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Preliminary group classification of general two-dimensional quasi-linear elliptic type equations

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В статті розглядається задача групової класифікації квазілінійних рівнянь еліптичного типу в двовимірному просторі. Отримано переліки усіх рівнянь цього класу, які допускають напівпрості алгебри Лі операторів симетрії та алгебри Лі операторів симетрії з нетривіальним розкладом Леві.

In the paper the problem of group classification of quasi-linear elliptic type equations in two-dimensional space is considered. We obtain the list of all equations of this type admitting semisimple Lie algebras of symmetry operators and Lie algebras of symmetry operators with non-trivial Levi decomposition.

We consider quasilinear two-dimensional equations of elliptic type

$$\Delta u = f(x, y, u, u_x, u_y). \tag{1}$$

In (1) and below $\Delta = \partial_{xx} + \partial_{yy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a two-dimensional Laplace operator, $u = u(x, y)$, F is an arbitrary smooth function in some domain of the space $W = \mathbb{R}^2 \times \underset{1}{V} = \langle x, y\rangle \times \langle u, u_x, u_y\rangle$, that is nonlinear, at least, with respect to one variable u, u_x, u_y.

Statement 1. *The Lie invariance group of equation* (1) *is generated by the infinitesimal operator*

$$\mathrm{v} = a(x, y)\partial_x + b(x, y)\partial_y + c(x, y, u)\partial_u, \tag{2}$$

where functions a, b, c, F satisfy the following system of equations:

$$\begin{aligned} & a_y + b_x = 0, \quad a_x - b_y = 0, \\ & c_{xx} + c_{yy} + 2u_x c_{xu} + 2u_y c_{yu} + (u_x^2 + u_y^2)c_{uu} + (c_u - 2a_x)F = \\ & \quad = aF_x + bF_y + cF_u + [c_x + u_x(c_u - a_x) - u_y b_x]F_{u_x} + \\ & \quad + [c_y + u_y(c_u - b_y) - u_x a_y]F_{u_y}. \end{aligned} \tag{3}$$

It is not difficult to see that the first two equations are Cauchy–Riemann conditions, that means that functions a and b are harmonic ones.

The group $\mathcal{E}$ is formed by those transformations

$$\bar{x} = \alpha(x, y, u), \quad \bar{y} = \beta(x, y, u), \quad v = \gamma(x, y, u), \quad \frac{D(\bar{x}, \bar{y}, v)}{D(x, y, u)} \neq 0,$$

preserving differential structure of equation (1), that transform it to an equation of the form

$$v_{\bar{x}\bar{x}} + v_{\bar{y}\bar{y}} = \Phi(\bar{x}, \bar{y}, v, v_{\bar{x}}, v_{\bar{y}}).$$

Statement 2. *The group $\mathcal{E}$ of equation* (1) *is formed by the following transformations:*

$$\bar{x} = \alpha(x, y), \quad \bar{y} = \beta(x, y), \quad v = \gamma(x, y, u), \tag{4}$$

where

$$\alpha_x = \epsilon\beta_y, \quad \alpha_y = -\epsilon\beta_x \quad (\epsilon = \pm 1),$$
$$\alpha_x^2 + \alpha_y^2 = \beta_x^2 + \beta_y^2 \neq 0, \quad \gamma_u \neq 0.$$

Lemma 1. *There exist such transformations from the group $\mathcal{E}$ that reduce operator* (2) *to one of the following operators:*

$$\mathrm{v} = \partial_x, \quad \mathrm{v} = \partial_u. \tag{5}$$

We start the group classification from the description of equations invariant with respect to Lie algebras with a non-trivial Levi decomposition.

First we will consider the following two equations of form (1):

$$\Delta u = f(u)(u_x^2 + u_y^2), \quad f \neq 0; \tag{6}$$
$$\Delta u = \lambda e^{\gamma u}, \quad \lambda, \gamma \in \mathbb{R}, \quad \lambda\gamma \neq 0. \tag{7}$$

These equations are invariant with respect to groups of infinitesimal transformation with infinite number of parameters that are generated by operators of form (2).

Equation (6) can be reduced by the substitution

$$v = \int^u \Phi^{-1}(\xi) d\xi, \quad \Phi(u) = \exp\left(\int^u f(\eta) d\eta\right)$$

to the two-dimensional Laplace equation. Equation (7) is connected to the Laplace equation by the Bäcklund transformation.

