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Matrices with all minors of some fixed order being equal: the rank, dimension and characteristic property

У статті досліджується клас $\mathfrak{M}$ матриць (над довільним полем), в яких всі мінори деякого фіксованого порядку k – рівні і відмінні від 0. Встановлено, що ранг таких матриць дорівнює k. Знайдено можливі значення для розмірності матриці з класу $\mathfrak{M}$. Дано також необхідну і достатню умову для того, щоб матриця належала до класу $\mathfrak{M}$.

Investigated in this paper is a class $\mathfrak{M}$ of matrices (over an arbitrary field) in which all minors of some fixed order k are equal and nonzero. It is established that the rank of such matrices equals to k. The possible values for the dimension of a matrix in $\mathfrak{M}$ are found. A necessary and sufficient condition for a matrix to belong to the class $\mathfrak{M}$ is also given.

1. Introduction

Matrices with all principal minors of some fixed order being equal were studied by R.C.Thompson in [1] and [2]. In [1] a classification was obtained for symmetric matrices having all principal minors of order t equal, for three consecutive values of t less than the rank of A. A similar result, a classification for real symmetric matrices such that all principal minors of order t are equal and all nonprincipal minors are of fixed sign for two consecutive values of t less than the rank of A, is presented in [2]. The paper [2] also characterizes square matrices A over an arbitrary field in which the condition on the principal minors of A is weakened: it is required that all principal minors of order t are equal for one fixed value of t less then the rank of A; while the condition on nonprincipal minors of order t is strengthened: it is required that they are also equal.

Investigated in this paper is a class $\mathfrak{M}$ of matrices (not only square and over an arbitrary field) in which all minors of some fixed order k are equal and nonzero. It is established that the rank of such matrices equals to k. The possible values for the dimension of a matrix in $\mathfrak{M}$ are found. A necessary and sufficient condition for a matrix to belong to the class $\mathfrak{M}$ is also given. As an example illustrating main results, a classification is found for matrices that have all minors of order 2 equal and nonzero.

2. Notation

Let A be a $m \times n$-matrix over an arbitrary field. For A, we use $A^{\top}$, A^{*}, $\operatorname{rank} A$, $\det A$ to stand for the transpose matrix, the adjoint matrix, the rank and the determinant of A, respectively.

By A^j we mean j-th column of A ($j \in \overline{1,n}$) and A_i is used to denote i-th row ($i \in \overline{1,m}$). In addition, we use the notation $(A^{j_1}A^{j_2}...A^{j_s})$ for the submatrix formed by selecting from A a subset of columns $A^{j_1}, A^{j_2}, ..., A^{j_s}$ in the same relative position.

Remark that a class $\mathfrak{M}$ of matrices over a field in which all minors of some fixed order k are equal to $w \neq 0$ is closed under taking submatrices and transposes.

3. Main results

Теорема 1. *Let P be a field and A be a $m \times n$-matrix over P in which all minors of order k are equal and nonzero. Then:*

(i) $\operatorname{rank} A = k$*;*

(ii) $k \leq m, n \leq k + 1$.

Доведення. (i) Let all minors of order k of the matrix A be equal to w. Since, by theorem's condition, $w \neq 0$, obviously, $\operatorname{rank} A \geq k$.

If $k = 1$ then $A = (a_{ij})$ where $a_{ij} = w$. In the case when $w \neq 0$ the rank of the matrix A equals to 1 and the assertion of the theorem is valid.

Let $k > 1$. Assume that the rank of the matrix A is greater than k. Then there exist $(k+1)$ linearly independent rows and $(k+1)$ linearly independent columns in A such that the corresponding square submatrix B of order $k+1$ of the matrix A is nonsingular: $B = (b_{ij})$, $1 \leq i \leq k+1$, $1 \leq j \leq k+1$. In this case, for the matrix B, there exists an inverse matrix B^{-1}:

$$B^{-1} = (\det B)^{-1} B^*$$

where B^* is an adjoint matrix to the matrix B. Since all minors of order k of the matrix B are equal to w,

$$B^{-1} = (\det B)^{-1} \begin{pmatrix} w & -w & w & \ldots & (-1)^{k+1}w \\ -w & w & -w & \ldots & (-1)^{k+2}w \\ w & -w & w & \ldots & (-1)^{k+3}w \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (-1)^{k+1}w & (-1)^{k+2}w & (-1)^{k+3}w & \ldots & (-1)^{2k}w \end{pmatrix}.$$

