1}^{n}\mathcal{L}(j) + \sum_{j_1<j_2=1}^{n}\mathcal{L}_{\mathrm{int}}(j_1,j_2),
$$

where the operator $\mathcal{L}(j)$ defined on the set $C_{n,0}$ of continuously differentiable functions with compact supports is the Liouville operator of free evolution of the j hard sphere and for $t \geqslant 0$ the operators $\mathcal{L}(j)$ and $\mathcal{L}_{\mathrm{int}}(j_1,j_2)$ are defined by the formulas [22, 83]:

$$
\begin{aligned}
&\mathcal{L}(j) \doteq \langle p_j, \frac{\partial}{\partial q_j}\rangle, \\
&\mathcal{L}_{\mathrm{int}}(j_1,j_2)b_n \doteq \sigma^2\int_{\mathbb{S}_+^2} d\eta\langle\eta,(p_{j_1}-p_{j_2})\rangle\,\delta(q_{j_1}-q_{j_2}+\sigma\eta)\times \\
&\qquad\times\big(b_n(x_1,\dots,q_{j_1},p_{j_1}^*,\dots,q_{j_2},p_{j_2}^*,\dots,x_n) - b_n(x_1,\dots,x_n)\big),
\end{aligned}
\tag{1.3}
$$

respectively. In formulas (1.3) the symbol $\langle\cdot,\cdot\rangle$ denotes a scalar product, δ is the Dirac measure, $\mathbb{S}_+^2 \doteq \{\eta\in\mathbb{R}^3 \mid |\eta| = 1, \langle\eta,(p_{j_1}-p_{j_2})\rangle > 0\}$ and the post-collision momenta: $p_{j_1}^*, p_{j_2}^*$ are defined by the equalities

$$
\begin{aligned}
p_{j_1}^* &\doteq p_{j_1} - \eta\,\langle\eta,(p_{j_1}-p_{j_2})\rangle, \\
p_{j_2}^* &\doteq p_{j_2} + \eta\,\langle\eta,(p_{j_1}-p_{j_2})\rangle.
\end{aligned}
\tag{1.4}
$$

For $t < 0$ operator (1.3) is defined by the corresponding expression [14].

It should be noted that the structure of generator (1.3) is determined, on the one hand, by the singular interaction potential of hard spheres and, on the other, by the fact that the group of operators (1.2) is defined for pairwise collisions outside the set $\mathbb{M}_n^0$ of the zero Lebesgue measure defined above in (1.2).

As mentioned above, the evolution of observables for many hard spheres, i.e., the sequences of functions $A_n(t) = S_n(t)A_n^0$, $n \geqslant 1$, is governed by the Cauchy problem for the sequence of the weak formulation of the Liouville equations with these generators [51]:

$$\frac{\partial}{\partial t}A_n(t) = \Big(\sum_{j=1}^{n}\mathcal{L}(j) \; + \sum_{j_1<j_2=1}^{n}\mathcal{L}_{\mathrm{int}}(j_1,j_2)\Big)A_n(t),$$
$$A_n(t)|_{t=0} = A_n^0, \quad n \geqslant 1. \tag{1.5}$$

For mean value functional (1.1) the following representation holds

$$(A(t), D(0)) = (A(0), D(t)),$$

where the sequence $D(t) = (1, D_1(t,x_1), \dots, D_n(t,x_1,\dots,x_n), \dots)$ of distribution functions is defined as follows:

$$D(t) = S^*(t)D(0),$$

and the mapping $S^*(t)$ is an adjoint operator to operator (1.2) in the sense of mean value functional (1.1). We emphasize that this equality is a consequence of a fundamental property of Hamiltonian systems, namely, the validity of the Liouville theorem for phase trajectories [14], i.e., isometry of the groups of operators (1.2).

On the space $L^1_\alpha = \bigoplus_{n=0}^{\infty} \alpha^n L^1_n$ of sequences of integrable functions, the group of operators $S^*(t) = \bigoplus_{n=0}^{\infty} S^*_n(t)$ is an adjoint to the group of operators (1.2) in the sense of functional (1.1) and is defined as follows [83]:

$$S^*(t) = S(-t). \tag{1.6}$$

On the space L^1_α, one-parameter mapping (1.6) is an isometric strong continuous group of operators, i.e., it is a C_0-group [19]. The infinitesimal generator $\mathcal{L}^* = \oplus_{n=0}^{\infty}\mathcal{L}^*_n$ of this group of operators has the structure:

$$\mathcal{L}^*_n \doteq \sum_{j=1}^{n}\mathcal{L}^*(j) \; + \sum_{j_1<j_2=1}^{n}\mathcal{L}^*_{\mathrm{int}}(j_1,j_2),$$

and for $t > 0$ the operators $\mathcal{L}^*(j)$ and $\mathcal{L}^*_{\mathrm{int}}(j_1, j_2)$ are defined by the formulas:

$$\begin{aligned}
&\mathcal{L}^*(j)f_n \doteq -\langle p_j, \frac{\partial}{\partial q_j}\rangle f_n,\\
&\mathcal{L}^*_{\mathrm{int}}(j_1, j_2)f_n \doteq \sigma^2 \int\limits_{\mathbb{S}^2_+} d\eta \langle \eta, (p_{j_1} - p_{j_2})\rangle \times\\
&\qquad \times \big(f_n(x_1, \ldots, p^*_{j_1}, q_{j_1}, \ldots, p^*_{j_2}, q_{j_2}, \ldots, x_n)\delta(q_{j_1} - q_{j_2} + \sigma\eta) -\\
&\qquad\qquad - f_n(x_1, \ldots, x_n)\delta(q_{j_1} - q_{j_2} - \sigma\eta)\big),
\end{aligned} \tag{1.7}$$

respectively, the pre-collision momenta $p^*_{j_1}, p^*_{j_2}$ are defined by relations (1.4) and notations accepted in formula (1.3) are used. For $t < 0$ these operators are defined by the corresponding expressions [64].

We note that the evolution of the state, i.e., the sequence of probability distribution functions $D_n(t) = S^*_n(t)D^0_n$, $n \geqslant 1$, describes by the Cauchy problem for the sequence of the weak formulation of the Liouville equations for many hard spheres with these generators [83]:

$$\begin{aligned}
&\frac{\partial}{\partial t}D_n(t) = \big(\sum_{j=1}^{n} \mathcal{L}^*(j) \ + \sum_{j_1<j_2=1}^{n} \mathcal{L}^*_{\mathrm{int}}(j_1, j_2)\big)D_n(t),\\
&D_n(t)|_{t=0} = D^0_n, \quad n \geqslant 1.
\end{aligned} \tag{1.8}$$

Thus, the evolution of finitely many colliding particles is governed by the fundamental evolution equations, such as the Liouville equation for observables (1.5) or its dual equation for a state (1.8).

To formulate another representation of the mean value functional (1.1) in terms of sequences of so-called reduced observables and reduced distribution functions, on sequences of bounded continuous functions we introduce an analog of the creation operator

$$(\mathfrak{a}^+ b)_s(x_1, \ldots, x_s) \doteq \sum_{j=1}^{s} b_{s-1}((x_1, \ldots, x_s)\backslash(x_j)), \tag{1.9}$$

and on sequences of integrable functions, we introduce an adjoint operator to operator (1.9) in the sense of mean value functional (1.1) which is an analogue of the annihilation operator

$$(\mathfrak{a} f)_n(x_1, \ldots, x_n) \ = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} f_{n+1}(x_1, \ldots, x_n, x_{n+1})dx_{n+1}. \tag{1.10}$$

Then as a consequence of the validity of equalities:

$$(b, f) = (e^{\mathfrak{a}^+}e^{-\mathfrak{a}^+}b, f) = (e^{-\mathfrak{a}^+}b, e^{\mathfrak{a}}f),$$

for mean value functional (1.1) the following representation holds:

$$\langle A\rangle(t)=(I,D(0))^{-1}(A(t),D(0))=(B(t),F(0)), \tag{1.11}$$

where a sequence of the reduced observables is defined by the formula

$$B(t)=e^{-\mathfrak{a}^{+}}S(t)A(0), \tag{1.12}$$

and a sequence of so-called reduced distribution functions is defined as follows (known as the non-equilibrium grand canonical ensemble [82])

$$F(0)=(I,D(0))^{-1}e^{\mathfrak{a}}D(0),$$

respectively.

Thus, according to the definition of the operator $e^{-\mathfrak{a}^{+}}$, the sequence of reduced observables (1.12) in component-wise form is represented by the expansions:

$$B_s(t,x_1,\ldots,x_s)= \\ =\sum_{n=0}^{s}\frac{(-1)^n}{n!}\sum_{j_1\neq\ldots\neq j_n=1}^{s}(S(t)A(0))_{s-n}\big((x_1,\ldots,x_s)\backslash(x_{j_1},\ldots,x_{j_n})\big), \quad s\geqslant 1, \tag{1.13}$$

The mean value functional (1.11) also has the following representation:

$$(B(t),F(0))=(B(0),F(t)). \tag{1.14}$$

The sequence $F(t)=(1,F_1(t,x_1),\ldots,F_n(t,x_1,\ldots,x_n),\ldots)$ of reduced distribution functions is defined as follows (known as the non-equilibrium grand canonical ensemble [82])

$$F(t)=(I,D(0))^{-1}e^{\mathfrak{a}}S^{*}(t)D(0), \tag{1.15}$$

where the mapping $S^{*}(t)$ is an adjoint operator (1.6) to operator (1.2). According to the definition of the operator $e^{\mathfrak{a}}$, the sequence of reduced distribution functions (1.15) in component-wise form is represented by the series:

$$F_s(t,x_1,\ldots,x_s)=(I,D(0))^{-1}\times \\ \times\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n}(S^{*}(t)D(0))_{s+n}(x_1,\ldots,x_{s+n})dx_{s+1}\cdots dx_{s+n},\ s\geqslant 1,$$

where the coefficient $(I,D(0))$ is the normalizing factor as above.

We emphasize that a widely used approach to the description of the evolution of many hard spheres is based on the evolution of a state determined by the BBGKY hierarchy for reduced distribution functions [14]. An equivalent approach to describing evolution is based on reduced observables (1.12) governed by the dual hierarchy of evolution equations [52].

2. Hierarchies of evolution equations for colliding particles

As is well known, hierarchies of evolution equations for sequences of reduced functions of observables and, accordingly, of a state for a finitely many hard spheres are equivalent to the Liouville equations. Their advantages consist in the possibility of rigorously describing the evolution of infinitely many hard spheres whose collective behavior exhibits thermodynamic (statistical) features, namely, the existence of an equilibrium state in such a system as well as the kinetic or hydrodynamic behavior in corresponding scaling approximations [14, 21, 64].

An alternative approach to the description of the evolution of the state of a hard-sphere system is based on functions determined by the cluster expansions of the probability distribution functions. The cumulants of probability distribution functions are interpreted as correlation functions and are governed by the Liouville hierarchy. The following outlines the approach to the description of the evolution of a state by means of both reduced distribution functions and reduced correlation functions, which is based on the dynamics of correlations [53]. It should be emphasized that on a microscopic scale, the macroscopic characteristics of fluctuations of observables are directly determined by the reduced correlation functions.

2.1. Hierarchy of evolution equations for reduced observables.

The motivation for describing the evolution of many-particle systems in terms of reduced observables is related to possible equivalent representations of the mean value functional (mathematical expectation) of observables, namely as (1.11) compared to the traditionally used form (1.14).

The evolution of sequence (1.12) of reduced observables of many hard spheres is determined by the Cauchy problem of the following abstract hierarchy of evolution equations [10, 51]:

$$\frac{d}{dt}B(t) = \mathcal{L}B(t) + \big[\mathcal{L}, \mathfrak{a}^{+}\big]B(t), \tag{2.1}$$

$$B(t)\big|_{t=0} = B(0), \tag{2.2}$$

where the operator $\mathcal{L}$ is generator (1.3) of the group of operators (1.2) for hard spheres, the symbol $\big[\cdot,\cdot\big]$ denotes the commutator of operators, which in equation (2.1) has the following component-wise form:

$$\big(\big[\mathcal{L}, \mathfrak{a}^{+}\big]b\big)_{s}(x_1,\ldots,x_s) = \\ = \sum_{j_1\neq j_2=1}^{s}\mathcal{L}_{\mathrm{int}}(j_1,j_2)b_{s-1}(t,(x_1,\ldots,x_s)\backslash x_{j_1}), \quad s\geqslant 1.$$

In a component-wise form the hierarchy of evolution equations (2.1)) for hard-sphere fluids, in fact, is a sequence of recurrence evolution equations (in literature it is known as the dual BBGKY hierarchy [52]). We adduce the simplest examples of recurrent evolution equations (2.1):

$$\frac{\partial}{\partial t}B_1(t,x_1)=\mathcal{L}(1)B_1(t,x_1),$$
$$\frac{\partial}{\partial t}B_2(t,x_1,x_2)=\Big(\sum_{i=1}^{2}\mathcal{L}(j)+\mathcal{L}_{\mathrm{int}}(1,2)\Big)B_2(t,x_1,x_2)+$$
$$+\mathcal{L}_{\mathrm{int}}(1,2)\big(B_1(t,x_1)+B_1(t,x_2)\big),$$

where the generators of these equations are defined by formula (1.3).

The non-perturbative solution of the Cauchy problem of the dual BBGKY hierarchy (2.1),(2.2) for hard spheres is a sequence of reduced observables represented by the following expansions [65, 66]:

$$B_s(t,x_1,\ldots,x_s)=\big(e^{\mathfrak{a}^+}\mathfrak{A}(t)B(0)\big)_s(x_1,\ldots,x_s)=$$
$$=\sum_{n=0}^{s}\frac{1}{n!}\sum_{j_1\neq\cdots\neq j_n=1}^{s}\mathfrak{A}_{1+n}\big(t,\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\},j_1,\ldots,j_n\big)\times \tag{2.3}$$
$$\times B^0_{s-n}(x_1,\ldots,x_{j_1-1},x_{j_1+1},\ldots,x_{j_n-1},x_{j_n+1},\ldots,x_s),\ s\geqslant 1,$$

where the mappings $\mathfrak{A}_{1+n}(t)$, $n\geqslant 0$, are the generating operators which are represented as cumulant expansions with respect of groups of operators (1.2). The simplest examples of reduced observables (2.3) are given by the following expansions:

$$B_1(t,x_1)=\mathfrak{A}_1(t,1)B^0_1(x_1),$$
$$B_2(t,x_1,x_2)=\mathfrak{A}_1(t,\{1,2\})B^0_2(x_1,x_2)+\mathfrak{A}_2(t,1,2)(B^0_1(x_1)+B^0_1(x_2)).$$

To determine the generating operators of expansions of reduced observables (2.3), we will introduce the notion of dual cluster expansions of groups of operators (1.2) in terms of operators interpreted as their cumulants. For this end on sequences of one-parametric mappings

$$\mathfrak{u}(t)=(0,\mathfrak{u}_1(t),\ldots,\mathfrak{u}_n(t),\ldots)$$

we define the following $\star$-product [90]

$$(\mathfrak{u}(t)\star\tilde{\mathfrak{u}}(t))_s(1,\ldots,s)\ =\sum_{Y\subset(1,\ldots,s)}\mathfrak{u}_{|Y|}(t,Y)\,\tilde{\mathfrak{u}}_{s-|Y|}(t,(1,\ldots,s)\backslash Y), \tag{2.4}$$

where $\sum_{Y\subset(1,\ldots,s)}$ is the sum over all subsets Y of the set $(1,\ldots,s)$.

Using the definition of the $\star$-product (2.4), the dual cluster expansions of groups of operators (1.2) are represented by the mapping $\mathbb{E}\mathrm{xp}_\star$ in the form

$$S(t)=\mathbb{E}\mathrm{xp}_\star\,\mathfrak{A}(t)=\mathbb{I}+\sum_{n=1}^{\infty}\frac{1}{n!}\mathfrak{A}(t)^{\star n},$$

where $S(t)=(0,S_1(t,1),\ldots,S_n(t,1,\ldots,n),\ldots)$ and $\mathbb{I}=(1,0,\ldots,0,\ldots)$. In component-wise form the dual cluster expansions are represented by the following recursive relations:

$$S_s(t,(1,\ldots,s)\backslash(j_1,\ldots,j_n),j_1,\ldots,j_n)= \\ =\sum_{\mathrm{P}:\,(\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\},\,j_1,\ldots,j_n)=\bigcup_i X_i}\ \prod_{X_i\subset\mathrm{P}}\mathfrak{A}_{|X_i|}(t,X_i),\ n\geqslant 0, \tag{2.5}$$

where the set consisting of one element of indices $(1,\ldots,s)\backslash(j_1,\ldots,j_n)$ we denoted by the symbol $\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\}$ and the symbol $\sum_{\mathrm{P}}$ means the sum over all possible partitions P of the set

$$(\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\},j_1,\ldots,j_n)$$

into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i\subset(1,\ldots,s)$.

The solution of recursive relations (2.5) are represented by the inverse mapping $\mathbb{L}\mathrm{n}_*$ in the form of the cumulant expansion

$$\mathfrak{A}(t)=\mathbb{L}\mathrm{n}_\star(\mathbb{I}+S(t))=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}S(t)^{\star n}.$$

Then the $(1+n)th$-order dual cumulant of groups of operators (1.2) is defined by the following expansion:

$$\mathfrak{A}_{1+n}(t,\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\},j_1,\ldots,j_n)\doteq \\ \doteq\sum_{\mathrm{P}:\,(\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\},j_1,\ldots,j_n)=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}S_{|\theta(X_i)|}(t,\theta(X_i)), \tag{2.6}$$

where the above notation is used and the declusterization mapping θ is defined by the formula:

$$\theta(\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\})=(1,\ldots,s)\backslash(j_1,\ldots,j_n).$$

The dual cumulants (2.6) of the first two orders have the form:

$$\mathfrak{A}_1(t,\{1,\ldots,s\})=S_s(t,1,\ldots,s),$$

$$\mathfrak{A}_{1+1}(t,\{(1,\ldots,s)\backslash(j)\},j)= \\ =S_s(t,1,\ldots,s)-S_{s-1}(t,(1,\ldots,s)\backslash(j))S_1(t,j).$$

If $b_s\in\mathcal{C}_s$, then for $(1+n)th$-order cumulant (2.6) of groups of operators (1.2) the estimate is valid

$$\big\|\mathfrak{A}_{1+n}(t)b_s\big\|_{\mathcal{C}_s}\leqslant\sum_{\mathrm{P}:\,(\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\},j_1,\ldots,j_n)=\bigcup_i X_i}(|\mathrm{P}|-1)!\big\|b_s\big\|_{\mathcal{C}_s}\leqslant \\ \leqslant\sum_{k=1}^{n+1}\mathrm{s}(n+1,k)(k-1)!\big\|b_s\big\|_{\mathcal{C}_s}\leqslant n!e^{n+2}\big\|b_s\big\|_{\mathcal{C}_s}, \tag{2.7}$$

where $\mathrm{s}(n+1,k)$ are the Stirling numbers of the second kind. Then according to this estimate (2.7) for the generating operators of expansions (2.3) provided that $\gamma < e^{-1}$ the inequality valid

$$\|B(t)\|_{C_\gamma} \leqslant e^2(1-\gamma e)^{-1}\|B(0)\|_{C_\gamma}. \tag{2.8}$$

In fact, the following criterion holds.

Criterion. *A solution of the Cauchy problem of the dual BBGKY hierarchy* (2.1),(2.2) *is represented by expansions* (2.3) *if and only if the generating operators of expansions* (2.3) *are solutions of cluster expansions* (2.5) *of the groups of operators* (1.2) *of the Liouville equations for hard spheres.*

The necessity condition means that cluster expansions (2.5) hold for groups of operators (1.2). These recurrence relations are derived from definition (1.13) of reduced observables, provided that they are represented as expansions (2.3) for the solution of the Cauchy problem of the dual BBGKY hierarchy (2.1),(2.2).

The sufficient condition means that the infinitesimal generator of one-parameter mapping (2.3) coincides with the generator of the sequence of recurrence evolution equations (2.1). Indeed, in the space C_γ the following existence theorem is true [51].

Theorem. *A non-perturbative solution of the Cauchy problem* (2.1),(2.2) *is represented by expansions* (2.3) *in which the generating operators are cumulants of the corresponding order* (2.6) *of groups of operators* (1.2)*:*

$$\begin{aligned} B_s(t,x_1,\ldots,x_s) = &\sum_{n=0}^{s}\frac{1}{n!}\sum_{j_1\neq\cdots\neq j_n=1}^{s}\ \sum_{\mathrm{P}:\,(\{(1,\ldots,s)\setminus(j_1,\ldots,j_n)\},j_1,\ldots,j_n)=\bigcup_i X_i} \\ &\Big((-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}} S_{|\theta(X_i)|}(t,\theta(X_i))\times \\ &\times B^0_{s-n}(x_1,\ldots,x_{j_1-1},x_{j_1+1},\ldots,x_{j_n-1},x_{j_n+1},\ldots,x_s)\Big),\ s\geqslant 1. \end{aligned} \tag{2.9}$$

Under the condition $\gamma < e^{-1}$ for initial data $B(0)\in C^0_\gamma$ of finite sequences of infinitely differentiable functions with compact supports sequence (2.9) *is a unique global classical solution, and for arbitrary initial data $B(0)\in C_\gamma$ is a unique global generalized solution.*

We note that the one component sequences $B^{(1)}(0) = (0, b_1(x_1), 0, \ldots)$ of reduced observables correspond to the additive-type observable, and the sequences

$$B^{(k)}(0) = (0,\ldots,b_k(x_1,\ldots,x_k),0,\ldots)$$

of reduced observables correspond to the k-ary-type observables [10].

If initial data (2.2) is specified by the additive-type reduced observable, then the structure of solution expansion (2.9) is simplified and attains the form

$$B_s^{(1)}(t,x_1,\ldots,x_s)=\mathfrak{A}_s(t,1,\ldots,s)\sum_{j=1}^{s}b_1(x_j),\quad s\geqslant 1,\tag{2.10}$$

where the generating operator $\mathfrak{A}_s(t)$ is the sth-order cumulant (2.6) of the groups of operators (1.2).