In the case $w \neq 0$, the rank of the matrix B^{-1} equals to 1. Since $k > 1$, it implies that B^{-1} is singular, which contradicts to the choice of B. Hence, the assumption is not valid and $\operatorname{rank} A = k$.

(ii) Let now show that the number of columns (as well as the number of rows) of the matrix A is equal to k or $k+1$. Obviously, $k \leq m, n$.

Let $m \leq n$. Assume $n \geq k+2$ and consider $k \times (k+2)$-submatrix C of the matrix A. All minors of order k of the matrix C are equal to $w \neq 0$, therefore, in view of (i), $\operatorname{rank} C = k$.

Denote by C^j the j-th column of C.

Since $\operatorname{rank} C = k$ and the determinant of the matrix $(C^1 C^2 ... C^k)$, obtained by deleting from C both column $(k+1)$ and column $(k+2)$, is equal to w, we get that the system of vectors $C^1, C^2, ..., C^k$ is linearly independent and is the basis of the system of vectors $C^1, C^2, ..., C^k, C^{k+1}, C^{k+2}$. Therefore, vector-columns C^{k+1} and C^{k+2} can be expressed as the linear combinations of $C^1, C^2, ..., C^k$:

$$C^{k+1} = \sum_{i=1}^{k} s_i C^i, \quad C^{k+2} = \sum_{i=1}^{k} l_i C^i, \qquad s_i, l_i \in P.$$

Consider the determinant of the matrix, obtained by deleting from C both column k and column $(k+2)$:

$$\det(C^1 C^2 ... C^{k-1} C^{k+1}) = \det(C^1 C^2 ... C^{k-1} (\sum_{i=1}^{k} s_i C^i)) =$$
$$= \det(C^1 C^2 ... C^{k-1} (s_k C^k)) = s_k \det(C^1 C^2 ... C^{k-1} C^k).$$

Since both $\det(C^1 C^2 ... C^{k-1} C^{k+1})$ and $\det(C^1 C^2 ... C^{k-1} C^k)$ are minors of order k of the matrix C, they are equal to $w \neq 0$, hence, $s_k = 1$.

Consider now the determinant of the matrix, obtained by deleting from C both column $(k-1)$ and column $(k+2)$:

$$\det(C^1 C^2 ... C^{k-2} C^k C^{k+1}) = \det(C^1 C^2 ... C^{k-2} C^k (\sum_{i=1}^{k} s_i C^i)) =$$
$$= \det(C^1 C^2 ... C^{k-2} C^k (s_{k-1} C^{k-1})) = s_{k-1} \det(C^1 C^2 ... C^{k-2} C^k C^{k-1}) =$$

$$= -s_{k-1}\det(C^1C^2...C^{k-2}C^{k-1}C^k).$$

Since both $\det(C^1C^2...C^{k-2}C^kC^{k+1})$ and $\det(C^1C^2...C^{k-2}C^{k-1}C^k)$ are minors of order k of the matrix C, they are equal to $w \neq 0$, hence, $s_{k-1} = -1$.

In a similar way, we get $s_{k-2} = 1$, $s_{k-3} = -1$, ..., $s_1 = (-1)^{k+1}$. Then

$$C^{k+1} = (-1)^{k+1}C^1 + (-1)^kC^2 + (-1)^{k-1}C^3 + ... - C^{k-1} + C^k = \\ = \sum_{i=1}^{k}(-1)^{k+2-i}C^i.$$

Repeating the same considerations for the column C^{k+2} of the matrix C, we get:

$$C^{k+2} = (-1)^{k+1}C^1 + (-1)^kC^2 + (-1)^{k-1}C^3 + ... - C^{k-1} + C^k = \\ = \sum_{i=1}^{k}(-1)^{k+2-i}C^i,$$

hence, $C^{k+1} = C^{k+2}$. But then the determinant $\det(C^3...C^kC^{k+1}C^{k+2})$ of order k of the matrix, obtained by deleting from C both column 1 and column 2, is necessarily equal to 0, which contradicts the condition of the theorem. Therefore, assumption is not valid and $n \leq k+1$.