An example of the additive-type observables is a number of particles, i.e., the sequence $N^{(1)}(0)=(0,1,0,\ldots)$, then

$$N_s^{(1)}(t)=\mathfrak{A}_s(t,1,\ldots,s)s\;=\sum_{\mathrm{P}:\,(1,\ldots,s)=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\sum_{j=1}^{s}1=$$

$$=\sum_{k=1}^{s}(-1)^{k-1}\mathrm{s}(s,k)(k-1)!s=s\delta_{s,1}=N_s^{(1)}(0),$$

where $\mathrm{s}(s,k)$ are the Stirling numbers of the second kind and $\delta_{s,1}$ is a Kronecker symbol. Consequently, the observable of a number of hard spheres is an integral of motion and, in particular, the average number of particles is preserving in time.

In the case of initial k-ary-type, $k\geqslant 2$, reduced observables solution expansion (2.9) takes the form

$$\begin{aligned}
&B_s^{(k)}(t)=0,\quad 1\leqslant s<k,\\
&B_s^{(k)}(t,x_1,\ldots,x_s)=\frac{1}{(s-k)!}\times\\
&\quad\times\sum_{j_1\neq\cdots\neq j_{s-k}=1}^{s}\mathfrak{A}_{1+s-k}\big(t,\{(1,\ldots,s)\setminus(j_1,\ldots,j_{s-k})\},j_1,\ldots,j_{s-k}\big)\times\\
&\qquad\times b_k(x_1,\ldots,x_{j_1-1},x_{j_1+1},\ldots,x_{j_{s-k}-1},x_{j_{s-k}+1},\ldots,x_s),\\
&\qquad\qquad s\geqslant k,
\end{aligned}\tag{2.11}$$

where the generating operator $\mathfrak{A}_{1+s-k}(t)$ is the $(1+s-k)th$-order cumulant (2.6) of the groups of operators (1.2).

We emphasize that cluster expansions (2.5) of the groups of operators (1.2) underlie of the classification of possible solution representations of the Cauchy problem (2.1),(2.2) of the dual BBGKY hierarchy. Indeed, using cluster expansions (2.5) of the groups of operators (1.2), other solution representations can be constructed.

For example, let us express the cumulants $\mathfrak{A}_{1+n}(t)$, $n\geqslant 2$, of groups of operators (1.2) with respect to the $1st$-order and $2nd$-order cumulants. The equalities are true:

$$\mathfrak{A}_{1+n}(t,\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\},j_1,\ldots,j_n) = \\ = \sum_{\varnothing\neq Y\subset(j_1,\ldots,j_n)} \mathfrak{A}_2(t,\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\},\{Y\})\times \\ \times \sum_{\mathrm{P}:\,(j_1,\ldots,j_n)\backslash Y=\bigcup_i X_i}(-1)^{|\mathrm{P}|}\,|\mathrm{P}|!\prod_{i=1}^{|\mathrm{P}|}\mathfrak{A}_1(t,\{X_i\}),\quad n\geqslant 2,$$

where $\sum\limits_{\varnothing\neq Y\subset(j_1,\ldots,j_n)}$ is a sum over all nonempty subsets $Y\subset(j_1,\ldots,j_n)$. Then, taking into account the identity

$$\sum_{\mathrm{P}:\,(j_1,\ldots,j_n)\backslash Y=\bigcup_i X_i}(-1)^{|\mathrm{P}|}|\mathrm{P}|!\prod_{i=1}^{|\mathrm{P}|}\mathfrak{A}_1(t,\{X_i\})B^0_{s-n}((x_1,\ldots,x_s)\backslash(x_{j_1},\ldots,x_{j_n})) = \\ = \sum_{\mathrm{P}:\,(j_1,\ldots,j_n)\backslash Y=\bigcup_i X_i}(-1)^{|\mathrm{P}|}\,|\mathrm{P}|!\,B^0_{s-n}((x_1,\ldots,x_s)\backslash(x_{j_1},\ldots,x_{j_n})), \tag{2.12}$$

and the equalities

$$\sum_{\mathrm{P}:\,(j_1,\ldots,j_n)\backslash Y=\bigcup_i X_i}(-1)^{|\mathrm{P}|}\,|\mathrm{P}|! = (-1)^{|(j_1,\ldots,j_n)\backslash Y|},\ Y\subset(j_1,\ldots,j_n), \tag{2.13}$$

for solution expansions (2.3) of the dual BBGKY hierarchy we derive the following representation:

$$B_s(t,x_1,\ldots,x_s) = \mathfrak{A}_1(t,\{1,\ldots,s\})B^0_s(x_1,\ldots,x_s)+ \\ +\sum_{n=1}^{s}\frac{1}{n!}\sum_{j_1\neq\cdots\neq j_n=1}^{s}\ \sum_{\substack{Y\subset(1,\ldots,s)\backslash(j_1,\ldots,j_n),\\ Y\neq\varnothing}}(-1)^{|(j_1,\ldots,j_n)\backslash Y|}\,\mathfrak{A}_2(t,\{j_1,\ldots,j_n\},\{Y\})\times \\ \times B^0_{s-n}((x_1,\ldots,x_s)\backslash(x_{j_1},\ldots,x_{j_n})),\ s\geqslant 1,$$

where notations accepted above are used.

Taking into account that initial reduced observables depend only from the certain phase space arguments, we deduce the reduced representation of expansions (2.9):

$$B(t)=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\,\frac{n!}{k!(n-k)!}\,(\mathfrak{a}^{+})^{n-k}S(t)(\mathfrak{a}^{+})^{k}B(0)=$$
$$=S(t)B(0)+\sum_{n=1}^{\infty}\frac{1}{n!}\big[\ldots\big[S(t),\underbrace{\mathfrak{a}^{+}\big],\ldots,\mathfrak{a}^{+}}_{\text{n-times}}\big]B(0)= \tag{2.14}$$
$$=e^{-\mathfrak{a}^{+}}S(t)e^{\mathfrak{a}^{+}}B(0).$$

Therefore, in component-wise form the generating operators of these expansions represented as expansions (2.3) are the following reduced cumulants of groups of operators (1.2):

$$U_{1+n}(t,\{1,\ldots,s-n\},s-n+1,\ldots,s)=$$
$$=\sum_{k=0}^{n}(-1)^{k}\frac{n!}{k!(n-k)!}S_{s-k}(t,1,\ldots,s-k). \tag{2.15}$$

Indeed, solutions of the recursive relations (2.5) with respect to first-order cumulants can be represented as expansions in terms of cumulants acting on variables on which the initial reduced observables depend, and in terms of cumulants not acting on these variables

$$\mathfrak{A}_{1+n}(t,\{(1,\ldots,s)\backslash(j_1,\ldots,j_n)\},j_1,\ldots,j_n)=$$
$$=\sum_{Y\subset(j_1,\ldots,j_n)}\mathfrak{A}_1(t,\{(1,\ldots,s)\backslash((j_1,\ldots,j_n)\cup Y)\})\times$$
$$\times\sum_{\mathrm{P}:\,(j_1,\ldots,j_n)\backslash Y=\bigcup_i X_i}(-1)^{|\mathrm{P}|}\,|\mathrm{P}|!\,\prod_{i=1}^{|\mathrm{P}|}\mathfrak{A}_1(t,\{X_i\}),$$

where $\sum_{Y\subset(j_1,\ldots,j_n)}$ is the sum over all possible subsets $Y\subset(j_1,\ldots,j_n)$. Then taking into account the identity (2.13) and the equalities (2.12) we derive expansions (2.14) over reduced cumulants (2.15).

We note that traditionally the solution of the BBGKY hierarchy for states of many hard spheres is represented by perturbation series [14,29,83]. The expansions (2.14) can also be represented as expansions (iterations) of perturbation theory [10]:

$$B(t)=\sum_{n=0}^{\infty}\int_0^{t}dt_1\cdots\int_0^{t_{n-1}}dt_n\,S(t-t_1)\big[\mathcal{L},\mathfrak{a}^{+}\big]\times$$
$$\times\,S(t_1-t_2)\cdots S(t_{n-1}-t_n)\big[\mathcal{L},\mathfrak{a}^{+}\big]S(t_n)B(0).$$

Indeed, as a result of applying of analogs of the Duhamel equation to generating operators (2.6) of expansions (2.3) we derive in component-wise form, for examples,

$$
\begin{aligned}
&U_1(t,\{1,\ldots,s\}) = S_s(t,1,\ldots,s),\\
&U_2(t,\{(1,\ldots,s)\backslash(j_1)\},j_1) =\\
&\quad = \int_0^t dt_1\, S_s(t-t_1,1,\ldots,s)\sum_{\substack{j_2=1,\\ j_2\neq j_1}}^{s} \mathcal{L}_{\mathrm{int}}(j_1,j_2)S_{s-1}(t_1,(1,\ldots,s)\backslash j_1).
\end{aligned}
$$

Recall that the mean value functional (1.11) exists if $B(0)\in C_\gamma$ and $F(0)\in L^1_\alpha$. In the case of the observable of a number of hard spheres $N^{(1)}(t)=(0,1,0,\ldots)$, this means that

$$
\big|(N^{(1)}(t),F(0))\big| \;=\; \Big|\int_{\mathbb{R}^3\times\mathbb{R}^3} F_1^0(x_1)dx_1\Big| \;\leqslant\; \|F_1^0\|_{L^1_1} \;<\; \infty. \tag{2.16}
$$

Consequently, the states of a finite number of hard spheres are described by sequences of functions from the space L^1_α. To describe an infinite number of hard spheres, it is necessary to consider reduced distribution functions from appropriate function spaces, for example, from the space of sequences of bounded functions with respect of the configuration variables [58,79,83].

2.2. **The BBGKY hierarchy for reduced distribution functions.** As mentioned already, the evolution of systems of many particles is traditionally described as the evolution of the state of a system based on the representation of the mean value functional for observables (1.14). In this case, the sequence of reduced distribution functions is determined by the hierarchy of evolution equations, known as the BBGKY hierarchy, whose generator is the operator adjoint to the generator of the hierarchy of evolution equations for reduced observables (2.1) in the sense of mean value functional (1.11).

The evolution of sequence (1.15) of reduced distribution functions from the space L^1_α is governed by the Cauchy problem of the BBGKY hierarchy for many hard spheres [6,14,83]:

$$
\frac{d}{dt}F(t) = \mathcal{L}^*F(t) + \big[\mathfrak{a},\mathcal{L}^*\big]F(t), \tag{2.17}
$$

$$
F(t)\big|_{t=0} = F(0), \tag{2.18}
$$

where the symbol $\big[\cdot,\cdot\big]$ denotes the commutator of operator (1.10) and of the Liouville operator (1.7), which is the generator of the isometric group of operators (1.6). Thus, in evolution equation (2.17), the second term has the following component-wise form:

$$\big([\mathfrak{a},\mathcal{L}^*]f\big)_s(x_1,\ldots,x_s)= \\ =\sum_{i=1}^{s}\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}dx_{s+1}\,\mathcal{L}^*_{\mathrm{int}}(i,s+1)f_{s+1}(t,x_1,\ldots,x_{s+1}),\quad s\geqslant 1.$$

For $t>0$ in a one-dimensional space, i.e., for gas of hard rods, this term of a generator has the form [82]:

$$\sum_{i=1}^{s}\int\limits_{\mathbb{R}\times\mathbb{R}}dx_{s+1}\mathcal{L}^*_{\mathrm{int}}(i,s+1)f_{s+1}(t)= \\ =\sum_{i=1}^{s}\int\limits_{0}^{\infty}dP\,P\Big(f_{s+1}(t,x_1,\ldots,q_i,p_i-P,\ldots,x_s,q_i-\sigma,p_i)- \\ -f_{s+1}(t,x_1,\ldots,x_s,q_i-\sigma,p_i+P)+ \\ +f_{s+1}(t,x_1,\ldots,q_i,p_i+P,\ldots,x_s,q_i+\sigma,p_i)- \\ -f_{s+1}(t,x_1,\ldots,x_s,q_i+\sigma,p_i-P)\Big), \tag{2.19}$$

and for $t<0$ this collision integral has the corresponding form [83].

It should be noted that for the system of a fixed, finite number of hard spheres, the BBGKY hierarchy is an equation system for a finite sequence of reduced distribution functions. Such an equation system is equivalent to the Liouville equation for the distribution function, which describes all possible states of finitely many hard spheres. For a system of an infinite number of hard spheres, the BBGKY hierarchy is an infinite chain of evolution equations, which can be derived as the thermodynamic limit of the BBGKY hierarchy of a fixed finite number of hard spheres [14]. We note that since a sequence of functions can be determined based on a generating functional, the corresponding hierarchy of evolution equations can also be formulated as the evolution equation for such a generating functional [44].

A non-perturbative solution of the Cauchy problem of the BBGKY hierarchy (2.17),(2.18) is a sequence of reduced distribution functions represented by the following expansions [52, 67]:

$$F_s(t,x_1,\ldots,x_s)=(e^{\mathfrak{a}}\mathfrak{A}^*(t)F(0))_s(x_1,\ldots,x_s)= \\ =\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n}\mathfrak{A}^*_{1+n}(t,\{1,\ldots,s\},s+1,\ldots,s+n)\times \\ \times F^0_{s+n}(x_1,\ldots,x_{s+n})dx_{s+1}\cdots dx_{s+n},\ s\geqslant 1, \tag{2.20}$$

where the mappings $\mathfrak{A}^*_{1+n}(t)$, $n \geqslant 0$, are the generating operators which are represented by the cumulant expansions with respect to the group

$$S^*(t) = \mathop{\oplus}_{n=0}^{\infty} S^*_n(t)$$

of operators (1.6).

Using the definition of the $\star$-product (2.4), the cluster expansions of the groups of operators (1.6) are represented by the mapping $\mathbb{E}\mathrm{xp}_\star$ in the form

$$S^*(t) = \mathbb{E}\mathrm{xp}_\star\, \mathfrak{A}^*(t).$$

In component-wise form cluster expansions are represented by the following recursive relations:

$$\begin{aligned} &S^*_{s+n}(t, 1, \ldots, s, s+1, \ldots, s+n) = \\ &\qquad = \sum_{\mathrm{P}:\, (\{1,\ldots,s\}, s+1,\ldots,s+n) = \bigcup_i X_i} \prod_{X_i \subset \mathrm{P}} \mathfrak{A}^*_{|X_i|}(t, X_i), \quad n \geqslant 0, \end{aligned} \tag{2.21}$$

where the set consisting of one element of indices $(1, \ldots, s)$ we denoted by the symbol $\{(1, \ldots, s)\}$ and the symbol $\sum_{\mathrm{P}}$ means the sum over all possible partitions P of the set

$$(\{1, \ldots, s\}, s+1, \ldots, s+n)$$

into $|\mathrm{P}|$ nonempty mutually disjoint subsets X_i.

The solution of recursive relations (2.21) are represented by the inverse mapping $\mathbb{L}\mathrm{n}_\star$ in the form of the cumulant expansion

$$\mathfrak{A}^*(t) = \mathbb{L}\mathrm{n}_\star(\mathbb{I} + S^*(t)).$$

Then the $(1+n)th$-order cumulant of the group $S^*(t) = \mathop{\oplus}_{n=0}^{\infty} S^*_n(t)$ of operators (1.6) is defined by the following expansion:

$$\begin{aligned} &\mathfrak{A}^*_{1+n}(t, \{1, \ldots, s\}, s+1, \ldots, s+n) \doteq \\ &\qquad \doteq \sum_{\mathrm{P}:\, (\{1,\ldots,s\}, s+1,\ldots,s+n) = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i \subset \mathrm{P}} S^*_{|\theta(X_i)|}(t, \theta(X_i)), \end{aligned} \tag{2.22}$$

where the declusterization mapping θ is defined by the formula:

$$\theta(\{1, \ldots, s\}) = (1, \ldots, s)$$

and the above notation is used. The simplest examples of cumulants (2.22) of the groups of operators (1.6) have the form:

$$\mathfrak{A}_1^*(t,\{1,\ldots,s\}) \doteq S_s^*(t,1,\ldots,s),$$

$$\mathfrak{A}_{1+1}^*(t,\{1,\ldots,s\},s+1) \doteq S_{s+1}^*(t,1,\ldots,s+1) - S_s^*(t,1,\ldots,s)S_1^*(t,s+1),$$

$$\begin{aligned}\mathfrak{A}_{1+2}^*(t,\{1,\ldots,s\},s+1,s+2) &\doteq S_{s+2}^*(t,1,\ldots,s+2)-\\ &- S_{s+1}^*(t,1,\ldots,s+1)S_1^*(t,s+2) - S_{s+1}^*(t,1,\ldots,s,s+2)S_1^*(t,s+1)-\\ &- S_s^*(t,1,\ldots,s)S_2^*(t,s+1,s+2) + 2!S_s^*(t,1,\ldots,s)S_1^*(t,s+1)S_1^*(t,s+2).\end{aligned}$$

If $f_s \in L_s^1$, then taking into account that $\|S_n^*(t)\|_{L_n^1} = 1$, for the $(1+n)th$-order cumulant (2.22) the following estimate is valid:

$$\begin{aligned}\big\|\mathfrak{A}_{1+n}^*(t)f_{s+n}\big\|_{L_{s+n}^1} &\leqslant \sum_{\mathrm{P}:\,(\{1,\ldots,s\},s+1,\ldots,s+n)=\bigcup_i X_i} (|\mathrm{P}|-1)!\big\|f_{s+n}\big\|_{L_{s+n}^1} \leqslant \\ &\leqslant \sum_{k=1}^{n+1} \mathrm{s}(n+1,k)(k-1)!\big\|f_{s+n}\big\|_{L_{s+n}^1} \leqslant n!e^{n+2}\big\|f_{s+n}\big\|_{L_{s+n}^1},\end{aligned} \tag{2.23}$$

where $\mathrm{s}(n+1,k)$ are the Stirling numbers of the second kind.

Then, according to this estimate (2.23) for the generating operators of expansions (2.20), provided that $\alpha > e$ series (2.20) converges on the norm of the space L_α^1, and the inequality holds

$$\|F(t)\|_{L_\alpha^1} \leqslant c_\alpha \|F(0)\|_{L_\alpha^1},$$

where $c_\alpha = e^2(1-\frac{e}{\alpha})^{-1}$. The parameter α can be interpreted as the value inverse to the average number of hard spheres.

In fact, the following criterion holds.

Criterion. *A solution of the Cauchy problem of the BBGKY hierarchy* (2.17), (2.18) *is represented by expansions* (2.20) *if and only if the generating operators of expansions* (2.20) *are solutions of cluster expansions* (2.21) *of the groups of operators* (1.6).

The necessity condition means that cluster expansions (2.21) are take place for groups of operators (1.6). These recurrence relations are derived from definition (1.15) of reduced distribution functions, provided that they are represented as expansions (2.20) for the solution of the Cauchy problem of the BBGKY hierarchy (2.17),(2.18).

The sufficient condition means that the infinitesimal generator of one-parameter mapping (2.20) coincides with the generator of the BBGKY hierarchy (2.17). Indeed, in the space L_α^1 the following existence theorem is true [65].

Theorem. *If $\alpha > e$, a non-perturbative solution of the Cauchy problem of the BBGKY hierarchy* (2.17),(2.18) *is represented by series expansions* (2.20) *in which the generating operators are cumulants of the corresponding order* (2.22) *of groups of operators* (1.6)*:*

$$F_s(t, x_1, \dots, x_s) = \\ = \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} \sum_{\mathrm{P}:\, (\{1,\dots,s\}, s+1,\dots,s+n) = \bigcup_i X_i} (-1)^{|\mathrm{P}|-1} (|\mathrm{P}|-1)! \times \\ \times \prod_{X_i \subset \mathrm{P}} S^*_{|\theta(X_i)|}(t, \theta(X_i)) F^0_{s+n}(x_1, \dots, x_{s+n}) dx_{s+1} \cdots dx_{s+n}, \ s \geqslant 1. \tag{2.24}$$

For initial data $F(0) \in L^1_0$ of finite sequences of infinitely differentiable functions with compact supports sequence (2.24) *is a unique global classical solution and for arbitrary initial data $F(0) \in L^1_\alpha$ is a unique global generalized solution.*

We observe that cluster expansions (2.21) of the groups of operators (1.6) underlie the classification of possible solution representations (2.20) of the Cauchy problem of the BBGKY hierarchy (2.17),(2.18). In a particular case, non-perturbative solution (2.24) of the BBGKY hierarchy for many hard spheres can be represented in the form of the perturbation (iteration) series as a result of applying analogs of the Duhamel equation to cumulants (2.22) of groups of operators.

Indeed, let us put groups of operators in the expression of cumulant (2.22) into a new order with respect to the groups of operators which act on the variables $(x_1, \dots, x_s)$

$$\mathfrak{A}^*_{1+n}(t, \{1, \dots, s\}, s+1, \dots, s+n) = \\ = \sum_{Y \subset (s+1,\dots,s+n)} S^*_{s+|Y|}(t, (1, \dots, s) \cup Y) \times \\ \times \sum_{\mathrm{P}:(s+1,\dots,s+n)\setminus Y = \bigcup_i Y_i} (-1)^{|\mathrm{P}|} |\mathrm{P}|! \prod_{Y_i \subset \mathrm{P}} S^*_{|Y_i|}(t, Y_i). \tag{2.25}$$

If $Y_i \subset (s+1, \dots, s+n)$, then for the integrable functions F^0_{s+n} and the unitary group of operators $S^*(t) = \oplus_{n=0}^{\infty} S^*_n(t)$ the equality is valid

$$\int\limits_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \cdots dx_{s+n} \prod_{Y_i \subset \mathrm{P}} S^*_{|Y_i|}(t; Y_i) F^0_{s+n}(x_1, \dots, x_{s+n}) = \\ = \int\limits_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \cdots dx_{s+n} F^0_{s+n}(x_1, \dots, x_{s+n}).$$

Then, taking into account the validity for arbitrary $Y \subset (s+1,\ldots,s+n)$ the following equality:

$$\sum_{\mathrm{P}\,:(s+1,\ldots,s+n)\setminus Y=\bigcup_i Y_i} (-1)^{|\mathrm{P}|}|\mathrm{P}|! \;=\; (-1)^{|(s+1,\ldots,s+n)\setminus Y|},$$

according to expression (2.25) for series expansions (2.24) of the BBGKY hierarchy, we obtain

$$\begin{aligned} &F_s(t,x_1,\ldots,x_s) = \\ &\quad = \sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} U^*_{1+n}(t,\{1,\ldots,s\},s+1,\ldots,s+n)\times \\ &\qquad\qquad \times F^0_{s+n}(x_1,\ldots,x_{s+n})dx_{s+1}\cdots dx_{s+n},\;\; s\geqslant 1, \end{aligned} \tag{2.26}$$

where $U^*_{1+n}(t)$ is the $(1+n)th$-order reduced cumulant of the groups of operators (1.6)

$$\begin{aligned} &U^*_{1+n}(t,\{1,\ldots,s\},s+1,\ldots,s+n) = \\ &\qquad = \sum_{Y\subset(s+1,\ldots,s+n)} (-1)^{|(s+1,\ldots,s+n)\setminus Y|}\, S^*_{|(1,\ldots,s)\cup Y|}(t,(1,\ldots,s)\cup Y). \end{aligned}$$

Using the symmetry property of initial reduced distribution functions, for integrand functions in every term of series (2.26) the following equalities are valid

$$\begin{aligned} &\sum_{Y\subset(s+1,\ldots,s+n)} (-1)^{|(s+1,\ldots,s+n)\setminus Y)|}\, S^*_{|(1,\ldots,s)\cup Y|}(t,(1,\ldots,s)\cup Y)F^0_{s+n} = \\ &\quad = \sum_{k=0}^{n}(-1)^k \sum_{i_1<\ldots<i_{n-k}=s+1}^{s+n} S^*_{s+n-k}(t,1,\ldots,s,i_1,\ldots,i_{n-k})F^0_{s+n} = \\ &\quad = \sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}S^*_{s+n-k}(t,1,\ldots,s+n-k)F^0_{s+n}(x_1,\ldots,x_{s+n}). \end{aligned}$$

Thus, the $(1+n)th$-order reduced cumulant represents by the following expansion [80]:

$$\begin{aligned} &U^*_{1+n}(t,\{1,\ldots,s\},s+1,\ldots,s+n) = \\ &\qquad = \sum_{k=0}^{n}(-1)^k\frac{n!}{k!(n-k)!}S^*_{s+n-k}(t,1,\ldots,s+n-k), \end{aligned}$$

and consequently, we derive the representation for series expansions of a solution of the BBGKY hierarchy [82] which is can be written down in terms of an analogue of the annihilation operator (1.10):

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \mathfrak{a}^{n-k} S^*(t) \mathfrak{a}^k F(0) = \\ &= S^*(t)F(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \big[\underbrace{\mathfrak{a}, \ldots, \big[\mathfrak{a}}_{\text{n-times}}, S^*(t)\big] \ldots \big] F(0) = \\ &= e^{\mathfrak{a}} S^*(t) e^{-\mathfrak{a}} F(0). \end{aligned} \tag{2.27}$$

Finally, in view of the validity of the equality

$$S^*(t-\tau)\big[\mathfrak{a}, \mathcal{L}^*\big] S^*(\tau) F(0) = \frac{d}{d\tau} S^*(t-\tau) \mathfrak{a} S^*(\tau) F(0),$$

expansion (2.27) is represented in the form of perturbation (iteration) series of the BBGKY hierarchy (2.17) for many hard spheres

$$\begin{aligned} F(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n S^*(t-t_1)\big[\mathfrak{a}, \mathcal{L}^*\big] S^*(t_1 - t_2) \ldots \times \\ \times S^*(t_{n-1} - t_n)\big[\mathfrak{a}, \mathcal{L}^*\big] S^*(t_n) F(0), \end{aligned}$$

or in component-wise form [6, 58, 79]:

$$\begin{aligned} F_s(t, x_1, \ldots, x_s) = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{(\mathbb{R}^3 \times \mathbb{R}^3)^n} dx_{s+1} \cdots dx_{s+n}\, S_s^*(t-t_1) \times \\ \times \sum_{j_1=1}^{s} \mathcal{L}^*_{\mathrm{int}}(j_1, s+1) S^*_{s+1}(t_1 - t_2) \cdots S^*_{s+n-1}(t_{n-1} - t_n) \times \\ \times \sum_{j_n=1}^{s+n-1} \mathcal{L}^*_{\mathrm{int}}(j_n, s+n) S^*_{s+n}(t_n) F^0_{s+n}(x_1, \ldots, x_{s+n}),\ s \geqslant 1. \end{aligned} \tag{2.28}$$

Let us make some comments concerning the existence of solutions to the Cauchy problem of the BBGKY hierarchy for initial data from various function spaces.

In the spaces of sequences of integrable functions, the existence and uniqueness of a global in time non-perturbative solution was proved in the papers [67, 82] (see also book [14]). It should be noted that the first few terms of the (2.24) series were established in papers [17, 18, 74, 75] as an analog of cluster expansions of the reduced equilibrium distribution functions.

The BBGKY hierarchy describes both the non-equilibrium and equilibrium states. Non-equilibrium states are described by the solution of the initial value problem for this hierarchy, and, correspondingly, equilibrium states are solutions of the steady BBGKY hierarchy. The existence of equilibrium solutions to the steady BBGKY hierarchy has been reviewed in books [14, 64].

As is known, to describe the evolution of a state of infinitely many particles, the suitable functional space is the space of sequences of functions bounded with respect to the configuration variables and decreasing with respect to the momentum ones; in particular, the equilibrium distribution functions belong to this space. For the solution extension from the space of sequences of integrable functions to this space, the method of the thermodynamic limit was developed [14, 57, 58].

For one-dimensional many-particle systems with short-range potential, using the method of interaction region developed by Petrina [81] for solution representation (2.26), the existence theorem for the BBGKY hierarchy was proved for the first time in this functional space. By a similar method for the initial reduced distribution functions from such a space, the existence of a mean value functional for solution (2.9) of the dual BBGKY hierarchy (2.1) was established in the paper [91].

As mentioned above, for a solution representation of the Cauchy problem of the BBGKY hierarchy for hard spheres is widely used in the representation as a series of perturbation theory (2.28) (an iteration series over the evolution of the state of selected groups of particles) [6, 15, 58, 79, 95]. In this form, the solution is applied to construct its Boltzmann–Grad asymptotics, which is governed by the Boltzmann kinetic equation (see section 3.1). The justification of a solution represented as an iteration series for hard spheres is based on giving a rigorous mathematical meaning to every term of the iteration series and on the proof of its convergence. The main difficulty in this problem is that the phase trajectories of particles for a system with a singular interaction potential are defined almost everywhere in the phase space, and initial distribution functions in the iteration series are concentrated on lower-dimensional manifolds. It is necessary to ensure that the trajectories are defined on these manifolds. This problem was completely solved in the papers [61, 83].

In the case of infinitely many hard spheres a local in time solution [58] of the Cauchy problem of the BBGKY hierarchy is represented by iteration series for arbitrary initial data from the space of sequences of functions bounded with respect to configuration variables and for initial data close to the equilibrium state it is a global in time solution [83]. For such initial data in a one-dimensional space for hard sphere system the existence of global in time solution was proved in the paper [34].

In addition, we remark that the correlation decay property, known as the Bogolyubov correlation weakening principle [6], for the solution of the BBGKY hierarchy for hard spheres was proved in [68].

2.3. **The Liouville hierarchy for correlation functions.** An alternative approach to the description of a state of finitely many hard spheres consists in the employment of functions determined by the cluster expansions of the probability distribution functions. The solutions to such cluster expansions are cumulants (semi-invariants) of probability distribution functions and are interpreted, from a physical point of view, as correlations of a state or correlation functions. The evolution of correlation functions is governed by the so-called Liouville hierarchy [31]. Historically, there have been several approaches to describing correlations in many-particle systems. Among them, we mention the well-known approach to the dynamics of correlations by I. Prigogine [86] and R. Balescu [1] and its applications in plasma theory.

Further, it will be established that the constructed dynamics of correlation underlie the description of the dynamics of infinitely many hard spheres governed by the BBGKY hierarchy for reduced distribution functions or the hierarchy of nonlinear evolution equations for reduced correlation functions, i.e., of the cumulants of reduced distribution functions.

We introduce a sequence of correlation functions $g(t) = (1, g_1(t, x_1), \ldots, g_s(t, x_1, \ldots, x_s), \ldots)$ by means of cluster expansions of the probability distribution functions $D(t) = (1, D_1(t, x_1), \ldots, D_n(t, x_1, \ldots, x_n), \ldots)$, defined on the set of allowed configurations $\mathbb{R}^{3n} \backslash \mathbb{W}_n$ as follows:

$$D_n(t, x_1, \ldots, x_n) = g_n(t, x_1, \ldots, x_n) + \\ + \sum_{\substack{\mathrm{P}:(x_1,\ldots,x_n)=\bigcup_i X_i, \\ |\mathrm{P}|>1}} \prod_{X_i \subset \mathrm{P}} g_{|X_i|}(t, X_i), \quad n \geqslant 1, \tag{2.29}$$

where $\sum_{\mathrm{P}:(x_1,\ldots,x_n)=\bigcup_i X_i, \, |\mathrm{P}|>1}$ is the sum over all possible partitions P of the set of the arguments $(x_1, \ldots, x_n)$ into $|\mathrm{P}| > 1$ nonempty mutually disjoint subsets $X_i \subset (x_1, \ldots, x_n)$.

On the set $\mathbb{R}^{3n} \backslash \mathbb{W}_n$ solutions of recursion relations (2.29) are given by the following expansions:

$$g_s(t, x_1, \ldots, x_s) = D_s(t, x_1, \ldots, x_s) + \\ + \sum_{\substack{\mathrm{P}:(x_1,\ldots,x_s)=\bigcup_i X_i, \\ |\mathrm{P}|>1}} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i \subset \mathrm{P}} D_{|X_i|}(t, X_i), \ s \geqslant 1. \tag{2.30}$$

The structure of expansions (2.30) is such that the correlation functions can be treated as cumulants (semi-invariants) of the probability distribution functions [24]. Such an interpretation of these functions is due to the fact that the probability distribution function of statistically independent hard spheres on allowed configurations is described by the product of single-particle correlation functions (probability distribution functions of each particle):

$$g_s(t,x_1,\ldots,x_s)=\prod_{i=1}^{s}g_1(t,x_i)\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s}\delta_{s,1},\ \ s\geqslant 1.$$

The evolution of the sequence of correlation functions (2.30) of many hard spheres is determined by the Cauchy problem of the weak formulation of the Liouville hierarchy of the following evolution equations [53]:

$$\begin{aligned}\frac{\partial}{\partial t}g_s(t,x_1,\ldots,x_s)&=\mathcal{L}^*_s(1,\ldots,s)g_s(t,x_1,\ldots,x_s)+\\&+\sum_{\mathrm{P}:\,(x_1,\ldots,x_s)=X_1\bigcup X_2}\ \sum_{i_1\in\widehat{X}_1}\sum_{i_2\in\widehat{X}_2}\mathcal{L}^*_{\mathrm{int}}(i_1,i_2)g_{|X_1|}(t,X_1)g_{|X_2|}(t,X_2),\end{aligned}\tag{2.31}$$

$$g_s(t,x_1,\ldots,x_s)\big|_{t=0}=g^0_s(x_1,\ldots,x_s),\quad s\geqslant 1,\tag{2.32}$$

where $\sum_{\mathrm{P}:\,(x_1,\ldots,x_s)=X_1\bigcup X_2}$ is the sum over all possible partitions P of the set $(x_1,\ldots,x_s)$ into two nonempty mutually disjoint subsets X_1 and X_2, the symbol $\widehat{X}_i$ means the set of indexes of the set X_i of phase space coordinates and the operator $\mathcal{L}^*_s$ is defined on the subspace $L^1_0\subset L^1$ by formulas (1.7). It should be noted that the Liouville hierarchy (2.31) is the evolution recurrence equations set.

For $t\geqslant 0$ we give a few examples of recurrence equations set (2.31) for a system of hard spheres:

$$\begin{aligned}&\frac{\partial}{\partial t}g_1(t,x_1)=-\langle p_1,\frac{\partial}{\partial q_1}\rangle g_1(t,x_1),\\&\frac{\partial}{\partial t}g_2(t,x_1,x_2)=-\sum_{j=1}^{2}\langle p_j,\frac{\partial}{\partial q_j}\rangle g_2(t,x_1,x_2)+\\&\qquad+\sigma^2\int_{\mathbb{S}^2_+}d\eta\langle\eta,(p_1-p_2)\rangle\Big(g_2(t,q_1,p^*_1,q_2,p^*_2)\delta(q_1-q_2+\sigma\eta)-\\&\qquad\qquad-g_2(t,x_1,x_2)\delta(q_1-q_2-\sigma\eta)\Big)+\\&\qquad+\sigma^2\int_{\mathbb{S}^2_+}d\eta\langle\eta,(p_1-p_2)\rangle\Big(g_1(t,q_1,p^*_1)g_1(t,q_2,p^*_2)\delta(q_1-q_2+\sigma\eta)-\\&\qquad\qquad-g_1(t,x_1)g_1(t,x_2)\delta(q_1-q_2-\sigma\eta)\Big),\end{aligned}$$

where it was used notations accepted above in definition (1.7).

Thus, in terms of correlation functions (2.30), the evolution of the states of a finite number of hard spheres is described by an equivalent method compared to probability distribution functions, namely, within the framework of the dynamics of correlations.

We note that because the Liouville hierarchy (2.31) is the recurrence evolution equations set, we can construct a solution of the Cauchy problem (2.31),(2.32), integrating each equation of the hierarchy as the inhomogeneous Liouville equation. For example, as a result of the integration of the first two equations of the Liouville hierarchy (2.31), we obtain the following equalities:

$$\begin{aligned}
&g_1(t,x_1)=S_1(-t,1)g_1^0(x_1),\\
&g_2(t,1,2)=S_2(-t,1,2)g_2^0(x_1,x_2)+\\
&\quad=\int_0^t dt_1 S_2(t_1-t,1,2)\mathcal{L}^*_{\mathrm{int}}(1,2)S_1(-t_1,1)S_1(-t_1,2)g_1^0(x_1)g_1^0(x_2).
\end{aligned}$$

Then for the corresponding term on the right-hand side of the second equality, an analog of the Duhamel equation holds

$$\begin{aligned}
&\int_0^t dt_1 S_2(t_1-t,1,2)\mathcal{L}^*_{\mathrm{int}}(1,2)S_1(-t_1,1)S_1(-t_1,2)=\\
&\qquad=-\int_0^t dt_1\frac{d}{dt_1}\big(S_2(t_1-t,1,2)S_1(-t_1,1)S_1(-t_1,2)\big)=\\
&\qquad\qquad=S_2(-t,1,2)-S_1(-t,1)S_1(-t,2)=\mathfrak{A}_2^*(t,1,2),
\end{aligned}$$

where $\mathfrak{A}_2^*(t)$ is the second-order cumulant (2.22) of groups of operators (1.6). As a result of similar transformations for $s>2$, the solution of the Cauchy problem (2.31),(2.32), constructed using an iterative procedure, can be represented as expansions in cumulants of groups of operators (1.6).

If the initial state is specified by the sequence

$$g(0)=(1,g_1^0(x_1),\ldots,g_n^0(x_1,\ldots,x_n),\ldots),$$

of correlation functions $g_n^0\in L_n^1$, $n\geqslant 1$, then the evolution of all possible states, i.e., the sequence

$$g(t)=(1,g_1(t,x_1),\ldots,g_s(t,x_1,\ldots,x_s),\ldots)$$

of the correlation functions $g_s(t)$, $s \geqslant 1$, is represented by the following expansions [53]:

$$g_s(t,x_1,\ldots,x_s) = \\ = \sum_{\mathrm{P}:\,(x_1,\ldots,x_s)=\bigcup_j X_j} \mathfrak{A}^*_{|\mathrm{P}|}(t,\{\widehat{X}_1\},\ldots,\{\widehat{X}_{|\mathrm{P}|}\}) \prod_{X_j\subset \mathrm{P}} g^0_{|X_j|}(X_j), \ s \geqslant 1, \tag{2.33}$$

where the symbol $\sum_{\mathrm{P}:\,(x_1,\ldots,x_s)=\bigcup_j X_j}$ denotes the sum over all possible partitions P of the set $(x_1,\ldots,x_s)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets X_j, the symbol $\widehat{X}$ means the set of indexes of the set X of phase space coordinates and the set $(\{\widehat{X}_1\},\ldots,\{\widehat{X}_{|\mathrm{P}|}\})$ consists of elements that are subsets $\widehat{X}_j \subset (1,\ldots,s)$, i.e., $|(\{\widehat{X}_1\},\ldots,\{\widehat{X}_{|\mathrm{P}|}\})| = |\mathrm{P}|$. The generating operator $\mathfrak{A}^*_{|\mathrm{P}|}(t)$ of expansions (2.33) is the $|\mathrm{P}|th$-order cumulant of the groups of operators (1.6) which is defined by the expansion

$$\mathfrak{A}^*_{|\mathrm{P}|}(t,\{\widehat{X}_1\},\ldots,\{\widehat{X}_{|\mathrm{P}|}\}) \doteq \\ \doteq \sum_{\mathrm{P}':\,(\{\widehat{X}_1\},\ldots,\{\widehat{X}_{|\mathrm{P}|}\})=\bigcup_k Z_k} (-1)^{|\mathrm{P}'|-1}(|\mathrm{P}'|-1)! \prod_{Z_k\subset \mathrm{P}'} S^*_{|\theta(Z_k)|}(t,\theta(Z_k)), \tag{2.34}$$

where the symbol θ is the declusterization mapping: $\theta(\{\widehat{X}_i\}) \doteq (\widehat{X}_i)$. The simplest examples of correlation functions (2.33) are given as follows:

$$g_1(t,x_1) = \mathfrak{A}^*_1(t,1)g^0_1(x_1),$$
$$g_2(t,x_1,x_2) = \mathfrak{A}^*_1(t,\{1,2\})g^0_2(x_1,x_2) + \mathfrak{A}^*_2(t,1,2)g^0_1(x_1)g^0_1(x_2).$$

The structure of expansions (2.33) is established as a result of the permutation of the terms of cumulant expansions (2.30) for correlation functions and cluster expansions (2.29) for initial probability distribution functions. Thus, the cumulant origin of correlation functions induces the cumulant structure of their dynamics (2.33).

In particular, in the absence of correlations between hard spheres at the initial moment (initial state satisfying the chaos condition [14, 95]) the sequence of the initial correlation functions on allowed configurations has the form $g^{(c)}(0) = (0, g^0_1(x_1), 0, \ldots, 0, \ldots)$. In terms of a sequence of the probability distribution functions, the chaos condition means that initial data is specified in the form

$$D^{(c)}(0) = \Big(1, D^0_1(x_1), D^0_1(x_1)D^0_1(x_2)\mathcal{X}_{\mathbb{R}^6\setminus\mathbb{W}_2}, \ldots, \prod_{i=1}^n D^0_1(x_i)\mathcal{X}_{\mathbb{R}^{3n}\setminus\mathbb{W}_n}, \ldots\Big),$$

where the function $\mathcal{X}_{\mathbb{R}^{3n}\setminus\mathbb{W}_n}$ is the Heaviside step function of allowed configurations of n hard spheres. In this case for $(x_1,\ldots,x_s) \in \mathbb{R}^{3s}\times(\mathbb{R}^{3s}\setminus\mathbb{W}_s)$

expansions (2.33) are represented as follows:

$$g_s(t,x_1,\ldots,x_s)=\mathfrak{A}_s^*(t,1,\ldots,s)\prod_{i=1}^s g_1^0(x_i)\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s},\quad s\geqslant 1, \tag{2.35}$$

where the generating operator $\mathfrak{A}_s^*(t)$ of this expansion is the sth-order cumulant of groups of operators (1.6) defined by the expansion

$$\mathfrak{A}_s^*(t,1,\ldots,s)\;=\sum_{\mathrm{P}:\,(1,\ldots,s)=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}S_{|X_i|}^*(t,X_i), \tag{2.36}$$

with notations accepted in formula (2.33). From the structure of series (2.35) it is clear that in case of the absence of correlations at the initial instant the correlations generated by the dynamics of a system of hard spheres are completely determined by cumulants (2.36) of the groups of operators (1.6).

We note that in the case of initial data $g^{(c)}(0)$ expansions (2.35) can be rewritten in another representation that explains their physical meaning. Indeed, for $n=1$ we have

$$g_1(t,x_1)=\mathfrak{A}_1^*(t,1)g_1^0(x_1)=g_1^0(p_1,q_1-p_1t).$$

Then, according to formula (2.35) and the definition of the first-order cumulant $\mathfrak{A}_1^*(t)=S_1(-t)$, and its inverse group of operators $S_1^{-1}(-t)=S_1(t)$, we express the correlation functions $g_s(t)$, $s\geqslant 2$, in terms of the one-particle correlation function $g_1(t)$. Therefore, for $s\geqslant 2$ expansions (2.35) are represented in the following form:

$$g_s(t,x_1,\ldots,x_s)=\widehat{\mathfrak{A}}_s^*(t,1,\ldots,s)\prod_{i=1}^s g_1(t,x_i),\quad s\geqslant 2,$$

where $\widehat{\mathfrak{A}}_s^*(t,1,\ldots,s)$ is the s-order cumulant (2.36) of the scattering operators

$$\widehat{S}_n(t,1,\ldots,n)\doteq S_n(-t,1,\ldots,n)\mathcal{X}_{\mathbb{R}^{3n}\setminus\mathbb{W}_n}\prod_{i=1}^n S_1(t,i),\quad n\geqslant 1.$$

On the subspace $L^1_{n,0}$ a generator of the scattering operator $\widehat{S}_n(t,1,\ldots,n)$ is determined by the operator:

$$\frac{d}{dt}\widehat{S}_n(t,1,\ldots,n)\mid_{t=0}\;=\sum_{j_1<j_2=1}^n\mathcal{L}^*_{\mathrm{int}}(j_1,j_2),$$

where for $t\geqslant 0$ the operator $\mathcal{L}^*_{\mathrm{int}}(j_1,j_2)$ is defined by formula (1.7).