It remains to consider the case $n \leq m$. Since all minors of order k of the transpose matrix $A^\top$ are also equal to $w \neq 0$, applying the proven result to $A^\top$ gives us that the number of columns of $A^\top$ does not exceed $k+1$, hence, $m \leq k+1$.

The theorem is proven. □

Наслідок 2. *Let A be a $k \times (k+1)$-matrix over the field P. All minors of order k of the matrix A are equal and nonzero iff the following conditions 1)-2) hold:*

1) rank $A = k$*;*

2) $(k+1)$*-th column* A^{k+1} *of the matrix* A *is expressed as the linear combination:*

$$A^{k+1} = \sum_{j=1}^{k} (-1)^{k+2-j} A^j =$$
$$= (-1)^{k+1} A^1 + (-1)^k A^2 + (-1)^{k-1} A^3 + ... - A^{k-1} + A^k$$

where A^j *is a* j*-th column of the matrix* A*,* $1 \le j \le k$*.*

Доведення. Necessity immediately follows from the proof of Theorem.

Sufficiency. Let the conditions 1)-2) hold for the matrix A and $\det(A^1 A^2 ... A^k) = w \neq 0$. Let M be an arbitrary minor of order k of the matrix A. Then M is a determinant of a matrix, obtained by deleting from A some column A^j, $j \in \overline{1, k+1}$.

If $j = k+1$ then $M = \det(A^1 A^2 ... A^k) = w$. Let $1 \le j \le k$. Then

$$M = \det(A^1 ... A^{j-1} A^{j+1} ... A^k A^{k+1}) =$$
$$= \det(A^1 ... A^{j-1} A^{j+1} ... A^k (\sum_{j=1}^{k} (-1)^{k+2-j} A^j)) =$$
$$= \det(A^1 ... A^{j-1} A^{j+1} ... A^k ((-1)^{k+2-j} A^j)) =$$
$$= (-1)^{k+2-j} \det(A^1 ... A^{j-1} A^{j+1} ... A^k A^j) =$$
$$= (-1)^{k+2-j} (-1)^{k-j} \det(A^1 ... A^{j-1} A^j A^{j+1} ... A^k) = (-1)^{2(k+1-j)} w = w.$$

The corollary is proven. □

The next proposition follows immediately from Corollary 1, in view of the fact that the class $\mathfrak{M}$ of matrices with all minors of some fixed order k being equal and nonzero is closed by taking inverse matrices and submatrices.

Наслідок 3. *Let* A *be a* $(k+1) \times (k+1)$*-matrix over the field* P*. All minors of order* k *of the matrix* A *are equal and nonzero iff the following conditions 1)-2) hold:*

1) $\operatorname{rank} A = k$*;*

2) $(k+1)$*-th column* A^{k+1} *of the matrix* A *is expressed as the linear combination:*

$$A^{k+1} = \sum_{j=1}^{k} (-1)^{k+2-j} A^j =$$
$$= (-1)^{k+1} A^1 + (-1)^k A^2 + (-1)^{k-1} A^3 + ... - A^{k-1} + A^k$$

where A^j *is a* j*-th column of the matrix* A*,* $1 \le j \le k$*.*

3) $(k+1)$*-th row* A_{k+1} *of the matrix* A *is expressed as the linear combination:*

$$A_{k+1} = \sum_{i=1}^{k} (-1)^{k+2-i} A^i =$$
$$= (-1)^{k+1} A_1 + (-1)^k A_2 + (-1)^{k-1} A_3 + ... - A_{k-1} + A_k$$

where A_i *is a* i*-th row of the matrix* A*,* $1 \le i \le k$*.*

Зауваження 1. *For arbitrary given field* P*,* $w \in P \backslash \{0\}$ *and positive integer* k*, there exist matrices over* P *of the dimensions* $k \times k$*,* $k \times (k+1)$*,* $(k+1) \times k$*,* $(k+1) \times (k+1)$ *having all minors of order* k *equal to* w*. Indeed, one can always indicate a square matrix* B *of order* k*, which determinant is equal to* w*, e.g.,*

$$A = \begin{pmatrix} w & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}.$$