If $g_n^0 \in L_n^1$, $n \geqslant 1$, one-parameter mapping (2.33) generates strong continuous group of nonlinear operators

$$\mathcal{G}(t; 1, \ldots, s \mid g(0)) \doteq g_s(t, x_1, \ldots, x_s), \tag{2.37}$$

and it is bounded, and the following estimate holds:

$$\big\|\mathcal{G}(t; 1, \ldots, s \mid g)\big\|_{L_s^1} \leqslant s! c^s,$$

where

$$c \equiv \max\Big(1, \max_{\mathrm{P}:\,(1,\ldots,s)=\bigcup_i X_i} \|g_{|X_i|}\|_{L^1_{|X_i|}}\Big).$$

For $g_n \in L_{n,0}^1$, $n \geqslant 1$, the infinitesimal generator of this group of nonlinear operators has the following structure

$$\begin{aligned}\mathcal{L}(1, \ldots, s \mid g) \doteq{}& \mathcal{L}_s^*(1, \ldots, s) g_s(x_1, \ldots, x_s) + \\ &+ \sum_{\mathrm{P}:\,(x_1,\ldots,x_s)=X_1 \bigcup X_2} \sum_{i_1 \in \widehat{X}_1} \sum_{i_2 \in \widehat{X}_2} \mathcal{L}_{\mathrm{int}}^*(i_1, i_2) g_{|X_1|}(X_1) g_{|X_2|}(X_2),\end{aligned} \tag{2.38}$$

where we used the notation adopted above in expansions (2.33).

The following statement is true [53].

Theorem. *If $t \in \mathbb{R}$, a unique solution of the Cauchy problem of the Liouville hierarchy (2.31),(2.32) is represented by a sequence of expansions (2.33). For $g_n^0 \in L_{n,0}^1 \subset L_n^1$, $n \geqslant 1$, a sequence of functions (2.33) is a classical solution and for arbitrary initial data $g_n^0 \in L_n^1$, $n \geqslant 1$, one has a generalized solution.*

The proof of the theorem is similar to the proof of the existence theorem for the BBGKY hierarchy in the space of sequences of integrable functions [14, 67]. Indeed, if the initial data is $g_n^0 \in L_{n,0}^1$, $n \geqslant 1$, then the infinitesimal generator of the group of nonlinear operators (2.37) coincides with the operator (2.38) and hence the Cauchy problem (2.31),(2.32) has a classical (strong) solution (2.33).

We remark that a steady solution of the Liouville hierarchy (2.31) is a sequence of the Ursell functions on the allowed configurations of hard spheres, i.e., it is the sequence $g^{(eq)} = (0, e^{-\beta \frac{p_1^2}{2}}, 0, \ldots,)$, where β is a parameter inversely proportional to temperature [64].

Finally, we emphasize that the dynamics of correlations, that is, the fundamental equations (2.31) describing the evolution of correlations of states of hard spheres, can be used as a basis for describing the evolution of the state of both a finite and an infinite number of hard spheres instead of the Liouville equations (1.8).

In what follows, we outline an approach to describing the evolution of a state using reduced distribution functions based on the dynamics of correlations in a system of many hard spheres governed by the Liouville hierarchy for correlation functions (2.31).

Remind that reduced distribution functions are defined by means of sequence (1.15) of the probability distribution functions:

$$F_s(t,x_1,\ldots,x_s) \doteq (I,D(t))^{-1}\times \\ \times\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1}\cdots dx_{s+n}\, D_{s+n}(t,x_1,\ldots,x_{s+n}),\ \ s\geqslant 1, \tag{2.39}$$

where the normalizing factor

$$(I,D(t)) \doteq \sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_1\cdots dx_n D_n(t,x_1,\ldots,x_n)$$

is a grand canonical partition function. The possibility of redefining of the reduced distribution functions naturally arises as a result of dividing the series in expression (2.39) by the series of the normalization factor.

A definition of reduced distribution functions equivalent to definition (2.39) is formulated on the basis of correlation functions (2.33) of a system of hard spheres by means of the following series expansion [53]:

$$F_s(t,x_1,\ldots,x_s) \doteq \sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1}\cdots dx_{s+n}\times \\ \times g_{1+n}(t,\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n}),\ \ s\geqslant 1, \tag{2.40}$$

where on the set of allowed configurations $\mathbb{R}^{3(s+n)}\setminus\mathbb{W}_{s+n}$ the correlation functions of clusters of hard spheres $g_{1+n}(t), n\geqslant 0$, are determined by the expansions:

$$g_{1+n}(t,\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n}) = \\ = \sum_{\substack{\mathrm{P}:(\{x_1,\ldots,x_s\},\\ x_{s+1},\ldots,x_{s+n})=\bigcup_i X_i}}\mathfrak{A}^*_{|\mathrm{P}|}\big(t,\{\theta(\widehat{X}_1)\},\ldots,\{\theta(\widehat{X}_{|\mathrm{P}|})\}\big)\prod_{X_i\subset\mathrm{P}} g^0_{|X_i|}(X_i),\ \ n\geqslant 0. \tag{2.41}$$

We remind that in expansions (2.41) the generating operator $\mathfrak{A}^*_{|\mathrm{P}|}(t)$ is the $|\mathrm{P}|th$-order cumulant (2.34) of the groups of operators (1.6), and the symbol $\sum_{\mathrm{P}:(\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n})=\bigcup_i X_i}$ means the sum over all possible partitions P of the set $(\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n})$ into nonempty mutually disjoint subsets X_i.

On allowed configurations the correlation functions of particle clusters in series (2.40), i.e., the functions $g_{1+n}(t,\{x_1,\ldots,x_s\},\,x_{s+1},\ldots,x_{s+n})$, $n\geqslant 0$, are defined as solutions of generalized cluster expansions of a sequence of solutions of the Liouville equations:

$$D_{s+n}(t,x_1,\ldots,x_{s+n})= \\ =\sum_{\substack{\mathrm{P}:(\{x_1,\ldots,x_s\},\\ x_{s+1},\ldots,x_{s+n})=\bigcup_i X_i}}\ \prod_{X_i\subset\mathrm{P}} g_{|X_i|}(t,X_i),\quad s\geqslant 1,\,n\geqslant 0, \tag{2.42}$$

namely,

$$g_{1+n}(t,\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n})= \\ =\sum_{\substack{\mathrm{P}:(\{x_1,\ldots,x_s\},\\ x_{s+1},\ldots,x_{s+n})=\bigcup_i X_i}}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}} D_{|\theta(X_i)|}(t,\theta(X_i)),\quad s\geqslant 1,\,n\geqslant 0,$$

where θ is the declusterization mapping defined in formula (2.34), the probability distribution function $D_{|\theta(X_i)|}(t,\theta(X_i))$ is a solution of the Liouville equation.

The correlation functions of particle clusters satisfy the Liouville hierarchy of evolution equations with the following generator

$$\mathcal{L}(\{1,\ldots,s\},s+1,\ldots,s+n\mid\mathfrak{d}_{\{Y\}}g(t))\doteq \\ \doteq\mathcal{L}^*_{s+n}(1,\ldots,s+n)g_{1+n}(t,X)+ \\ +\sum_{\mathrm{P}:\,X=X_1\bigcup X_2}\ \sum_{i_1\in\theta(\hat{X}_1)}\ \sum_{i_2\in\theta(\hat{X}_2)}\mathcal{L}^*_{\mathrm{int}}(i_1,i_2)g_{|X_1|}(t,X_1)g_{|X_2|}(t,X_2),\ n\geqslant 0, \tag{2.43}$$

where $X\equiv(\{Y\},x_{s+1},\ldots,x_{s+n})\equiv(\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n})$, the sequence of solutions of generalized cluster expansions (2.42) is denoted by means of the mapping

$$(\mathfrak{d}_{\{Y\}}g)_n(x_1,\ldots,x_n)\doteq g_{1+n}(\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n}),\quad n\geqslant 0,$$

and we also used the notations adopted above in expansion (2.33).

We note that on the allowed configurations the correlation functions of hard-sphere clusters can be expressed through correlation functions of hard spheres (2.33) by the following relations:

$$g_{1+n}(t,\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n})= \\ =\sum_{\substack{\mathrm{P}:(\{x_1,\ldots,x_s\},\\ x_{s+1},\ldots,x_{s+n})=\bigcup_i X_i}}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\times \\ \times\prod_{X_i\subset\mathrm{P}}\ \sum_{\mathrm{P}':\,\theta(X_i)=\bigcup\limits_{j_i} Z_{j_i}}\ \prod_{Z_{j_i}\subset\mathrm{P}'} g_{|Z_{j_i}|}(t,Z_{j_i}),\ n\geqslant 0. \tag{2.44}$$

In particular case $n=0$, i.e., the correlation function of a cluster of the s hard spheres, these relations take the form

$$g_{1+0}(t,\{x_1,\ldots,x_s\}) \;=\sum_{\mathrm{P}:\,\theta(\{x_1,\ldots,x_s\})=\bigcup_i X_i}\;\prod_{X_i\subset\mathrm{P}} g_{|X_i|}(t,X_i).$$

As a consequence of these relations, for the initial state satisfying the chaos condition, from (2.41) the following generalization of expansions (2.35) holds:

$$\begin{aligned} &g_{s+n}(t,\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n}) = \\ &\quad=\mathfrak{A}_{1+n}(t,\{1,\ldots,s\},s+1,\ldots,s+n)\times \\ &\qquad\times\prod_{i=1}^{s+n} g_1^0(x_i)\mathcal{X}_{\mathbb{R}^{3(s+n)}\setminus\mathbb{W}_{s+n}},\;\; s\geqslant 1,\, n\geqslant 0. \end{aligned} \tag{2.45}$$

As we noted above, the possibility of the description of the evolution of a state based on the dynamics of correlations (2.40) occurs naturally in consequence of dividing the series of expressions (2.39) by the series of the normalizing factor. To provide evidence of this statement, we will introduce the necessary notions and prove the validity of some auxiliary equalities.

On sequences of functions $f,\widetilde{f}\in L^1\oplus_{n=0}^{\infty}L^1_n$ we define the following $*$-product [90]

$$(f*\widetilde{f})_s(x_1,\ldots,x_s) \;=\sum_{Z\subset(x_1,\ldots,x_s)} f_{|Z|}(Z)\,\widetilde{f}_{s-|Z|}((x_1,\ldots,x_s)\setminus Z), \tag{2.46}$$

where $\sum_{Z\subset(x_1,\ldots,x_s)}$ is the sum over all subsets Z of the set $(x_1,\ldots,x_s)$. Using the definition of the $*$-product (2.46), we introduce the mapping $\mathbb{E}\mathrm{xp}_*$ and the inverse mapping $\mathbb{L}\mathrm{n}_*$ on sequences

$$h=(0,h_1(x_1),\ldots,h_n(x_1,\ldots,x_n),\ldots)$$

of functions $h_n\in L^1_n$ by the expansions:

$$\begin{aligned} (\mathbb{E}\mathrm{xp}_*\,h)_s(x_1,\ldots,x_s) &= \Big(\mathbb{I}+\sum_{n=1}^{\infty}\frac{h^{*n}}{n!}\Big)_s(x_1,\ldots,x_s) = \\ &= \delta_{s,0}+\sum_{\mathrm{P}:\,(x_1,\ldots,x_s)=\bigcup_i X_i}\;\prod_{X_i\subset\mathrm{P}} h_{|X_i|}(X_i), \end{aligned} \tag{2.47}$$

where we used the notations accepted in formula (2.29), $\mathbb{I}=(1,0,\ldots,0,\ldots)$ and $\delta_{s,0}$ is the Kronecker symbol, and respectively,

$$\begin{aligned}(\mathbb{L}\mathrm{n}_*(\mathbb{I}+h))_s(x_1,\ldots,x_s) &= \Big(\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{h^{*n}}{n}\Big)_s(x_1,\ldots,x_s) = \\ &= \sum_{\mathrm{P}:\,(x_1,\ldots,x_s)=\bigcup\limits_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}h_{|X_i|}(X_i).\end{aligned}\tag{2.48}$$

Therefore in terms of sequences of functions recursion relations (2.29) are rewritten in the form

$$D(t)=\mathbb{E}\mathrm{xp}_*\,g(t),$$

where $D(t)=\mathbb{I}+(0,D_1(t,x_1),\ldots,D_n(t,x_1,\ldots,x_n),\ldots)$. As a result, we get

$$g(t)=\mathbb{L}\mathrm{n}_*\,D(t).$$

Thus, according to definition (2.46) of the $*$-product and mapping (2.48), in the component-wise form solutions of recursion relations (2.29) are represented by expansions (2.30).

For arbitrary $f=(f_0,f_1,\ldots,f_n,\ldots)\in L^1$ and the set $Y\equiv(x_1,\ldots,x_s)$ we define the linear mapping $\mathfrak{d}_Y: f\rightarrow\mathfrak{d}_Y f$, by the formula

$$(\mathfrak{d}_Y f)_n(x_1,\ldots,x_n)\doteq f_{s+n}(x_1,\ldots,x_s,x_{s+1},\ldots,x_{s+n}),\ n\geqslant 0.\tag{2.49}$$

For the set $\{Y\}$ consisting of the one element $Y=(x_1,\ldots,x_s)$, we have, respectively

$$(\mathfrak{d}_{\{Y\}}f)_n(x_1,\ldots,x_n)\doteq f_{1+n}(\{x_1,\ldots,x_s\},x_{s+1},\ldots,x_{s+n}),\ n\geqslant 0.\tag{2.50}$$

On sequences $\mathfrak{d}_Y f$ and $\mathfrak{d}_{Y'}\widetilde{f}$ we introduce the $*$-product

$$(\mathfrak{d}_Y f*\mathfrak{d}_{Y'}\widetilde{f})_{|X|}(X)\doteq\sum_{Z\subset X}f_{|Z|+|Y|}(Y,Z)\,\widetilde{f}_{|X\backslash Z|+|Y'|}(Y',X\backslash Z),$$

where X,Y,Y' are the sets, which characterize clusters of hard spheres, and $\sum_{Z\subset X}$ is the sum over all subsets Z of the set X. In particular case $Y=\varnothing$, $Y'=\varnothing$, this definition reduces to definition of $*$-product (2.46).

Let us establish some properties of introduced mappings (2.47) and (2.50).

If $f_n\in L^1_n$, $n\geqslant 1$ for the sequences $f=(0,f_1,\ldots,f_n,\ldots)$, according to definitions of mappings (2.47) and (2.50), the following equality holds

$$\mathfrak{d}_{\{Y\}}\mathbb{E}\mathrm{xp}_*f=\mathbb{E}\mathrm{xp}_*f*\mathfrak{d}_{\{Y\}}f,\tag{2.51}$$

and for mapping (2.49), respectively

$$\mathfrak{d}_Y\mathbb{E}\mathrm{xp}_*f=\mathbb{E}\mathrm{xp}_*f\ \ *\sum_{\mathrm{P}:\,Y=\bigcup_i X_i}\mathfrak{d}_{X_1}f*\ldots*\mathfrak{d}_{X_{|\mathrm{P}|}}f,$$

where $\sum_{\mathrm{P}:\,Y=\bigcup_i X_i}$ denotes the sum over all possible partitions P of the set $Y \equiv (x_1,\ldots,x_s)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i \subset Y$.

Hence, in terms of mappings (2.49) and (2.50) generalized cluster expansions (2.42) take the form

$$\mathfrak{d}_Y D(t) = \mathfrak{d}_{\{Y\}}\mathbb{E}\mathrm{xp}_*\, g(t). \tag{2.52}$$

On sequences of functions $f \in L^1 = \oplus_{n=0}^{\infty} L^1_n$ we also define the analogue of the annihilation operator

$$(\mathfrak{a}f)_n(x_1,\ldots,x_n) \;=\; \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx_{n+1} f_{n+1}(x_1,\ldots,x_n,x_{n+1}). \tag{2.53}$$

Then for sequences $f, \widetilde{f} \in L^1$, the following equality holds

$$(e^{\mathfrak{a}} f * \widetilde{f})_0 = (e^{\mathfrak{a}} f)_0 (e^{\mathfrak{a}} \widetilde{f})_0, \tag{2.54}$$

where such a notation was used

$$(e^{\mathfrak{a}} f)_0 = \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_1 \cdots dx_n\, f_n(x_1,\ldots,x_n). \tag{2.55}$$

Now let us prove the equivalence of definition (2.39) of the reduced distribution functions and their definition (2.40) within the framework of the dynamics of correlations.

In terms of mapping (2.49) and notation (2.55) the definition of reduced distribution functions (2.39) is written as follows

$$F_s(t,x_1,\ldots,x_s) = (e^{\mathfrak{a}} D(t))_0^{-1} (e^{\mathfrak{a}} \mathfrak{d}_Y D(t))_0.$$

Using generalized cluster expansions (2.52), and as a consequence of equalities (2.51) and (2.54), we find

$$(e^{\mathfrak{a}} \mathfrak{d}_Y D(t))_0 = (e^{\mathfrak{a}} \mathfrak{d}_{\{Y\}}\mathbb{E}\mathrm{xp}_*\, g(t))_0 = \\ (e^{\mathfrak{a}}\mathbb{E}\mathrm{xp}_* g(t) * \mathfrak{d}_{\{Y\}} g(t))_0 = (e^{\mathfrak{a}}\mathbb{E}\mathrm{xp}_* g(t))_0 (e^{\mathfrak{a}} \mathfrak{d}_{\{Y\}} g(t))_0.$$

Taking into account that, according to the particular case $Y = \varnothing$, of cluster expansions (2.42), the equality holds

$$(e^{\mathfrak{a}}\mathbb{E}\mathrm{xp}_* g(t))_0 = (e^{\mathfrak{a}} D(t))_0,$$

as a result, we establish the following representation for the reduced distribution functions

$$F_s(t,x_1,\ldots,x_s) = (e^{\mathfrak{a}} \mathfrak{d}_{\{Y\}} g(t))_0.$$

Therefore, in componentwise-form we obtain relation (2.40).

Since the correlation functions $g_{1+n}(t)$, $n \geqslant 0$, are governed by the corresponding Liouville hierarchy for the cluster of hard spheres and hard spheres, the reduced distribution functions (2.40) are governed by the BBGKY hierarchy for hard spheres

$$\frac{\partial}{\partial t}F(t) = e^{\mathfrak{a}}\mathcal{L}(\{\cdot\},\cdot \mid e^{-\mathfrak{a}}F(t)), \tag{2.56}$$

where the operator $\mathcal{L}(\{\cdot\},\cdot \mid f)$ is generator (2.43) of the Liouville hierarchy for a cluster of hard spheres and hard spheres. For a generator of this hierarchy of evolution equations takes place the following representation:

$$e^{\mathfrak{a}}\mathcal{L}(\{\cdot\},\cdot \mid e^{-\mathfrak{a}}F(t)) = e^{\mathfrak{a}}\mathcal{L}^{*}e^{-\mathfrak{a}}F(t),$$

where the operator $\mathcal{L}^{*} = \oplus_{n=0}^{\infty}\mathcal{L}_n^{*}$ is a direct sum of the Liouville operators and the operator $\mathfrak{a}$ is defined by formula (2.53). Due to the fact that pairwise collisions occur during the evolution, a generator of this hierarchy is reduced to the operator of such a structure [14]

$$e^{\mathfrak{a}}\mathcal{L}^{*}e^{-\mathfrak{a}} = \mathcal{L}^{*} + [\mathfrak{a},\mathcal{L}^{*}],$$

where as above the bracket $[\cdot,\cdot]$ is the commutator of operators.

We note that for the first time the BBGKY hierarchy for many hard spheres (2.56) was mathematically justified in paper [83] (see also [14]).

In consequence of definition (2.40) and the cumulant structure of representation of a solution (2.33) for the Liouville hierarchy (2.31), if initial state specified by the sequence of reduced distribution functions

$$F(0) = \big(1, F_1^0(x_1), \ldots, F_n^0(x_1,\ldots,x_n),\ldots\big),$$

then the evolution of all possible states, i.e., a sequence of the reduced distribution functions $F_s(t)$, $s \geqslant 1$, is determined by the series expansions (2.24).

We remark that the representation (2.24) is directly established for the initial states satisfying the chaos condition due to the validity in this case of the representation (2.45) for the correlation functions of the hard-sphere cluster and of the hard spheres.

Consequently, as follows from the above, the cumulant structure of generating operators of expansions for correlation functions (2.33) or (2.41) induces the cumulant structure (2.22) of generating operators of series expansions for reduced distribution functions (2.24) or in other words, the evolution of the state of a system of an infinite number of hard spheres is governed by the dynamics of correlations on a microscopic scale.

Thus, we have established relation (2.40) between the reduced distribution functions and correlation functions governed by the Liouville hierarchy.

2.4. **The hierarchy of nonlinear evolution equations for reduced correlation functions.** As is known, on a microscopic scale, the macroscopic characteristics of fluctuations of observables are directly determined by means of the reduced correlation functions. Assuming as a basis an alternative approach to the description of the evolution of states of a hard-sphere system within the framework of correlation functions (2.33), then the reduced correlation functions are defined by means of a solution of the Cauchy problem of the Liouville hierarchy (2.31),(2.32) as follows [53]:

$$G_s(t,x_1,\ldots,x_s)\doteq$$
$$\doteq\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_{s+1}\cdots dx_{s+n}\,g_{s+n}(t,x_1,\ldots,x_{s+n}),\ \ s\geqslant1, \tag{2.57}$$

where the generating function $g_{s+n}(t,x_1,\ldots,x_{s+n})$ is defined by expansion (2.33), or in terms of mapping (2.49) and notation (2.55) this definition takes the form

$$G_s(t,x_1,\ldots,x_s)=(e^{\mathfrak{a}}\mathfrak{d}_Y g(t))_0,$$

or in terms of sequences of functions this expression has the form

$$G(t)=e^{\mathfrak{a}}g(t).$$

We emphasize that nth term of expansions (2.57) of the reduced correlation functions are determined by the $(s{+}n)th$-particle correlation function (2.33) in contrast with the expansions of reduced distribution functions (2.40) which are determined by the $(1+n)th$-particle correlation function of clusters of hard spheres (2.41).

Such a representation for reduced correlation functions (2.57) can be derived as a result of the fact that the reduced correlation functions are cumulants of reduced distribution functions (2.40). Indeed, traditionally reduced correlation functions are introduced by means of the cluster expansions of the reduced distribution functions similar to the cluster expansions of the probability distribution functions (2.29) and on the set of allowed configurations $\mathbb{R}^{3n}\backslash\mathbb{W}_n$ they have the form:

$$F_s(t,x_1,\ldots,x_s)=\sum_{\mathrm{P}:(x_1,\ldots,x_s)=\bigcup_i X_i}\ \prod_{X_i\subset\mathrm{P}}G_{|X_i|}(t,X_i),\ \ s\geqslant1, \tag{2.58}$$

where as above the symbol $\sum_{\mathrm{P}:(x_1,\ldots,x_s)=\bigcup_i X_i}$ is the sum over all possible partitions P of the set $(x_1,\ldots,x_s)$ into $|\mathrm{P}|$ nonempty mutually disjoint subsets $X_i\subset(x_1,\ldots,x_s)$. As a consequence of this, the solution of recurrence relations (2.58) are represented through reduced distribution functions as follows:

$$G_s(t,x_1,\ldots,x_s)=$$
$$=\sum_{\mathrm{P}:(x_1,\ldots,x_s)=\bigcup_i X_i}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\prod_{X_i\subset\mathrm{P}}F_{|X_i|}(t,X_i),\ \ s\geqslant1. \tag{2.59}$$

Functions (2.59) are interpreted as the functions which describe the correlations of hard-sphere states. The structure of expansions (2.59) is such that the reduced correlation functions are cumulants (semi-invariants) of the reduced distribution functions (2.24).

Thus, taking into account representation (2.40) of the reduced distribution functions, in consequence of the validity of relations (2.44) we derive representation (2.57) of the reduced correlation functions through correlation functions

$$G_s(t, x_1, \ldots, x_s) = \\ = \sum_{\mathrm{P}:(x_1,\ldots,x_s)=\bigcup_i X_i} (-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)! \prod_{X_i \subset \mathrm{P}} (e^{\mathfrak{a}} \mathfrak{d}_{\{X_i\}} g(t)) = (e^{\mathfrak{a}} \mathfrak{d}_Y g(t))_0.$$

Since the correlation functions $g_{s+n}(t)$, $n \geqslant 0$, are governed by the Liouville hierarchy for hard spheres (2.31), the reduced correlation functions defined as (2.57) are governed by the hierarchy of nonlinear equations for hard spheres (the nonlinear BBGKY hierarchy) [53]:

$$\begin{aligned} \frac{\partial}{\partial t} G_s(t, x_1, \ldots, x_s) &= \mathcal{L}^*_s G_s(t, x_1, \ldots, x_s) + \\ &+ \sum_{\mathrm{P}:\,(x_1,\ldots,x_s)=X_1 \bigcup X_2} \sum_{i_1 \in \widehat{X}_1} \sum_{i_2 \in \widehat{X}_2} \mathcal{L}^*_{\mathrm{int}}(i_1, i_2) G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2)) + \\ &+ \int_{\mathbb{R}^3 \times \mathbb{R}^3} dx_{s+1} \Big(\sum_{i=1}^{s} \mathcal{L}^*_{\mathrm{int}}(i, s+1) G_{s+1}(t, x_1, \ldots, x_{s+1}) + \\ &+ \sum_{\mathrm{P}:\,(x_1,\ldots,x_{s+1})=X_1 \bigcup X_2} \sum_{\substack{i \in \widehat{X}_1 \\ s+1 \in \widehat{X}_2}} \mathcal{L}^*_{\mathrm{int}}(i, s+1) G_{|X_1|}(t, X_1) G_{|X_2|}(t, X_2) \Big), \end{aligned} \tag{2.60}$$

$$G_s(t, x_1, \ldots, x_s)\big|_{t=0} = G^0_s(x_1, \ldots, x_s), \quad s \geqslant 1, \tag{2.61}$$

where the symbol $\sum_{\mathrm{P}:(x_1,\ldots,x_{s+1})=X_1 \bigcup X_2}$ means the sum over all possible partitions of the set $(x_1, \ldots, x_{s+1})$ into two mutually disjoint subsets X_1 and X_2, the sum over the index i which takes values from the subset $\widehat{X}_1$ provided that the index $s+1$ belongs to the subset $\widehat{X}_2$ is denoted by $\sum_{i \in \widehat{X}_1, s+1 \in \widehat{X}_2}$ and notations accepted in the Liouville hierarchy (2.31) are used.

A generator of this hierarchy of nonlinear evolution equations has the following structure:

$$\frac{\partial}{\partial t} G(t) = e^{\mathfrak{a}} \mathcal{L}(\cdot \mid e^{-\mathfrak{a}} G(t)),$$

where the operator $\mathcal{L}(\cdot \mid f) = \oplus_{n=0}^{\infty} \mathcal{L}(1, \ldots, n \mid f)$ is a direct sum of generators (2.38) of the Liouville hierarchy (2.31). Here are some component-wise examples of hierarchy (2.60):

$$\frac{\partial}{\partial t}G_1(t,x_1)=\mathcal{L}^*_1(1)G_1(t,x_1)+$$
$$+\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}dx_2\mathcal{L}^*_{\mathrm{int}}(1,2)\big(G_2(t,x_1,x_2)+G_1(t,x_1)G_1(t,x_2)\big),$$
$$\frac{\partial}{\partial t}G_2(t,x_1,x_2)=\mathcal{L}^*_2(1,2)G_2(t,x_1,x_2)+\mathcal{L}^*_{\mathrm{int}}(1,2)G_1(t,x_1)G_1(t,x_2)+$$
$$+\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}dx_3\Big(\sum_{i=1}^{2}\mathcal{L}^*_{\mathrm{int}}(i,3)\big(G_3(t,x_1,x_2,x_3)+G_2(t,x_1,x_2)G_1(t,x_3)\big)+$$
$$+\mathcal{L}^*_{\mathrm{int}}(2,3)G_2(t,x_1,x_3)G_1(t,x_2)+\mathcal{L}^*_{\mathrm{int}}(1,3)G_2(t,x_2,x_3)G_1(t,x_1)\Big),$$

where it was used notations accepted above in definition (1.7).

If $G(0)=(1,G^0_1(x_1),\ldots,G^0_s(x_1,\ldots,x_s),\ldots)$ is a sequence of reduced correlation functions at initial instant, then by means of mappings (2.37) the evolution of all possible states, i.e., the sequence of the reduced correlation functions $G_s(t)$, $s\geqslant 1$, is determined by the following series expansions:

$$G_s(t,x_1,\ldots,x_s)=$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_{s+1}\cdots dx_{s+n}\times \tag{2.62}$$
$$\times\,\mathfrak{A}_{1+n}\big(t;\{1,\ldots,s\},s+1,\ldots,s+n\mid G(0)\big),\quad s\geqslant 1,$$

where the generating operator $\mathfrak{A}_{1+n}(t;\{1,\ldots,s\},s+1,\ldots,s+n\mid G(0))$ of this series is the $(1+n)th$-order cumulant of groups of nonlinear operators (2.33):

$$\mathfrak{A}_{1+n}(t;\{1,\ldots,s\},s+1,\ldots,s+n\mid G(0))\doteq$$
$$\doteq\sum_{\mathrm{P}:\,(\{1,\ldots,s\},s+1,\ldots,s+n)=\bigcup_k X_k}(-1)^{|\mathrm{P}|-1}(|\mathrm{P}|-1)!\times \tag{2.63}$$
$$\times\,\mathcal{G}\big(t;\theta(X_1)\mid\ldots\mathcal{G}(t;\theta(X_{|\mathrm{P}|})\mid G(0))\ldots\big),\;\; n\geqslant 0,$$

and where the composition of mappings (2.33) of the corresponding noninteracting groups of particles was denoted by

$$\mathcal{G}(t;\theta(X_1)\mid\ldots\mathcal{G}(t;\theta(X_{|\mathrm{P}|})\mid G(0))\ldots),$$

for example,

$$\mathcal{G}\big(t;1\mid\mathcal{G}(t;2\mid G(0))\big)=\mathfrak{A}_1(t,1)\mathfrak{A}_1(t,2)G^0_2(x_1,x_2),$$
$$\mathcal{G}\big(t;1,2\mid\mathcal{G}(t;3\mid G(0))\big)=\mathfrak{A}_1(t,\{1,2\})\mathfrak{A}_1(t,3)G^0_3(x_1,x_2,x_3)+$$

$$+\mathfrak{A}_2(t,1,2)\mathfrak{A}_1(t,3)\big(G_1^0(x_1)G_2^0(x_2,x_3)+G_1^0(x_2)G_2^0(x_1,x_3)\big).$$

We will adduce examples of expansions (2.63). The first order cumulant of the groups of nonlinear operators (2.33) is the group of these nonlinear operators

$$\mathfrak{A}_1(t;\{1,\ldots,s\}\mid G(0))=\mathcal{G}(t;1,\ldots,s\mid G(0)).$$

In case of $s=2$ the second order cumulant of nonlinear operators (2.33) has the structure

$$\begin{aligned}
&\mathfrak{A}_{1+1}(t;\{1,2\},3\mid G(0))=\\
&\quad=\mathcal{G}(t;1,2,3\mid G(0))-\mathcal{G}\big(t;1,2\mid\mathcal{G}(t;3\mid G(0))\big)=\\
&\quad=\mathfrak{A}_{1+1}^*(t,\{1,2\},3)G_3^0(1,2,3)+\\
&\quad\;+\big(\mathfrak{A}_{1+1}^*(t,\{1,2\},3)-\mathfrak{A}_2(t,2,3)\mathfrak{A}_1^*(t,1)\big)G_1^0(x_1)G_2^0(x_2,x_3)+\\
&\quad\;+\big(\mathfrak{A}_{1+1}(t,\{1,2\},3)-\mathfrak{A}_2^*(t,1,3)\mathfrak{A}_1^*(t,2)\big)G_1^0(x_2)G_2^0(x_1,x_3)+\\
&\quad\;+\mathfrak{A}_{1+1}^*(t,\{1,2\},3)G_1^0(x_3)G_2^0(x_1,x_2)+\mathfrak{A}_3^*(t,1,2,3)G_1^0(x_1)G_1^0(x_2)G_1^0(x_3),
\end{aligned}$$

where the operator

$$\mathfrak{A}_3^*(t,1,2,3)=\mathfrak{A}_{1+1}^*(t,\{1,2\},3)-\mathfrak{A}_2^*(t,2,3)\mathfrak{A}_1^*(t,1)-\mathfrak{A}_2^*(t,1,3)\mathfrak{A}_1^*(t,2)$$

is the third-order cumulant (2.36) of groups of operators (1.6) of a system of hard spheres.

The following statement is true [53].

Theorem. *Let $G(0)\in\oplus_{n=0}^{\infty}L_n^1$, then for arbitrary $t\in\mathbb{R}$ provided that $\max\limits_{n\geqslant1}\|G_n^0\|_{L_n^1}<(2e^3)^{-1}$, the sequence of reduced correlation functions (2.62) is a unique solution of the Cauchy problem of nonlinear hierarchy (2.60), (2.61) for hard spheres.*

In the particular case of the initial state specified by the sequence of reduced correlation functions $G^{(c)}=(0,G_1^0,0,\ldots,0,\ldots)$ on the allowed configurations, that is, in the absence of correlations between hard spheres at the initial moment of time [14], according to definition (2.63) of the generating operators, reduced correlation functions (2.62) are represented by the following series expansions:

$$\begin{aligned}
G_s(t,x_1,\ldots,x_s)=&\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_{s+1}\cdots dx_{s+n}\,\times\\
&\times\,\mathfrak{A}_{s+n}^*(t;1,\ldots,s+n)\prod_{i=1}^{s+n}G_1^0(x_i)\mathcal{X}_{\mathbb{R}^{3(s+n)}\setminus\mathbb{W}_{s+n}},\quad s\geqslant1,
\end{aligned}\tag{2.64}$$

where the generating operator $\mathfrak{A}^*_{s+n}(t)$ is the $(s+n)th$-order cumulant (2.36) of the groups of operators (1.6).

We emphasize that in the absence of correlations of states of hard spheres on allowed configurations at the initial moment of time, the generators of expansions into a series of reduced correlation functions (2.64) and reduced distribution functions (2.24) differ only in the order of cumulants of groups of operators of hard spheres. Therefore, by means of such reduced distribution functions or reduced correlation functions, the process of creating correlations in a system of hard spheres is described.

We note that the reduced correlation functions give an equivalent approach to the description of the evolution of states of many hard spheres, along with the reduced distribution functions. Indeed, the macroscopic characteristics of fluctuations of observables are directly determined by the reduced correlation functions on the microscopic scale [6] for example, the functional of the dispersion of an additive-type observable, i.e., the sequence $A^{(1)} = (0, a_1(x_1), \ldots, \sum\limits_{i_1=1}^{n} a_1(x_{i_1}), \ldots)$, is represented by the formula

$$\langle (A^{(1)} - \langle A^{(1)} \rangle)^2 \rangle(t) = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 \, (a_1^2(x_1) - \langle A^{(1)} \rangle^2(t)) G_1(t, x_1) + \\ + \int\limits_{(\mathbb{R}^3 \times \mathbb{R}^3)^2} dx_1 dx_2 \, a_1(x_1) a_1(x_2) G_2(t, x_1, x_2),$$

where

$$\langle A^{(1)} \rangle(t) = \int\limits_{\mathbb{R}^3 \times \mathbb{R}^3} dx_1 \, a_1(x_1) G_1(t, x_1)$$

is the mean value functional of an additive-type observable.

3. Nonlinear kinetic equations for many hard spheres

The conventional philosophy of the description of kinetic evolution is that if the initial state is specified by a one-particle (reduced) distribution function, then at an arbitrary time the evolution of the state in an appropriate scaling limit can be effectively described by means of a one-particle distribution function that is governed by the nonlinear kinetic equation. Below, we give an answer to the question about the description of the kinetic evolution of colliding particles, not on the basis of a common interpretation but within the framework of the evolution of the observables of many hard spheres.

The problem of a rigorous description of the kinetic evolution by means of hard sphere observables will be considered by giving the example of the

Boltzmann–Grad asymptotics of a non-perturbative solution of the Cauchy problem of the dual BBGKY hierarchy [51].

3.1. **On the Boltzmann–Grad scaling approximation.** The present notion of the Boltzmann–Grad approximation was first introduced in Grad's paper [73]. From a physical point of view, this approximation means that we deal with a low-density gas in a situation where the diameter of a hard sphere, or, in other words, the radius of the short-range interaction potential, is sufficiently less in comparison with the average length of a free path of hard spheres.

In a dimensionless form, the generator of the BBGKY hierarchy for hard spheres contains a scaling parameter: the ratio of the diameter of hard spheres to their mean free path [35]. The finite value of the mean free path of hard spheres means that in this approximation the average number of particles tends to infinity; in other words, according to the definition of (2.16), the state must be described by functions from the appropriate function spaces, for example, from the space to which the sequences of reduced equilibrium distribution functions belong [64, 90]. In this case, the initial state is described by the reduced distribution functions from the space L^∞_ξ of sequences of functions bounded with respect to the configuration variables and decreasing with respect to the momentum ones, equipped with the norm

$$\|f\|_{L^\infty_\xi} = \sup_{n\geqslant 0} \xi^{-n} \sup_{x_1,\dots,x_n} |f_n(x_1,\dots,x_n)| \exp\Big(\beta \sum_{i=1}^{n} \frac{p_i^2}{2}\Big),$$

where $\xi > 0$ and $\beta > 0$ are parameters.

For such initial data, the Boltzmann–Grad asymptotics of a solution of the Cauchy problem of the BBGKY hierarchy for hard spheres are described by the so-called Boltzmann hierarchy [79]. As a consequence, for factorized initial data, i.e., for the initial state without correlations, which describes molecular chaos [14], the equation determining the evolution of an initial state is a closed equation for a one-particle distribution function, that is to say Boltzmann's kinetic equation [13].

The detailed analysis of the problem of the construction of such asymptotics for a solution of the Cauchy problem of the BBGKY hierarchy shows that the basic difficulty consists in proving the term-by-term convergence of the iteration series that represents this solution to the corresponding limit, that is, to the series representing the solution of the Cauchy problem of the Boltzmann hierarchy. This difficulty is related to the fact that the integrands in each term of the iteration series do not converge to the limit uniformly across the whole domain of integration. We note that in early works on the justification of the Boltzmann–Grad limit, attention was not properly paid to this property, and a precise mathematical meaning was not given to the individual terms of the iteration series representing a solution of the BBGKY hierarchy. In the papers [61, 83] a complete discussion of these problems was presented.

From a mathematical point of view, the existence of the Boltzmann–Grad asymptotics of a perturbative solution of the BBGKY hierarchy for hard spheres was discussed in Cercignani's paper [12] and later in Lanford's work [79]. A rigorous mathematical proof of the Boltzmann–Grad limit theorem has been given in a series of papers [57–59,61,83] by D. Ya. Petrina and V. I. Gerasimenko. The Boltzmann–Grad limit theorem for equilibrium states was proved in the paper [60].

Recently, there has been unflagging interest in the problem of deriving kinetic equations from the dynamics of many colliding particles as an asymptotic behavior of the BBGKY hierarchy in the scaling limits. In particular, progress in the rigorous solution of this problem on the basis of perturbation theory was achieved in the Boltzmann–Grad limit in the works [2–4, 20, 25, 28, 29, 87–89]; also see links therein.

3.2. **The Boltzmann–Grad limit of reduced observables.** To determine the scaling parameter, we rewrite the dual BBGKY hierarchy in dimensionless form. Then generator (1.3) of the hierarchy takes the form:

$$\begin{aligned}
&\mathcal{L}(j)b_n \doteq \langle p_j, \frac{\partial}{\partial q_j}\rangle b_n, \\
&\mathcal{L}_{\mathrm{int}}(j_1,j_2)b_n \doteq \epsilon^2 \int\limits_{\mathbb{S}^2_+} d\eta \langle \eta, (p_{j_1}-p_{j_2})\rangle \delta(q_{j_1}-q_{j_2}+\epsilon\eta)\times \\
&\qquad\times \big(b_n(x_1,\ldots,q_{j_1},p^*_{j_1},\ldots,q_{j_2},p^*_{j_2},\ldots,x_n)-b_n(x_1,\ldots,x_n)\big),
\end{aligned} \tag{3.1}$$

where the coefficient $\epsilon > 0$ is a scaling parameter, which is the ratio of the diameter $\sigma > 0$ to the mean free path of hard spheres. For $t \leqslant 0$, a generator of the dimensionless dual BBGKY hierarchy is determined by the corresponding expression [51].

Then the Boltzmann–Grad asymptotic behavior of dimensionless reduced observables (2.3) is described by the following statement [51].

Theorem. *Assume that for the initial data $B^{\epsilon,0}_n \in \mathcal{C}_n$, $n \geqslant 1$, there is a limit $b^0_n \in \mathcal{C}_n$ in the sense of $*$-weak convergence of space $\mathcal{C}_n$*

$$\mathrm{w}^*\!-\lim_{\epsilon\rightarrow 0}\big(\epsilon^{-2n}B^{\epsilon,0}_n - b^0_n\big) = 0. \tag{3.2}$$

Then, for an arbitrary finite time interval, the Boltzmann–Grad limit of dimensionless reduced observables (2.3) *exists in the same sense*

$$\mathrm{w}^*-\lim_{\epsilon\rightarrow 0}\big(\epsilon^{-2s}B_s(t)-b_s(t)\big)=0, \tag{3.3}$$

and it is determined by the expansions:

$$\begin{aligned}
b_s(t,x_1,\ldots,x_s)=&\sum_{n=0}^{s-1}\int\limits_0^t dt_1\cdots\int\limits_0^{t_{n-1}}dt_n\prod_{j\in(1,\ldots,s)}S_1(t-t_1,j)\times\\
&\times\sum_{i_1\neq j_1=1}^{s}\mathcal{L}^0_{\mathrm{int}}(i_1,j_1)\prod_{j\in(1,\ldots,s)\backslash(j_1)}S_1(t_1-t_2,j)\cdots\prod_{j\in(1,\ldots,s)\backslash(j_1,\ldots,j_{n-1})}S_1(t_{n-1}-t_n,j)\times\\
&\times\sum_{\substack{i_n\neq j_n=1,\\ i_n,j_n\neq(j_1,\ldots,j_{n-1})}}^{s}\mathcal{L}^0_{\mathrm{int}}(i_n,j_n)\prod_{j\in(1,\ldots,s)\backslash(j_1,\ldots,j_n)}S_1(t_n,j)b^0_{s-n}((x_1,\ldots,x_s)\backslash(x_{j_1},\ldots,x_{j_n})),\\
&\qquad\qquad s\geqslant 1,
\end{aligned} \tag{3.4}$$

where for the collision operator of point particles, the notation $\mathcal{L}^0_{\mathrm{int}}(j_1,j_2)$ *is used*

$$\begin{aligned}
\mathcal{L}^0_{\mathrm{int}}(j_1,j_2)b_n\doteq&\int_{\mathbb{S}^2_+}d\eta\langle\eta,(p_{j_1}-p_{j_2})\rangle\delta(q_{j_1}-q_{j_2})\times\\
&\times\big(b_n(x_1,\ldots,q_{j_1},p^*_{j_1},\ldots,q_{j_2},p^*_{j_2},\ldots,x_n)-b_n(x_1,\ldots,x_n)\big).
\end{aligned} \tag{3.5}$$

Let us make several comments on this theorem.