Consider a $k \times (k+1)$*-matrix* A *such that* B *is a submatrix of* A *obtained by deleting* $(k+1)$*-th row:* $B = (A^1 A^2 ... A^k)$*, and*

$$A^{k+1} = \sum_{j=1}^{k} (-1)^{k+2-j} A^j =$$

$$= (-1)^{k+1}A^1 + (-1)^k A^2 + (-1)^{k-1}A^3 + ... - A^{k-1} + A^k$$

In view of Corollary 1, all minors of order k of the matrix A are equal to w. In a similar way, one can construct matrices the dimensions $(k+1) \times k$, $(k+1) \times (k+1)$.

As an illustration to the Theorem, consider the following example classifying matrices in which all minors of order 2 are equal to some fixed $w \neq 0$.

Example 1. Let A be a matrix over a field P. All minors of order 2 of A are equal to $w \neq 0$ iff A is a matrix of one of the following types:

1) $A = \begin{pmatrix} a_1 & a_2 \\ -wa_2^{-1} & 0 \end{pmatrix}$ where $a_1, a_2 \in P$, $a_2 \neq 0$;

2) $A = \begin{pmatrix} (a_2a_3 + w)a_4^{-1} & a_2 \\ a_3 & a_4 \end{pmatrix}$ where $a_2, a_3, a_4 \in P$, $a_4 \neq 0$;

3) $A = \begin{pmatrix} a_1 & a_1 + a_2 & a_2 \\ -wa_2^{-1} & -wa_2^{-1} & 0 \end{pmatrix}$ where $a_1, a_2 \in P$, $a_2 \neq 0$;

4) $A = \begin{pmatrix} (a_2a_3 + w)a_4^{-1} & (a_2a_3 + w)a_4^{-1} + a_2 & a_2 \\ a_3 & a_3 + a_4 & a_4 \end{pmatrix}$
where $a_2, a_3, a_4 \in P$, $a_4 \neq 0$;

5) $A = \begin{pmatrix} a_1 & -wa_2^{-1} \\ a_1 + a_2 & -wa_2^{-1} \\ a_2 & 0 \end{pmatrix}$ where $a_1, a_2 \in P$, $a_2 \neq 0$;

6) $A = \begin{pmatrix} (a_2a_3 + w)a_4^{-1} & a_3 \\ (a_2a_3 + w)a_4^{-1} + a_2 & a_3 + a_4 \\ a_2 & a_4 \end{pmatrix}$ where $a_2, a_3, a_4 \in P$, $a_4 \neq 0$;

7) $A = \begin{pmatrix} a_1 & a_1+a_2 & a_2 \\ a_1 - wa_2^{-1} & a_1+a_2-wa_2^{-1} & a_2 \\ -wa_2^{-1} & -wa_2^{-1} & 0 \end{pmatrix}$ where $a_1, a_2 \in P$, $a_2 \neq 0$;

8) $A = \begin{pmatrix} (a_2a_3+w)a_4^{-1} & (a_2a_3+w)a_4^{-1}+a_2 & a_2 \\ (a_2a_3+w)a_4^{-1}+a_3 & (a_2a_3+w)a_4^{-1}+a_2+a_3+a_4 & a_2+a_4 \\ a_3 & a_3+a_4 & a_4 \end{pmatrix}$ where $a_2, a_3, a_4 \in P$, $a_4 \neq 0$.

Indeed, by Theorem, rank $A = 2$, the number of rows and columns is 2 or 3.