Consider the existence of the Boltzmann–Grad limit for a special case of reduced observables, namely additive-type reduced observables. Let us say that for the initial additive-type dimensionless reduced observable $B^{(1)}(0)=(0,b^{\epsilon}_1,0,\ldots)$ the following condition is satisfied:

$$\mathrm{w}^*-\lim_{\epsilon\rightarrow 0}\big(\epsilon^{-2}b^{\epsilon}_1-b^0_1\big)=0,$$

then, according to statement (3.3), for additive-type reduced observables (2.10) we derive

$$\mathrm{w}^*-\lim_{\epsilon\rightarrow 0}\big(\epsilon^{-2s}B^{(1)}_s(t)-b^{(1)}_s(t)\big)=0,$$

where the limit reduced observable $b_s^{(1)}(t)$ is determined as a special case of expansion (3.4):

$$b_s^{(1)}(t,x_1,\ldots,x_s)=\int\limits_0^t dt_1\cdots\int\limits_0^{t_{s-2}}dt_{s-1}\prod_{j\in(1,\ldots,s)}S_1(t-t_1,j)\times$$
$$\times\sum_{i_1\neq j_1=1}^{s}\mathcal{L}^0_{\mathrm{int}}(i_1,j_1)\prod_{j\in(1,\ldots,s)\setminus(j_1)}S_1(t_1-t_2,j)\cdots\prod_{j\in(1,\ldots,s)\setminus(j_1,\ldots,j_{s-2})}S_1(t_{s-2}-t_{s-1},j)\times \tag{3.6}$$
$$\times\sum_{\substack{i_{s-1}\neq j_{s-1}=1,\\ i_{s-1},j_{s-1}\neq(j_1,\ldots,j_{s-2})}}^{s}\mathcal{L}^0_{\mathrm{int}}(i_{s-1},j_{s-1})\prod_{j\in(1,\ldots,s)\setminus(j_1,\ldots,j_{s-1})}S_1(t_{s-1},j)b_1^0\big((x_1,\ldots,x_s)\setminus(x_{j_1},\ldots,x_{j_{s-1}})\big),$$
$$s\geqslant 1.$$

We make several examples of expansions (3.6) of the limit additive-type reduced observables:

$$b_1^{(1)}(t,x_1)=S_1(t,1)\,b_1^0(x_1),$$
$$b_2^{(1)}(t,x_1,x_2)=\int_0^t dt_1\prod_{i=1}^{2}S_1(t-t_1,i)\,\mathcal{L}^0_{\mathrm{int}}(1,2)\sum_{j=1}^{2}S_1(t_1,j)\,b_1^0(x_j).$$

Also suppose that the following condition is valid for the initial k-ary-type reduced observable $B^{(k)}(0)=(0,\ldots,b_k^{\epsilon},0,\ldots)$:

$$\mathrm{w}^*\!-\lim_{\epsilon\to 0}\big(\epsilon^{-2}b_k^{\epsilon}-b_k^0\big)=0,$$

then, according to statement (3.3), for k-ary-type dimensionless reduced observables (2.11), we derive

$$\mathrm{w}^*\!-\lim_{\epsilon\to 0}\big(\epsilon^{-2s}B_s^{(k)}(t)-b_s^{(k)}(t)\big)=0,$$

where the limit reduced observable $b_s^{(k)}(t)$ is determined as a special case of expansion (3.4):

$$b_s^{(k)}(t,x_1,\ldots,x_s)=\int\limits_0^t dt_1\cdots\int\limits_0^{t_{s-k-1}}dt_{s-k}\prod_{j\in(1,\ldots,s)}S_1(t-t_1,j)\sum_{i_1\neq j_1=1}^{s}\mathcal{L}^0_{\mathrm{int}}(i_1,j_1)\times$$
$$\times\prod_{j\in(1,\ldots,s)\setminus(j_1)}S_1(t_1-t_2,j)\cdots\prod_{j\in(1,\ldots,s)\setminus(j_1,\ldots,j_{s-k-1})}S_1(t_{s-k-1}-t_{s-k},j)\sum_{\substack{i_{s-k}\neq j_{s-k}=1,\\ i_{s-k},j_{s-k}\neq(j_1,\ldots,j_{s-k-1})}}^{s}\mathcal{L}^0_{\mathrm{int}}(i_{s-k},j_{s-k})\times \tag{3.7}$$
$$\times\prod_{j\in(1,\ldots,s)\setminus(j_1,\ldots,j_{s-k})}S_1(t_{s-k},j)b_k^0\big((x_1,\ldots,x_s)\setminus(x_{j_1},\ldots,x_{j_{s-k}})\big),\ 1\leqslant s\leqslant k.$$

If $b^0 \in \mathcal{C}_\gamma$, then the sequence $b(t) = (b_0, b_1(t), \ldots, b_s(t), \ldots)$ of limit reduced observables (3.4) is a generalized global solution of the Cauchy problem of the dual Boltzmann hierarchy with hard sphere collisions [51]:

$$\frac{\partial}{\partial t} b_s(t) = \sum_{j=1}^{s} \mathcal{L}(j)\, b_s(t) + \sum_{j_1 \neq j_2 = 1}^{s} \mathcal{L}^0_{\mathrm{int}}(j_1, j_2)\, b_{s-1}(t, (x_1, \ldots, x_s) \backslash (x_{j_1})), \quad (3.8)$$

$$b_s(t, x_1, \ldots, x_s)\mid_{t=0} = b^0_s(x_1, \ldots, x_s), \quad s \geqslant 1, \quad (3.9)$$

where it was used notations accepted in (3.4).

This fact is proved similar to the case of an iteration series of the dual BBGKY hierarchy [10].

It should be noted that equations set (3.8) has the structure of recurrence evolution equations. We make a few examples of the dual Boltzmann hierarchy with hard sphere collisions (3.8):

$$\frac{\partial}{\partial t} b_1(t, x_1) = \langle p_1, \frac{\partial}{\partial q_1} \rangle b_1(t, x_1),$$

$$\frac{\partial}{\partial t} b_2(t, x_1, x_2) = \sum_{j=1}^{2} \langle p_j, \frac{\partial}{\partial q_j} \rangle b_2(t, x_1, x_2) +$$

$$+ \int_{\mathbb{S}^2_+} d\eta \langle \eta, (p_1 - p_2) \rangle \big(b_1(q_1, p_1^*) - b_1(x_1) + b_1(q_2, p_2^*) - b_1(x_2) \big) \delta(q_1 - q_2).$$

Thus, in the Boltzmann–Grad scaling asymptotics, the kinetic evolution of hard sphere observables is described in terms of limit reduced observables (3.4) governed by the dual Boltzmann hierarchy (3.8) with hard sphere collisions.

3.3. **The Boltzmann kinetic equation.** We now establish the relationship between the constructed Boltzmann–Grad asymptotics of the reduced observables and the description of the kinetic evolution of states in terms of the one-particle reduced distribution function described by the Boltzmann kinetic equation.

In the case of the absence of correlations between particles at initial time, i.e., initial states satisfying a chaos condition [14], in dimensionless form, the sequence of initial reduced distribution functions for a system of hard spheres has the form

$$F^{(c)} \equiv \Big(1, F_1^{\epsilon,0}(x_1), \ldots, \prod_{i=1}^{s} F_1^{\epsilon,0}(x_i) \mathcal{X}_{\mathbb{R}^{3s} \backslash \mathbb{W}_s}, \ldots \Big), \quad (3.10)$$

where $\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s}$ is the Heaviside step function of the allowed configurations. This assumption about initial state is intrinsic for the kinetic theory, because in this case all possible states of gases are described by means of a one-particle distribution function.

Let $F_1^{0,\epsilon} \in L^\infty_\xi(\mathbb{R}^3 \times \mathbb{R}^3)$, i.e., the following inequality holds:

$$|F_1^{0,\epsilon}(x_i)| \leqslant \xi \exp(-\beta \frac{p_i^2}{2}),$$

where $\xi > 0, \beta \geqslant 0$ are parameters.

We assume that the Boltzmann–Grad limit of the initial one-particle (reduced) distribution function $F_1^{0,\epsilon} \in L^\infty_\xi(\mathbb{R}^3 \times \mathbb{R}^3)$ exists in the sense of a weak convergence of the space $L^\infty_\xi(\mathbb{R}^3 \times \mathbb{R}^3)$, namely,

$$\mathrm{w-}\lim_{\epsilon\rightarrow 0}(\epsilon^2 F_1^{0,\epsilon} - f_1^0) = 0, \tag{3.11}$$

then the Boltzmann–Grad limit of the initial state (3.10) satisfies a chaos property too, i.e.,

$$f^{(c)} \equiv \big(1, f_1^0(x_1), \ldots, \prod_{i=1}^s f_1^0(x_i), \ldots\big).$$

We note that assumption (3.11) with respect to the Boltzmann–Grad limit of initial states holds true for the equilibrium state [60].

If $b(t) \in \mathcal{C}_\gamma$ and $|f_1^0(x_i)| \leqslant \xi \exp(-\beta \frac{p_i^2}{2})$, then the Boltzmann–Grad limit of mean value functional $\big(B(t), F^{(c)}\big)$ exists under the condition that [83]:

$$t < t_0 \equiv \big(\mathrm{const}(\xi,\beta)\|f_1^0\|_{L^\infty_\xi(\mathbb{R}^3\times\mathbb{R}^3)}\big)^{-1},$$

and it is determined by the following series expansion:

$$\big(b(t), f^{(c)}\big) = \sum_{s=0}^\infty \frac{1}{s!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1 \cdots dx_s\, b_s(t, x_1, \ldots, x_s) \prod_{i=1}^s f_1^0(x_i).$$

For the limit of additive-type reduced observables (3.6) the following equality holds [51]:

$$\begin{aligned}\big(b^{(1)}(t), f^{(c)}\big) &= \sum_{s=0}^\infty \frac{1}{s!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1 \cdots dx_s\, b_s^{(1)}(t, x_1, \ldots, x_s) \prod_{i=1}^s f_1^0(x_i) = \\ &= \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx_1\, b_1^0(x_1) f_1(t, x_1),\end{aligned} \tag{3.12}$$

where function $b_s^{(1)}(t)$ is given by expansion (3.6) and the distribution function $f_1(t,x_1)$ is represented by the series

$$\begin{aligned} f_1(t,x_1) = &\sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \int_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_2 \cdots dx_{n+1}\, S_1^*(t-t_1,1)\times \\ &\times \mathcal{L}_{\mathrm{int}}^{0,*}(1,2) \prod_{j_1=1}^{2} S_1^*(t_1-t_2,j_1) \cdots \prod_{i_n=1}^{n} S_1^*(t_{n-1}-t_n,i_n)\times \\ &\times \sum_{k_n=1}^{n} \mathcal{L}_{\mathrm{int}}^{0,*}(k_n,n+1) \prod_{j_n=1}^{n+1} S_1^*(t_n,j_n) \prod_{i=1}^{n+1} f_1^0(x_i), \end{aligned} \tag{3.13}$$

and the following operator was introduced:

$$\begin{aligned} &\int_{\mathbb{R}^3\times\mathbb{R}^3} dx_{n+1} \mathcal{L}_{\mathrm{int}}^{0,*}(i,n+1) f_{n+1}(x_1,\ldots,x_{n+1}) \equiv \\ &\quad\equiv \int_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_{n+1} d\eta \,\langle \eta,(p_i-p_{n+1})\rangle\times \\ &\qquad\qquad \times \big(f_{n+1}(x_1,\ldots,q_i,p_i^*,\ldots,x_s,q_i,p_{n+1}^*)- \\ &\qquad\qquad\qquad -f_{n+1}(x_1,\ldots,x_s,q_i,p_{n+1})\big). \end{aligned} \tag{3.14}$$

A one-particle distribution function represented as a series (3.13) is a solution of the Cauchy problem of the Boltzmann kinetic equation:

$$\begin{aligned} \frac{\partial}{\partial t} f_1(t,x_1) = &-\langle p_1, \frac{\partial}{\partial q_1}\rangle f_1(t,x_1)+ \\ &+\int_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_2\, d\eta\, \langle \eta,(p_1-p_2)\rangle\times \\ &\qquad \times \big(f_1(t,q_1,p_1^*) f_1(t,q_1,p_2^*) - f_1(t,x_1) f_1(t,q_1,p_2)\big), \end{aligned} \tag{3.15}$$

$$f_1(t,x_1)_{|t=0} = f_1^0(x_1). \tag{3.16}$$

Thus, we establish that the dual Boltzmann hierarchy (3.8) for additive-type reduced observables and initial state (3.11) describe the evolution of hard sphere systems just as the Boltzmann kinetic equation (3.15).

We remark that in a one-dimensional space, the collision integral of the Boltzmann equation with elastic hard sphere collisions identically equals zero. In a one-dimensional space, the Boltzmann–Grad limit is not trivial in the case of hard sphere dynamics with inelastic collisions. In the paper [43] for one-dimensional granular gas, the process of the creation and propagation of correlations in the Boltzmann–Grad scaling limit was also described (see also section 5.1).

Correspondingly, if the initial state of hard spheres is given by a sequence of reduced distribution functions (3.10), then in the Boltzmann–Grad limit, the property of the propagation of initial chaos holds [51]. It is a result of the validity of the following equality for the limit k-ary reduced observables, i.e., for the sequences $b^{(k)}(0) = (0, \ldots, b_k^0(x_1, \ldots, x_k), 0, \ldots)$,

$$\begin{aligned}\big(b^{(k)}(t), f^{(c)}\big) &= \sum_{s=0}^{\infty} \frac{1}{s!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1 \cdots dx_s\, b_s^{(k)}(t, x_1, \ldots, x_s) \prod_{i=1}^{s} f_1^0(x_i) = \\ &= \frac{1}{k!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^k} dx_1 \cdots dx_k\, b_k^0(x_1, \ldots, x_k) \prod_{i=1}^{k} f_1(t, x_i), \ \ k \geqslant 2,\end{aligned} \tag{3.17}$$

where the limit one-particle reduced distribution function $f_1(t)$ is defined by expansion (3.13) and therefore it is governed by the Cauchy problem of the Boltzmann kinetic equation (3.15),(3.16).

Thus, in the Boltzmann–Grad scaling limit, an equivalent approach to the description of the kinetic evolution of hard spheres in terms of the Cauchy problem of the Boltzmann kinetic equation (3.15),(3.16) is given by the Cauchy problem of the dual Boltzmann hierarchy with hard sphere collisions (3.8),(3.9) for the additive-type reduced observables. In the case of non-additive-type reduced observables, a solution of the dual Boltzmann hierarchy with hard sphere collisions (3.8) is equivalent to the property of the propagation of initial chaos in the sense of equality (3.17).

3.4. **The Boltzmann kinetic equation with initial correlations.** We now consider the case of the more general initial state of a hard sphere system specified by the one-particle reduced distribution function $F_1^{0,\epsilon} \in L_\xi^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ in the presence of correlations, i.e., the initial state that is specified by the following sequence of reduced distribution functions:

$$F^{(cc)} = \Big(1, F_1^{0,\epsilon}(x_1), g_2^\epsilon \prod_{i=1}^{2} F_1^{0,\epsilon}(x_i), \ldots, g_n^\epsilon \prod_{i=1}^{n} F_1^{0,\epsilon}(x_i), \ldots\Big), \tag{3.18}$$

where the functions $g_n^\epsilon \equiv g_n^\epsilon(x_1, \ldots, x_n) \in C_n(\mathbb{R}^{3n} \times (\mathbb{R}^{3n} \setminus \mathbb{W}_n))$, $n \geqslant 2$, specify the initial correlations. Since many-particle systems in condensed states are characterized by correlations, sequence (3.18) describes the initial state of the kinetic evolution of hard sphere fluids.

We assume that the Boltzmann–Grad limit of the initial one-particle reduced distribution function $F_1^{0,\epsilon} \in L_\xi^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ exists in the sense as above, i.e., in the sense of a weak convergence the equality holds:

$$\lim_{\epsilon\to 0} (\epsilon^2 F_1^{0,\epsilon} - f_1^0) = 0,$$

and in the case of correlation functions, suppose that:

$$\lim_{\epsilon\rightarrow 0}(g_n^\epsilon - g_n) = 0,\ n \geqslant 2.$$

Then in the Boltzmann–Grad limit, initial state (3.18) is specified by the following sequence of the limit reduced distribution functions:

$$f^{(cc)} = \Big(1, f_1^0(x_1), g_2\prod_{i=1}^{2} f_1^0(x_i), \ldots, g_n\prod_{i=1}^{n} f_1^0(x_i), \ldots\Big). \tag{3.19}$$

We now consider relationships between the constructed Boltzmann–Grad asymptotic behavior of reduced observables and the nonlinear Boltzmann-type kinetic equation in the case of initial state specified by sequence (3.19).

For the limit additive-type reduced observables (3.6) and initial states (3.19) the following equality is true:

$$\begin{aligned}
&\big(b^{(1)}(t), f^{(cc)}\big) = \\
&\qquad = \sum_{s=0}^{\infty}\frac{1}{s!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s} dx_1\cdots dx_s\, b_s^{(1)}(t,x_1,\ldots,x_s)g_s(x_1,\ldots,x_s)\prod_{i=1}^{s} f_1^0(x_i) = \\
&\qquad = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx_1\, b_1^0(x_1)f_1(t,x_1),
\end{aligned}$$

where the functions $b_s^{(1)}(t)$ are represented by expansions (3.6) and the limit reduced distribution function $f_1(t)$ is represented by the following series expansion:

$$\begin{aligned}
&f_1(t,x_1) = \sum_{n=0}^{\infty}\int\limits_0^t dt_1\cdots\int\limits_0^{t_{n-1}} dt_n\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_2\cdots dx_{n+1}S_1^*(t-t_1,1)\times \\
&\times\mathcal{L}_{\mathrm{int}}^{0,*}(1,2)S_1^*(t_1-t_2,j_1)\cdots\prod_{i_n=1}^{n} S_1^*(t_n-t_n,i_n)\times \\
&\times\sum_{k_n=1}^{n}\mathcal{L}_{\mathrm{int}}^{0,*}(k_n,n+1)\prod_{j_n=1}^{n+1} S_1^*(t_n,j_n)g_{1+n}(x_1,\ldots,x_{n+1})\prod_{i=1}^{n+1} f_1^0(x_i).
\end{aligned} \tag{3.20}$$

Series (3.20) is uniformly convergent for a finite time interval under the condition as above.

The function $f_1(t)$ represented by series (3.20) is a weak solution of the Cauchy problem of the Boltzmann kinetic equation with initial correlations [30, 55]:

$$\begin{aligned}
&\frac{\partial}{\partial t}f_1(t,x_1)=-\langle p_1,\frac{\partial}{\partial q_1}\rangle f_1(t,x_1)+\int\limits_{\mathbb{R}^3\times\mathbb{S}^2_+}dp_2\,d\eta\,\langle\eta,(p_1-p_2)\rangle\times\\
&\quad\times\big(g_2(q_1-p_1^*t,p_1^*,q_2-p_2^*t,p_2^*)f_1(t,q_1,p_1^*)f_1(t,q_1,p_2^*)-\\
&\qquad-g_2(q_1-p_1t,p_1,q_2-p_2t,p_2)f_1(t,x_1)f_1(t,q_1,p_2)\big),
\end{aligned}\tag{3.21}$$

$$f_1(t,x_1)\big|_{t=0}=f_1^0(x_1).\tag{3.22}$$

This fact is proved similarly to the case of a perturbative solution of the BBGKY hierarchy for hard spheres represented by the iteration series [14, 58].

Thus, in the case of initial states specified by a one-particle reduced distribution function (3.19) we establish that the dual Boltzmann hierarchy with hard sphere collisions (3.8) for additive-type reduced observables describes the evolution of a hard sphere system just as the Boltzmann kinetic equation with initial correlations (3.21).

The property of the propagation of initial correlations is a consequence of the validity of the following equality for the mean value functional of the limit k-ary reduced observables in the case of $k\geqslant 2$

$$\begin{aligned}
&\big(b^{(k)}(t),f^{(cc)}\big)=\\
&=\sum_{s=0}^{\infty}\frac{1}{s!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s}dx_1\cdots dx_s\,b_s^{(k)}(t,x_1,\ldots,x_s)g_s(x_1,\ldots,x_s)\prod_{j=1}^{s}f_1^0(x_j)=\\
&=\frac{1}{k!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^k}dx_1\cdots dx_k\,b_k^0(x_1,\ldots,x_k)\times\\
&\quad\times\prod_{i_1=1}^{k}S_1^*(t,i_1)g_k(x_1,\ldots,x_k)\prod_{i_2=1}^{k}(S_1^*)^{-1}(t,i_2)\prod_{j=1}^{k}f_1(t,x_j),
\end{aligned}\tag{3.23}$$

where the one-particle reduced distribution function $f_1(t,x_j)$ is solution (3.20) of the Cauchy problem of the Boltzmann kinetic equation with initial correlations (3.21), (3.22), and the inverse group to the group of operators $S_1^*(t)$ we denote by

$$(S_1^*)^{-1}(t)=S_1^*(-t)=S_1(t).$$

This fact is proved similarly to the proof of a property on the propagation of initial chaos (3.17).

We note that, according to equality (3.23), in the Boltzmann–Grad limit, the reduced correlation functions are defined as cluster expansions of reduced distribution functions, namely,

$$f_s(t,x_1,\ldots,x_s) \ = \sum_{\mathrm{P}:(x_1,\ldots,x_s)=\bigcup_i X_i} \ \prod_{X_i\subset \mathrm{P}} g_{|X_i|}(t,X_i), \ \ s\geqslant 1,$$

and they have the explicit form:

$$\begin{aligned} &g_1(t,x_1)=f_1(t,x_1),\\ &g_s(t,x_1,\ldots,x_s)=\\ &\qquad\qquad =\tilde{g}_s(q_1-p_1t,p_1,\ldots,q_s-p_st,p_s)\prod_{j=1}^{s} f_1(t,x_j), \ \ s\geqslant 2, \end{aligned} \tag{3.24}$$

where for initial correlation functions (3.19) it is used the following notations:

$$\tilde{g}_s(x_1,\ldots,x_s) \ = \sum_{\mathrm{P}:(x_1,\ldots,x_s)=\bigcup_i X_i} \ \prod_{X_i\subset \mathrm{P}} g_{|X_i|}(X_i),$$

the symbol $\sum_{\mathrm{P}}$ means the sum over possible partitions P of the set of arguments $(x_1,\ldots,x_s)$ on $|\mathrm{P}|$ nonempty subsets X_i, and the one-particle reduced distribution function $f_1(t)$ is a solution of the Cauchy problem of the Boltzmann kinetic equation with initial correlations (3.21),(3.22).

Thus, in the case of the limit k-ary reduced observables, a solution of the dual Boltzmann hierarchy with hard sphere collisions (3.8) is equivalent to a property of the propagation of initial correlations for the k-particle reduced distribution function in the sense of equality (3.23) or in other words, the Boltzmann–Grad scaling dynamics does not create new correlations.

4. Origin of kinetic equations

One of the challenges of kinetic theory, as mentioned above, is understanding the nature of the possibility of describing the evolution of the state of a system of many hard spheres by means of the state of a typical hard sphere. More precisely, we further focus on the problem of the origin of the description of the evolution of the state of hard spheres by the Enskog-type kinetic equation.

In the circumstances where the initial state is specified by a one-particle reduced distribution function, for the mean value functional of observables at an arbitrary instant, the representation is also valid in terms of a one-particle reduced distribution function, the evolution of which is governed by a non-Markovian nonlinear evolution equation. In other words, for such initial data, the Cauchy problem of the BBGKY hierarchy for hard spheres is equivalent to the nonlinear Enskog-type kinetic equation and a sequence of reduced functionals determined by the solution of this evolution equation.

4.1. **The generalised Enskog kinetic equation.** In the case of initial state (3.10) the dual picture of the evolution to the picture described by employing observables governed by the dual BBGKY hierarchy (2.1) for hard spheres is the picture of the evolution of a state described by means of the non-Markovian Enskog kinetic equation and by a sequence of explicitly defined functionals of the solution of such a kinetic equation that describe the evolution of all possible correlations in a system of hard spheres [46,47].

In view of the fact that the initial state is completely specified by a one-particle reduced distribution function on allowed configurations (3.10), for mean value functional (1.11) the following representation holds [46]:

$$\big(B(t),F^{(c)}\big)=\big(B(0),F(t\mid F_1(t))\big), \tag{4.1}$$

where $F^{(c)}$ is the sequence of initial reduced distribution functions (3.10), and the sequence $F(t\mid F_1(t))=\big(1,F_1(t),F_2(t\mid F_1(t)),\ldots,F_s(t\mid F_1(t)),\ldots\big)$ is a sequence of the reduced functionals of the state $F_s(t,x_1,\ldots,x_s\mid F_1(t))$, $s\geqslant 2$, represented by the series expansions over the products of the one-particle distribution function $F_1(t)$, namely

$$\begin{aligned}F_s(t,x_1,\ldots,x_s\mid F_1(t))&\doteq\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n}dx_{s+1}\cdots dx_{s+n}\times\\&\times\mathfrak{V}_{1+n}(t,\{1,\ldots,s\},s+1,\ldots,s+n)\prod_{i=1}^{s+n}F_1(t,x_i),\;\; s\geqslant 2.\end{aligned}\tag{4.2}$$

where the generating operators of series (4.2) are the $(1+n)th$-order operators $\mathfrak{V}_{1+n}(t)$, $n\geqslant 0$, determined by the following expansions [47]:

$$\begin{aligned}&\mathfrak{V}_{1+n}\big(t,\{1,\ldots,s\},s+1,\ldots,s+n\big)=\\&=n!\sum_{k=0}^{n}(-1)^k\sum_{n_1=1}^{n}\cdots\sum_{n_k=1}^{n-n_1-\cdots-n_{k-1}}\frac{1}{(n-n_1-\cdots-n_k)!}\times\\&\quad\times\widehat{\mathfrak{A}}_{1+n-n_1-\cdots-n_k}(t,\{1,\ldots,s\},s+1,\ldots,s+n-n_1-\cdots-n_k)\times\\&\quad\times\prod_{j=1}^{k}\sum_{\substack{\mathrm{D}_j:Z_j=\bigcup_{l_j}X_{l_j}\\|\mathrm{D}_j|\leqslant s+n-n_1-\cdots-n_j}}\frac{1}{|\mathrm{D}_j|!}\times\\&\qquad\qquad\times\sum_{i_1\neq\ldots\neq i_{|\mathrm{D}_j|}=1}^{s+n-n_1-\ldots-n_j}\;\prod_{X_{l_j}\subset\mathrm{D}_j}\frac{1}{|X_{l_j}|!}\,\widehat{\mathfrak{A}}_{1+|X_{l_j}|}(t,i_{l_j},X_{l_j}),\end{aligned}\tag{4.3}$$

In expansion (4.3) the symbol $\sum_{\mathrm{D}_j:Z_j=\bigcup_{l_j}X_{l_j}}$ means the sum over all possible dissections of the linearly ordered set

$$Z_j\equiv(s+n-n_1-\ldots-n_j+1,\ldots,s+n-n_1-\ldots-n_{j-1})$$

on no more than $s+n-n_1-\ldots-n_j$ linearly ordered subsets, and the $(1+n)th$-order scattering cumulant we denoted by the operator:

$$\begin{aligned}&\widehat{\mathfrak{A}}_{1+n}(t,\{1,\ldots,s\},s+1,\ldots,s+n)\doteq\\&\doteq\mathfrak{A}^*_{1+n}(t,\{1,\ldots,s\},s+1,\ldots,s+n)\mathcal{X}_{\mathbb{R}^{3(s+n)}\setminus\mathbb{W}_{s+n}}\prod_{i=1}^{s+n}\mathfrak{A}^*_1(t,i)^{-1},\end{aligned}\tag{4.4}$$

where the operator $\mathfrak{A}^*_{1+n}(t)$ is the $(1+n)th$-order cumulant of the groups of operators (1.6) of hard spheres.

We provide some examples of expressions for the generating operators of series (4.2) for reduced functionals of the state:

$$\begin{aligned}\mathfrak{V}_1(t,\{1,\ldots,s\})&=\widehat{\mathfrak{A}}_1(t,\{1,\ldots,s\})\doteq\\&\doteq S^*_s(t,1,\ldots,s)\mathcal{X}_{\mathbb{R}^{3(s)}\setminus\mathbb{W}_s}\prod_{i=1}^{s}S^*_1(t,i)^{-1},\\\mathfrak{V}_2(t,\{1,\ldots,s\},s+1)&=\widehat{\mathfrak{A}}_2(t,\{1,\ldots,s\},s+1)-\\&-\widehat{\mathfrak{A}}_1(t,\{1,\ldots,s\})\sum_{i_1=1}^{s}\widehat{\mathfrak{A}}_2(t,i_1,s+1).\end{aligned}$$

The method of constructing reduced state functionals (4.2) is based on the application of the variation of cluster expansions, the so-called kinetic cluster expansions, to generating operators (2.22) of series representing solutions of hierarchies of evolution equations [47].

The one-particle distribution function $F_1(t)$ in dimensionless form, i.e., the first element of the sequence $F(t\mid F_1(t))$, is determined by series (2.20), namely

$$\begin{aligned}F_1(t,x_1)=\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n}&dx_2\cdots dx_{n+1}\times\\&\times\mathfrak{A}^*_{1+n}(t,1,\ldots,n+1)\prod_{i=1}^{n+1}F_1^{\epsilon,0}(x_i)\mathcal{X}_{\mathbb{R}^{3(1+n)}\setminus\mathbb{W}_{1+n}},\end{aligned}\tag{4.5}$$

where the generating operator $\mathfrak{A}^*_{1+n}(t)$ is the $(1+n)$th-order cumulant of the groups of operators (1.6).

Let us note that in particular case of initial data (2.2) specified by the additive-type reduced observables, according to solution expansion (2.10), equality (4.1) takes the form

$$\big(B^{(1)}(t),F(0)\big)=\int\limits_{\mathbb{R}^3\times\mathbb{R}^3}dx_1\,b_1^{\epsilon}(x_1)F_1(t,x_1), \tag{4.6}$$

where the one-particle distribution function $F_1(t)$ is determined by series (4.5). In the case of initial data (2.2) specified by the s-ary reduced observable $s\geqslant 2$, equality (4.1) has the form

$$\big(B^{(s)}(t),F(0)\big)=\frac{1}{s!}\int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^s}dx_1\cdots dx_s\,b_s^{\epsilon}(x_1,\ldots,x_s)F_s(t,x_1,\ldots,x_s\mid F_1(t)),$$

where the reduced functionals of the state $F_s(t,x_1,\ldots,x_s\mid F_1(t))$ are determined by series (4.2).

Thus, for the initial state specified by a one-particle distribution function, the evolution of all possible states of a system of many hard spheres can be described by means of the state of a typical particle without any scaling approximations. We emphasize that reduced functionals of the state(4.2) describe all possible correlations created during the evolution of many hard spheres in terms of the state of a typical hard sphere.

For $t\geqslant 0$ the one-particle distribution function (4.5) is a solution of the following Cauchy problem of the non-Markovian generalized Enskog kinetic equation [47, 53]:

$$\begin{aligned}\frac{\partial}{\partial t}F_1(t,q_1,p_1)&=-\langle p_1,\frac{\partial}{\partial q_1}\rangle F_1(t,q_1,p_1)+\epsilon^2\int\limits_{\mathbb{R}^3\times\mathbb{S}^2_+}dp_2d\eta\langle\eta,(p_1-p_2)\rangle\times\\&\times\big(F_2(t,q_1,p_1^*,q_1-\epsilon\eta,p_2^*\mid F_1(t))-F_2(t,q_1,p_1,q_1+\epsilon\eta,p_2\mid F_1(t))\big),\end{aligned} \tag{4.7}$$

$$F_1(t)\big|_{t=0}=F_1^{\epsilon,0}, \tag{4.8}$$

where the collision integral is determined by the reduced functional of the state (4.2) in the case of $s=2$ and the expressions p_1^* and p_2^* are the pre-collision momenta of hard spheres (1.4). The series on the right-hand side of this equation converges under the condition: $\|F_1(t)\|_{L^1(\mathbb{R}\times\mathbb{R})}<e^{-8}$.

Hence in the case of the additive-type reduced observables the generalized Enskog kinetic equation (4.7) is dual to the dual BBGKY hierarchy of hard spheres (2.1) with respect to bilinear form (1.11).

We observe that the structure of the collision integral of the generalized Enskog equation (4.7) is such that the first term of its expansion is the collision integral of the Boltzman–Enskog kinetic equation, and the next terms describe all possible correlations that are created by the dynamics of hard spheres and by the propagation of initial correlations connected with the forbidden configurations, indeed

$$\frac{\partial}{\partial t}F_1(t,x_1) = -\langle p_1, \frac{\partial}{\partial q_1}\rangle F_1(t,x_1) + \mathcal{I}_{GEE},$$

where the collision integral is determined by the following series expansion:

$$\mathcal{I}_{GEE} \doteq \epsilon^2 \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{\mathbb{R}^3\times\mathbb{S}^2_+} dp_2 d\eta \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_3 \cdots dx_{n+2} \langle \eta, (p_1 - p_2)\rangle \times$$
$$\times \big(\mathfrak{V}_{1+n}(t, \{1^*, 2^*_-\}, 3, \ldots, n{+}2) F_1(t, q_1, p^*_1) F_1(t, q_1{-}\epsilon\eta, p^*_2) \prod_{i=3}^{n+2} F_1(t, x_i) -$$
$$- \mathfrak{V}_{1+n}(t, \{1, 2_+\}, 3, \ldots, n+2) F_1(t, x_1) F_1(t, q_1 + \epsilon\eta, p_2) \prod_{i=3}^{n+2} F_1(t, x_i)\big),$$

and the notations adopted for the conventional notation of the Enskog collision integral were used: indices $(1^\sharp, 2^\sharp_\pm)$ denote that the evolution operator $\mathfrak{V}_{1+n}(t)$ acts on the corresponding phase points $(q_1, p^\sharp_1)$ and $(q_1 \pm \epsilon\eta, p^\sharp_2)$, and the $(n+1)$th-order evolution operator $\mathfrak{V}_{1+n}(t)$, $n \geqslant 0$, is determined by expansion (4.3) in the case of $s = 2$.

We note that in the work [47] for the initial-value problem (4.7),(4.8) the existence theorem was proved in the space of integrable functions. The accordance of the generalized Enskog equation (4.7) and of the Markovian Enskog-type kinetic equation was also established there. By the point, we remark that in the paper [97] the explicit soliton-like solutions of kinetic equation (4.7) were found.

Thus, if the initial state is specified by a one-particle distribution function on allowed configurations, then the evolution of many hard spheres governed by the dual BBGKY hierarchy (2.1) for reduced observables can be completely described by the generalized Enskog kinetic equation (4.7) and by a sequence of reduced functionals of the state (4.2).

We remark also that in the case of the initial state that involves correlations (3.18) considered approach permits to take into consideration the initial correlations in the kinetic equations [30, 52].

Further, we sketch out the Boltzmann–Grad scaling behavior of the non-Markovian Enskog kinetic equation (4.7) and reduced state functional (4.2).

Taking into account the validity of assumption (3.11) for the initial one-particle distribution function (3.10), in that case for a finite time interval, the Boltzmann–Grad limit of dimensionless solution (4.5) of the Cauchy problem of the non-Markovian Enskog kinetic equation (4.7),(4.8) exists in the same sense, namely

$$\lim_{\epsilon\rightarrow 0}\big(\epsilon^{2}F_{1}(t,x_{1})-f_{1}(t,x_{1})\big)=0,$$

where the limit one-particle distribution function $f_1(t)$ is a weak solution of the Cauchy problem of the Boltzmann kinetic equation (3.15),(3.16).

Taking into consideration the fact of the existence of the Boltzmann–Grad scaling limit of a solution of the non-Markovian Enskog kinetic equation (4.7), for reduced functionals of the state (4.2) the following statement holds [47]:

$$\lim_{\epsilon\rightarrow 0}\big(\epsilon^{2s}F_{s}\big(t,x_{1},\ldots,x_{s}\mid F_{1}(t)\big)-\prod_{j=1}^{s}f_{1}(t,x_{j})\big)=0,$$

where the limit one-particle distribution function $f_1(t)$ is governed by the Boltzmann kinetic equation with hard sphere collisions (3.15). Because all possible correlations of many hard spheres with elastic collisions are described by reduced functionals of the state (4.2), as noted above, this property means the propagation of the initial chaos in the Boltzmann–Grad limit.

The proof of these statements is based on the properties of cumulants of asymptotically perturbed groups of operators (1.6) and the explicit structure (4.3) of the generating operators of series expansions (4.2) for reduced functional of state and of series (4.5).

4.2. **Dynamics of correlations governed by kinetic equations.** Let the initial state be specified by a one-particle reduced correlation function, namely, the initial state specified by a sequence of reduced correlation functions satisfying the chaos property stated above, i.e., by the sequence

$$G^{(c)}=(G_{0},G_{1}^{0},0,\ldots,0,\ldots).$$

We note that such an assumption about initial states is intrinsic to the contemporary kinetic theory of many-particle systems [14, 15].

Since the initial data $G^{(c)}$ is completely specified only by a one-particle correlation function, the Cauchy problem (2.60),(2.61) of the nonlinear hierarchy for hard spheres is not a completely well-defined Cauchy problem because the initial data is not independent for every unknown function determined of the hierarchy of mentioned evolution equations. As a consequence, it becomes possible to reformulate such a Cauchy problem as a new Cauchy problem for a one-particle correlation function with independent initial data and explicitly defined functionals of the solution of this Cauchy problem for the kinetic equation.

We formulate such a restated Cauchy problem and the sequence of the suitable functionals. In the case under consideration, the reduced correlation functionals $G_s(t \mid G_1(t))$, $s \geqslant 2$, are represented with respect to the one-particle correlation function (2.64), i.e.,

$$G_1(t,x_1) = \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_2 \cdots dx_{1+n}\, \mathfrak{A}^*_{1+n}(t,1,\ldots,n+1) \times \\ \times \prod_{i=1}^{n+1} G_1^0(x_i)\mathcal{X}_{\mathbb{R}^{3(n+1)}\setminus\mathbb{W}_{n+1}}, \tag{4.9}$$

as the following series expansions:

$$G_s\big(t,x_1,\ldots,x_s \mid G_1(t)\big) = \\ = \sum_{n=0}^{\infty} \frac{1}{n!} \int\limits_{(\mathbb{R}^3\times\mathbb{R}^3)^n} dx_{s+1} \cdots dx_{s+n} \times \\ \times \mathfrak{V}_{s+n}\big(t,1,\ldots,s+n\big) \prod_{i=1}^{s+n} G_1(t,x_i), \;\; s \geqslant 2. \tag{4.10}$$

The generating operator $\mathfrak{V}_{s+n}(t)$, $n \geqslant 0$, of the $(s+n)th$-order of this series is determined by the following expansion:

$$\mathfrak{V}_{s+n}\big(t,1,\ldots,s,s+1,\ldots,s+n\big) = \\ = n! \sum_{k=0}^{n} (-1)^k \sum_{n_1=1}^{n} \cdots \sum_{n_k=1}^{n-n_1-\ldots-n_{k-1}} \frac{1}{(n-n_1-\ldots-n_k)!} \times \\ \times \hat{\mathfrak{A}}_{s+n-n_1-\ldots-n_k}(t,1,\ldots,s+n-n_1-\ldots-n_k) \times \\ \times \prod_{j=1}^{k} \sum_{\substack{\mathrm{D}_j : Z_j = \bigcup_{l_j} X_{l_j}, \\ |\mathrm{D}_j| \leqslant s+n-n_1-\cdots-n_j}} \frac{1}{|\mathrm{D}_j|!} \times \\ \times \sum_{i_1\neq\ldots\neq i_{|\mathrm{D}_j|}=1}^{s+n-n_1-\ldots-n_j} \prod_{X_{l_j}\subset \mathrm{D}_j} \frac{1}{|X_{l_j}|!} \hat{\mathfrak{A}}_{1+|X_{l_j}|}(t, i_{l_j}, X_{l_j}), \tag{4.11}$$

where $\sum_{\mathrm{D}_j : Z_j = \bigcup_{l_j} X_{l_j}}$ is the sum over all possible dissections of the linearly ordered set

$$Z_j \equiv (s+n-n_1-\ldots-n_j+1,\ldots,s+n-n_1-\ldots-n_{j-1})$$

on no more than $s+n-n_1-\ldots-n_j$ linearly ordered subsets, the $(s+n)th$-order scattering cumulant is defined by the formula

$$\hat{\mathfrak{A}}_{s+n}(t,1,\ldots,s+n) \doteq \mathfrak{A}^*_{s+n}(t,1,\ldots,s+n)\mathcal{X}_{\mathbb{R}^{3(s+n)}\setminus\mathbb{W}_{s+n}} \prod_{i=1}^{s+n} (\mathfrak{A}^*_1)^{-1}(t,i),$$

and notations accepted above were used.

We adduce simplest examples of generating operators (4.11):

$$\mathfrak{V}_s(t,1,\ldots,s)=\mathfrak{A}_s(t,1,\ldots,s)\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s}\prod_{i=1}^{s}(\mathfrak{A}_1^*)^{-1}(t,i),$$

$$\begin{aligned}\mathfrak{V}_{s+1}(t,1,\ldots,s,s{+}1)&=\mathfrak{A}_{s+1}(t,1,\ldots,s{+}1)\mathcal{X}_{\mathbb{R}^{3(s+1)}\setminus\mathbb{W}_{s+1}}\prod_{i=1}^{s+1}(\mathfrak{A}_1^*)^{-1}(t,i)-\\&-\mathfrak{A}_s(t,1,\ldots,s)\mathcal{X}_{\mathbb{R}^{3s}\setminus\mathbb{W}_s}\prod_{i=1}^{s}(\mathfrak{A}_1^*)^{-1}(t,i)\times\\&\times\sum_{j=1}^{s}\mathfrak{A}_2(t,j,s+1)\mathcal{X}_{\mathbb{R}^{6}\setminus\mathbb{W}_2}(\mathfrak{A}_1^*)^{-1}(t,j)(\mathfrak{A}_1^*)^{-1}(t,s+1).\end{aligned}$$

If $\|G_1(t)\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}<e^{-(3s+2)}$, for arbitrary $t\in\mathbb{R}$ series (4.10) converges in the norm of the space L^1_s [47].

We note that in the case of initial state specified by a one-particle correlation function the reduced correlation functionals (4.10) describe all possible correlations generated by the dynamics of many hard spheres in terms of a one-particle correlation function.

Thus, according to the representation (2.64) of reduced correlation functions, the cumulant structure of their generating operators induces a generalized cumulant structure of the generating operators for series (4.10) of reduced correlation functionals.

In this case, the method of constructing reduced correlation functionals (4.10) is based on the application of the variation of cluster expansions, the so-called kinetic cluster expansions [47], to generating operators (2.36) of series representing reduced correlation functions (2.64).