Case 1. Let A be a square matrix of order 2: $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $a_1a_4 - a_2a_3 = w \neq 0$, $a_1, a_2, a_3, a_4 \in P$. If $a_4 = 0$ then $a_2a_3 = -w$. Since $w \neq 0$, we have $a_2 \neq 0$ and $a_3 = -wa_2^{-1}$, hence, $A = \begin{pmatrix} a_1 & a_2 \\ -wa_2^{-1} & 0 \end{pmatrix}$ and A is of type 1). If $a_4 \neq 0$ then $a_1 = (a_2a_3 + w)a_4^{-1}$, hence,

$$A = \begin{pmatrix} (a_2a_3+w)a_4^{-1} & a_2 \\ a_3 & a_4 \end{pmatrix}$$

and A is of type 2).

Case 2. Let A be a 2×3-matrix. Then, by Corollary, its 2-nd column is a sum of the 1-st and 3-rd columns: $A = \begin{pmatrix} a_1 & a_1+a_2 & a_2 \\ a_3 & a_3+a_4 & a_4 \end{pmatrix}$ where $a_1a_4 - a_2a_3 = w$, $a_1, a_2, a_3, a_4 \in P$. If $a_4 = 0$ then $a_2 \neq 0$ and $a_3 = -wa_2^{-1}$, hence,

$$A = \begin{pmatrix} a_1 & a_1+a_2 & a_2 \\ -wa_2^{-1} & -wa_2^{-1} & 0 \end{pmatrix}$$

and A is of type 3). If $a_4 \neq 0$ then $a_1 = (a_2a_3 + w)a_4^{-1}$, hence,

$$A = \begin{pmatrix} (a_2a_3+w)a_4^{-1} & (a_2a_3+w)a_4^{-1}+a_2 & a_2 \\ a_3 & a_3+a_4 & a_4 \end{pmatrix}$$

and A is of type 4).

Case 3. Let A be a 3×2-matrix. Then the transpose matrix $A^\top$ is a matrix of type 3) or type 4), hence, A is a matrix of type 5) or type 6).

Case 4. Let A be a 3×3-matrix. Then, by Corollary 2, its 2-nd column is a sum of the 1-st and 3-rd columns, while its 2-nd row is a sum of the 1-st and 3-rd rows: $A = \begin{pmatrix} a_1 & a_1 + a_2 & a_2 \\ a_1 + a_3 & a_1 + a_2 + a_3 + a_4 & a_2 + a_4 \\ a_3 & a_3 + a_4 & a_4 \end{pmatrix}$ where $a_1, a_2, a_3, a_4 \in P$, $a_1a_4 - a_2a_3 = w \neq 0$. If $a_4 = 0$ then $a_2 \neq 0$, $a_3 = -wa_2^{-1}$, hence, $A = \begin{pmatrix} a_1 & a_1 + a_2 & a_2 \\ a_1 - wa_2^{-1} & a_1 + a_2 - wa_2^{-1} & a_2 \\ -wa_2^{-1} & -wa_2^{-1} & 0 \end{pmatrix}$ and A is a matrix of type 7). If $a_4 \neq 0$ then $a_1 = (a_2a_3 + w)a_4^{-1}$, hence, $A = \begin{pmatrix} (a_2a_3 + w)a_4^{-1} & (a_2a_3 + w)a_4^{-1} + a_2 & a_2 \\ (a_2a_3 + w)a_4^{-1} + a_3 & (a_2a_3 + w)a_4^{-1} + a_2 + a_3 + a_4 & a_2 + a_4 \\ a_3 & a_3 + a_4 & a_4 \end{pmatrix}$ and A is of type 8).

Corollaries 1,2 and direct calculations show that the matrices of types 1)-8) have all minors of order 2 equal to w.

4. Conclusion

In this paper, we have established that the rank of a matrix having all minors of order k equal and nonzero is equal to k. The number of columns of such matrices is k or $k + 1$ (as well as the number of rows). Using the necessary and sufficient condition for a matrix to have all minors of order k equal and nonzero, one can easily classify all matrices for fixed values of k. In this study, such classification is given for $k = 2$.

Література

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