Indeed, taking into account relations of kinetic cluster expansions for scattering cumulants (4.4):

$$\begin{aligned}\widehat{\mathfrak{A}}_{s+n}(t,1,\ldots,s+n)&=\sum_{n_1=0}^{n}\frac{n!}{(n-n_1)!}\mathfrak{V}_{s+n-n_1}\big(t,1,\ldots,s+n-n_1\big)\times\\&\times\sum_{\substack{\mathrm{D}:Z=\bigcup_l X_l,\\|\mathrm{D}|\leqslant s+n-n_1}}\frac{1}{|\mathrm{D}|!}\sum_{i_1\neq\ldots\neq i_{|\mathrm{D}|}=1}^{s+n-n_1}\prod_{X_l\subset\mathrm{D}}\frac{1}{|X_l|!}\,\widehat{\mathfrak{A}}_{1+|X_l|}(t,i_l,X_l),\end{aligned}$$

where $\sum_{\mathrm{D}:Z=\bigcup_l X_l,\,|\mathrm{D}|\leqslant s+n-n_1}$ is the sum over all possible dissections D of the linearly ordered set

$$Z\equiv(s+n-n_1+1,\ldots,s+n)$$

on no more than $s+n-n_1$ linearly ordered subsets, we derive the expansions of the reduced correlation functionals

$$G_s(t, x_1, \ldots, x_s \mid G_1(t)),\ s \geqslant 2,$$

on the basis of solution expansions (2.64) of the hierarchy of nonlinear evolution equations (2.60).

Note that the structure of kinetic cluster expansions of scattering cumulants of the groups of operators is similar to the structure of virial expansions of equilibrium distribution functions, i.e., as a power series in the density.

If initial data $G_1^0 \in L_1^1$, then for arbitrary $t \in \mathbb{R}$ one-particle correlation function (4.9) is a weak solution of the Cauchy problem of the non-Markovian Enskog kinetic equation [53]:

$$\begin{aligned}\frac{\partial}{\partial t}G_1(t,x_1) = \mathcal{L}^*(1)G_1(t,x_1) + \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx_2\, \mathcal{L}^*_{\mathrm{int}}(1,2)G_1(t,x_1)G_1(t,x_2) + \\ + \int\limits_{\mathbb{R}^3\times\mathbb{R}^3} dx_2\, \mathcal{L}^*_{\mathrm{int}}(1,2)G_2\big(t,x_1,x_2 \mid G_1(t)\big),\end{aligned} \tag{4.12}$$

$$G_1(t,x_1)\big|_{t=0} = G_1^0(x_1), \tag{4.13}$$

where the first part of the collision integral in equation (4.12) has the Boltzmann–Enskog structure, and the second part of the collision integral is determined in terms of the two-particle correlation functional represented by series expansion (4.10) which describes all possible correlations that are created by hard-sphere dynamics and by the propagation of initial correlations related to the forbidden configurations.

In the paper [47], similar statements were proved for the state evolution of a hard-sphere system described in terms of reduced distribution functions governed by the BBGKY hierarchy. We emphasize that the nth term of expansions (4.10) of the reduced correlation functionals are determined by the $(s+n)th$-order generating operator (4.3) in contradistinction to the expansions of reduced distribution functionals of the state constructed in [47] which are determined by the $(1+n)th$-order generating operator (4.3).

Thus, for the initial state specified by a one-particle correlation function, the evolution of all possible states of the system of hard spheres can be described without any approximations within the framework of a one-particle correlation function governed by the non-Markovian Enskog-type kinetic equation (4.12), and by a sequence of explicitly defined functionals (4.10) of its solution.

5. Conclusion

In conclusion, the challenges of the evolution of many hard spheres with inelastic collisions will be reviewed, as will some applications of the methods outlined above to complex systems of various natures.

5.1. **On the dynamics of inelastic collisions.** According to contemporary concept [96, 99] on a microscopic scale, the characteristic properties of granular media are determined by dissipative collisional dynamics and can be described as the evolution of a system of many hard spheres with inelastic collisions.

Since the characteristic features of the collective behavior of inelastically colliding particles in one-dimensional space reflect the main properties of granular gases, an approach to the rigorous derivation of the Boltzmann-type equation for one-dimensional granular gases will be presented below. We note that, in contrast to the system of hard rods with inelastic collisions, in one-dimensional space the evolution of hard rods with elastic collisions is trivial in the Boltzmann–Grad scaling limit; it is known as so-called free molecular motion or the Knudsen flow [14].

In the case of a one-dimensional granular gas for $t \geqslant 0$ in dimensionless form the Cauchy problem of the non-Markovian generalized Enskog kinetic equation (4.7),(4.8) takes the form [37, 42, 43]:

$$\begin{aligned}
&\frac{\partial}{\partial t}F_1(t,q_1,p_1) = -p_1\frac{\partial}{\partial q_1}F_1(t,q_1,p_1)+\\
&+\int_0^\infty dP\,P\Big(\frac{1}{(1-2\varepsilon)^2}\,F_2(t,q_1,p_1^\diamond(p_1,P),q_1-\epsilon,p_2^\diamond(p_1,P)\mid F_1(t))-\\
&\qquad\qquad -F_2(t,q_1,p_1,q_1-\epsilon,p_1+P\mid F_1(t))\Big)+\\
&+\int_0^\infty dP\,P\Big(\frac{1}{(1-2\varepsilon)^2}\,F_2(t,q_1,\tilde{p}_1^\diamond(p_1,P),q_1+\epsilon,\tilde{p}_2^\diamond(p_1,P)\mid F_1(t))-\\
&\qquad\qquad -F_2(t,q_1,p_1,q_1+\epsilon,p_1-P\mid F_1(t))\Big),
\end{aligned}\tag{5.1}$$

$$F_1(t)|_{t=0} = F_1^{\epsilon,0},\tag{5.2}$$

where $\varepsilon = \frac{1-e}{2} \in [0,\frac{1}{2})$ and $e \in (0,1]$ is a restitution coefficient, $\epsilon > 0$ is a scaling parameter (the ratio of a hard sphere diameter (the length) $\sigma > 0$ to the mean free path), the collision integral is determined by reduced functional (4.2) of the state $F_1(t)$ in the case of $s = 2$ and the expressions:

$$p_1^\diamond(p_1,P) = p_1 - P + \frac{\varepsilon}{2\varepsilon-1}\,P,$$
$$p_2^\diamond(p_1,P) = p_1 - \frac{\varepsilon}{2\varepsilon-1}\,P$$

and the expressions

$$\tilde{p}_1^{\diamond}(p_1,P)=p_1+P-\frac{\varepsilon}{2\varepsilon-1}\,P,$$
$$\tilde{p}_2^{\diamond}(p_1,P)=p_1+\frac{\varepsilon}{2\varepsilon-1}\,P,$$

are transformed pre-collision momenta of inelastically colliding particles in a one-dimensional space.

The solution of the Cauchy problem (5.1),(5.2) is represented by the following series:

$$F_1^{\epsilon}(t,x_1)=\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}dx_2\cdots dx_{n+1}\,\mathfrak{A}^{*}_{1+n}(t)\prod_{i=1}^{n+1}F_1^{\epsilon,0}(x_i)\mathcal{X}_{\mathbb{R}^{(1+n)}\setminus\mathbb{W}_{1+n}},\quad(5.3)$$

where the generating operator $\mathfrak{A}^{*}_{1+n}(t)$ is the $(1{+}n)$th-order cumulant (2.22) of the semigroups of operators (1.6) of inelastically colliding hard rods in a one-dimensional space. Let the initial one-particle distribution function satisfy the following condition: $|F_1^{\epsilon,0}(x_1)|\leqslant Ce^{-\frac{\beta}{2}p_1^2}$, where $\beta>0$ is a parameter and $C<\infty$ is some constant. Then every term of series (5.3) exists; for a finite time interval it is the uniformly convergent series with respect to x_1 from an arbitrary compact, and function (5.3) is a weak solution of the Cauchy problem (5.1),(5.2) of the non-Markovian Enskog-type equation with inelastic collisions.

We assume that, in the sense of weak convergence, there exists a limit

$$\mathrm{w-}\lim_{\epsilon\to 0}\big(F_1^{\epsilon,0}(x_1)-f_1^0(x_1)\big)=0.$$

Then, for a finite time interval, the Boltzmann–Grad limit of solution (5.3) of the Cauchy problem of the non-Markovian Enskog-type equation for a one-dimensional granular gas (5.1) exists in the sense of a weak convergence

$$\mathrm{w-}\lim_{\epsilon\to 0}\big(F_1^{\epsilon}(t,x_1)-f_1(t,x_1)\big)=0,\qquad(5.4)$$

where the limit of one-particle distribution function (5.3) is determined by the following series uniformly convergent on an arbitrary compact set

$$f_1(t,x_1)=\sum_{n=0}^{\infty}\frac{1}{n!}\int\limits_{\mathbb{R}^n\times\mathbb{R}^n}dx_2\cdots dx_{n+1}\,\mathfrak{A}^{0}_{1+n}(t)\prod_{i=1}^{n+1}f_1^0(x_i),\qquad(5.5)$$

and the generating operator $\mathfrak{A}^{0}_{1+n}(t)\equiv\mathfrak{A}^{0}_{1+n}(t,1,\ldots,n+1)$ is the $(n+1)th$-order cumulant of semigroups (1.6) of point particles with inelastic collisions. For $t\geqslant 0$ an infinitesimal generator of this semigroup of operators is determined by the operator:

$$(\mathcal{L}_n^{*,0} f_n)(x_1,\ldots,x_n) = -\sum_{j=1}^{n} p_j \frac{\partial}{\partial q_j} f_n(x_1,\ldots,x_n) +$$
$$+\sum_{j_1<j_2=1}^{n} |p_{j_2}-p_{j_1}| \delta(q_{j_1}-q_{j_2}) \times$$
$$\times \Big(\frac{1}{(1-2\varepsilon)^2} f_n(x_1,\ldots,x_{j_1}^{\diamond},\ldots,x_{j_2}^{\diamond},\ldots,x_n) - f_n(x_1,\ldots,x_n)\Big),$$

where $x_j^{\diamond} \equiv (q_j, p_j^{\diamond})$ and the pre-collision momenta $p_{j_1}^{\diamond}$, $p_{j_2}^{\diamond}$ of inelastically colliding particles are determined by the following expressions:

$$p_{j_1}^{\diamond} = p_{j_2} + \frac{\varepsilon}{2\varepsilon - 1}(p_{j_1} - p_{j_2}),$$
$$p_{j_2}^{\diamond} = p_{j_1} - \frac{\varepsilon}{2\varepsilon - 1}(p_{j_1} - p_{j_2}).$$

For $t \geqslant 0$ the limit one-particle distribution function represented by series (5.5) is a weak solution of the Cauchy problem of the Boltzmann-type kinetic equation of point particles with inelastic collisions [43]

$$\frac{\partial}{\partial t} f_1(t,q,p) = -p \frac{\partial}{\partial q} f_1(t,q,p) + \int_{-\infty}^{+\infty} dp_1 \, |p - p_1| \times$$
$$\times \Big(\frac{1}{(1-2\varepsilon)^2} f_1(t,q,p^{\diamond}) f_1(t,q,p_1^{\diamond}) - f_1(t,q,p)\, f_1(t,q,p_1)\Big) + \sum_{n=1}^{\infty} \mathcal{I}_0^{(n)}. \tag{5.6}$$

In kinetic equation (5.6) the remainder $\sum_{n=1}^{\infty} \mathcal{I}_0^{(n)}$ of the collision integral is determined by the following expressions:

$$\mathcal{I}_0^{(n)} \equiv \frac{1}{n!} \int_0^{\infty} dP\, P \int_{\mathbb{R}^n \times \mathbb{R}^n} dq_3 dp_3 \cdots dq_{n+2} dp_{n+2} \mathfrak{V}_{1+n}(t) \times$$
$$\times \Big(\frac{1}{(1-2\varepsilon)^2} f_1(t,q,p_1^{\diamond}(p,P)) f_1(t,q,p_2^{\diamond}(p,P)) - f_1(t,q,p) f_1(t,q,p+P)\Big) \times$$
$$\times \prod_{i=3}^{n+2} f_1(t,q_i,p_i) +$$
$$+ \int_0^{\infty} dP\, P \int_{\mathbb{R}^n \times \mathbb{R}^n} dq_3 dp_3 \cdots dq_{n+2} dp_{n+2} \mathfrak{V}_{1+n}(t) \times$$
$$\times \Big(\frac{1}{(1-2\varepsilon)^2} f_1(t,q,\tilde{p}_1^{\diamond}(p,P)) f_1(t,q,\tilde{p}_2^{\diamond}(p,P)) - f_1(t,q,p) f_1(t,q,p-P)\Big) \times$$
$$\times \prod_{i=3}^{n+2} F_1(t,q_i,p_i),$$

where the generating operators

$$\mathfrak{V}_{1+n}(t) \equiv \mathfrak{V}_{1+n}(t, \{1,2\}, 3, \ldots, n+2),\ n \geqslant 0,$$

of the series for a collision integral are represented by expansions (4.3) with respect to the cumulants of semigroups of scattering operators of point hard rods with inelastic collisions in a one-dimensional space

$$\widehat{S}^0_n(t, 1, \ldots, n) \doteq S^{*,0}_n(t, 1, \ldots, s) \prod_{i=1}^{n} (S^{*,0}_1)^{-1}(t, i).$$

In fact, series expansions for the collision integral of the non-Markovian Enskog equation for a granular gas (5.6) or solution (5.3) are represented as the power series over the density, so that the terms $\mathcal{I}^{(n)}_0$, $n \geqslant 1$, of the collision integral in kinetic equation (5.6) are corrections with respect to the density to the Boltzmann collision integral of one-dimensional granular gases formulated in the paper [96].

Since the scattering operator of point hard rods is an identity operator in the approximation of elastic collisions, namely, in the limit $\varepsilon \to 0$, the collision integral of the Boltzmann kinetic equation (5.6) in a one-dimensional space is identical to zero. In the quasi-elastic approximation [96] the limit one-particle distribution function (5.5)

$$\lim_{\varepsilon \to 0} \varepsilon f_1(t, q, p) = f^0(t, q, p),$$

satisfies the nonlinear friction kinetic equation for one-dimensional granular gases [96]:

$$\begin{aligned}\frac{\partial}{\partial t} f^0(t, q, p) = -p \frac{\partial}{\partial q} f^0(t, q, p) + \\ + \frac{\partial}{\partial p} \int_{-\infty}^{\infty} dp_1\, |p_1 - p|\, (p_1 - p)\, f^0(t, q, p_1) f^0(t, q, p).\end{aligned}$$

Taking into consideration the result (5.4) on the Boltzmann–Grad asymptotic behavior of the non-Markovian Enskog equation (5.1), for reduced functionals of the state (4.2) in a one-dimensional space, the following statement is true [43]:

$$\mathrm{w-}\lim_{\epsilon \to 0} \big(F_s\big(t, x_1, \ldots, x_s \mid F^{\epsilon}_1(t)\big) - f_s\big(t, x_1, \ldots, x_s \mid f_1(t)\big)\big) = 0,\ \ s \geqslant 2, \quad (5.7)$$

where in equality (5.7) the limit reduced functionals of the limit one-particle distribution function (5.5) are determined by the series expansions with a structure similar to series (4.2) and the generating operators represented by expansions (4.3) over the cumulants of semigroups of scattering operators of point hard rods with inelastic collisions in a one-dimensional space.

As mentioned above, in the case of a system of hard rods with elastic collisions, the limit reduced functionals of the state are the product of the limit one-particle distribution functions, describing the free motion of point particles.

Thus, the Boltzmann–Grad asymptotic behavior of solution (5.3) of the non-Markovian Enskog equation (5.1) is governed by the Boltzmann kinetic equation (5.6) for a one-dimensional granular gas.

We emphasize that the Boltzmann-type equation (5.6) describes the memory effects in a one-dimensional granular gas. In addition, the limit of reduced functionals of the state $f_s\big(t, x_1, \ldots, x_s \mid f_1(t)\big)$, $s \geqslant 2$, which are defined above, describe the process of the propagation of initial chaos in a one-dimensional granular gas, or, in other words, the process of creatiing correlations in a system of hard rods with inelastic collisions.

It should be noted that the Boltzmann–Grad asymptotic behavior of the non-Markovian Enskog equation with inelastic collisions in a multidimensional space is analogous to the Boltzmann–Grad asymptotic behavior of a hard sphere system with the elastic collisions [43], i.e., it is governed by the Boltzmann equation for a granular gas [98, 99], and the asymptotic behavior of the reduced functionals of the state (4.2) is described by the product of one-particle distribution functions of its solution, i.e., describes the propagation of initial chaos.

5.2. **Some bibliographic notes on collisional dynamics.** Above, it was studied systems of identical colliding particles, which are described by means of functions of observables and distribution functions, which are symmetrical with respect to arbitrary permutations of their arguments. In papers [32,33,67,70,82], the theory of the hierarchies of evolution equations for systems of many colliding particles described by non-symmetric functions was developed. An example of such a system is a one-dimensional system of particles interacting with their nearest neighbors, so-called non-symmetric systems of particles [32].

As is known, many-entity systems of active soft condensed matter are dynamic systems exhibiting a collective behavior that differs from the statistical behavior of ordinary gases. To describe the nature of entities (or self-propelled particles), in the paper [78], collision dynamics based on Markov jump processes, which should reflect the internal properties of living creatures, were proposed. In works [36, 45] an approach was developed to describe the collective behavior of complex systems of mathematical biology within the framework of the evolution of observables of many colliding stochastic processes, and the dual Vlasov hierarchy was constructed in the mean field approximation. This representation of the kinetic evolution seems, in fact, to be the direct mathematically fully consistent formulation modeling the kinetic evolution of biological systems, since the notion of the state is more subtle and is an implicit characteristic of populations of living creatures. In the paper [36] the processes of creation of correlations generated by the dynamics of active soft matter and propagation of initial correlations have also been described by means of the non-Markovian generalized kinetic equation with initial correlations, and, in particular, in the mean-field scaling approximation, the Vlasov-type kinetic equation for many colliding stochastic processes was constructed.

The study of systems of colliding particles in interaction with the environment, the so-called open systems, involves a number of unsolved fundamental problems. One of them is related to the challenge of the origin of stochastic behavior in dynamical systems of many particles. In papers [48–50], based on the approaches to the derivation of kinetic equations outlined above, a generalization of the Fokker–Planck equation for open systems of colliding particles was justified.

In previous decades, a lot of work has been performed on discrete-velocity models of the Boltzmann equation, which are of significant conceptual interest for the kinetic theory of gases and, at the same time, represent a fascinating mathematical subject [84]. In connection with this topic of research, we note the works [39–41, 54], in which the discrete-velocity model was studied, related to the problem of deriving a model of the Enskog discrete-velocity kinetic equation.

An overview of some modern applications of kinetic equations to the description of non-equilibrium processes in complex systems of various natures is presented in the monograph [23].

5.3. **Outlook.** The purpose of this review was to analyze the development and current advances of the theory of evolution equations for systems of many colliding particles, in particular, kinetic equations and their relations to the fundamental equations that describe the laws of nature.

The problem of constructing a solution to the Cauchy problem for hierarchies of evolution equations of observables (2.1) and the state (2.17) of a system of hard spheres with elastic collisions for initial data belonging to some functional spaces is considered. As was established, solutions of hierarchies of evolution equations are determined by groups of operators, which are represented by expansions over the groups of particles whose evolution is described by cumulants of the corresponding order of the groups of operators of the Liouville equations. Due to the fact that the cumulants of the groups of operators are determined by cluster expansions of the groups of operators of the Liouville equations, in the corresponding function spaces there are different representations for solutions to the hierarchies of evolution equations. These cluster expansions of the groups of operators underlie the classification of possible solution representations to the Cauchy problem for the hierarchies of evolution equations of many colliding particles.

To describe the evolution of the state of a many-particle system, there is an alternative approach that is based on the dynamics of correlations. In this approach, a state of finitely many hard spheres is described with the employment of functions determined by the cluster expansions of the probability distribution functions that are governed by the so-called Liouville hierarchy (2.31). It was above established that the constructed dynamics of correlation underlie the description of the dynamics of infinitely many hard spheres governed by the BBGKY hierarchy for reduced distribution functions (2.17) or the hierarchy of nonlinear evolution equations for reduced correlation functions (2.60), i.e., of the cumulants of reduced distribution functions. We emphasize the importance of the mathematical description of the processes of the creation and propagation of correlations, in particular, for numerous applications [2, 3, 86].

To describe the evolution of many hard spheres within the framework of the evolution of states for an initial state close to "kinetic," i.e., a state described in terms of the state of a typical particle, there is another possibility: by means of the so-called non-Markovian Enskog kinetic equation (4.7). In other words, the origin of the collective behavior of a hard-sphere system on a microscopic scale was examined above. As already mentioned, one of the advantages of such an approach to the derivation of kinetic equations from underlying collisional dynamics is the opportunity to construct the kinetic equations with initial correlations, which makes it possible to describe the creation of correlations and propagation of initial correlations. Another advantage of this approach is related to the rigorous derivation of the Boltzmann equation (3.15) with higher-order corrections to the canonical term of the collision integral.

Thus, the concept of cumulants of the groups of operators of Liouville equations underlies non-perturbative expansions of solutions to hierarchies of fundamental evolution equations that describe the evolution of observables and a state of many colliding particles, as well as underlies the kinetic description of their collective behavior. We note that for quantum many-particle systems the concept of cumulants of groups of operators is considered in review [38].

In the paper, possible approaches to the rigorous derivation [35] and justification [62, 63] of the kinetic equations for many colliding particles were considered. One of them is an approach to the description of the kinetic evolution within the framework of the evolution of the observables of many colliding particles [51]. The advances of the method based on the dual Boltzmann hierarchy (3.8) are the opportunity to construct kinetic equations (3.21), taking into account the correlations of particles of the initial state, and the description of the process of propagation of initial correlations in scaling approximations (3.24).

The paper [69] considered the challenge of deriving hydrodynamic equations from the dual BBGKY hierarchy for reduced observed microscopic phase densities. We notice that the rigorous derivation of hydrodynamic equations from the dynamics of many colliding particles is still an open problem. Regarding the classical problem of rigorous derivation of the hydrodynamic equations from the Boltzmann kinetic equation in scaling limits, we refer to the books [72, 93].

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