Invariance of equation (1) with respect to Lie algebras of symmetry operators with non-trivial Levi decomposition. It is well known in the theory of abstract Lie algebras [1] that arbitrary Lie algebra with a non-trivial Levi decomposition contains some simple (semisimple) Lie algebra as a subalgebra. For this reason, first of all, we will describe equations that are invariant with respect to simple (semisimple) Lie algebras of symmetry operators.

Theorem 1. *Up to $\mathcal{E}$-equivalence, there are two classes of quasilinear equations of form* (1) *admitting Lie algebras of symmetry operators that are different realizations of the algebra* $so(3)$*:*

$$\begin{aligned}
\text{I.}\quad & \Delta u = \operatorname{ch}^{-2} y \tilde{F}(u,\omega), \quad \omega = (u_x^2 + u_y^2)\operatorname{ch}^2 y : \\
& so^1(3) = \langle \partial_x, \operatorname{sh} y \cos x \partial_x - \operatorname{ch} y \sin x \partial_y, \\
& \qquad - \operatorname{sh} y \sin x \partial_x - \operatorname{ch} y \cos x \partial_y \rangle; \\
\text{II.}\quad & \Delta u = \operatorname{ch}^{-2} y \tilde{F}(\xi \operatorname{ch} y, \eta \operatorname{ch} y), \quad \xi = (u_x - \operatorname{th} y)\sin u + u_y \cos u, \\
& \eta = (u_x - \operatorname{th} y)\cos u - u_y \sin u : \\
& so^2(3) = \langle \partial_x, \operatorname{sh} y \cos x \partial_x - \operatorname{ch} y \sin x \partial_y + \operatorname{ch} y \cos x \partial_u, \\
& \qquad - \operatorname{sh} y \sin x \partial_x - \operatorname{ch} y \cos x \partial_y - \operatorname{ch} y \sin x \partial_u \rangle.
\end{aligned}$$

Theorem 2. *Up to $\mathcal{E}$-equivalence, there are two classes of quasilinear equations of form* (1) *admitting Lie algebras of symmetry operators that are different realizations of the algebra* $sl(2,\mathbb{R})$*:*

$$\begin{aligned}
\text{I.}\quad & \Delta u = y^{-2}\tilde{F}(u,\omega), \quad \omega = y^2(u_x^2 + u_y^2) : \\
& sl^1(2,\mathbb{R}) = \langle 2x\partial_x + 2y\partial_y, -(x^2 - y^2)\partial_x - 2xy\partial_y, \partial_x \rangle; \\
\text{II.}\quad & \Delta u = y^{-2}\tilde{F}(v,\omega), \quad v = (1 - 2yu_x)\cos 2u + 2yu_y \sin 2u, \\
& \omega = 2yu_y \cos 2u - (1 - 2yu_x)\sin 2u : \\
& sl^2(2,\mathbb{R}) = \langle 2x\partial_x + 2y\partial_y, -(x^2 - y^2)\partial_x - 2xy\partial_y + y\partial_u, \partial_x \rangle.
\end{aligned}$$

It is well known from the theory of abstract Lie algebra over the field $\mathbb{R}$ that there exist four types of classical simple algebras:

1) type $A_{n-1} (n > 1)$ that contains four real forms of the algebra $sl(n, C)$: $su(n)$, $sl(n,\mathbb{R})$, $su(p,q)$ $(p + q = n, p \geq q)$, $su^*(2n)$;

2) type D_n $(n > 1)$ that contains three real forms of the algebra $so(2n, C)$: $so(2n)$, $so(p, q)$ $(p + q = 2n, p \geq q)$, $so^*(2n)$;

3) type B_n $(n > 1)$ that contains two real forms of the algebra $so(2n+1, C)$: $so(2n + 1)$, $so(p, q)$ $(p + q = 2n + 1, p > q)$;

4) type C_n $(n \geq 1)$ that contains three real forms of the algebra $sp(n, C)$: $sp(n)$, $sp(n, \mathbb{R})$, $sp(p, q)$ $(p + q = n, p \geq q)$.

and the following special cases of semisimple real algebras: G_1, F_4, E_6, E_7, E_8.

Theorem 3. *Equations of form* (1) *invariant with respect to a symmetry algebra with a non-trivial Levi decomposition are exhausted by ones presented in theorems* 1 *and* 2.

Sketch of proof. 1. $so(4) = \langle e_i | i = 1, 2, 3\rangle \oplus \langle \bar{e}_i | i = 1, 2, 3\rangle$. Then in accordance to the results of theorem 2 the operators e_i $(i = 1, 2, 3)$ represent a basis of realizations $so^1(3)$ or $so^2(3)$. Direct calculations show that in such case the operators $\bar{e}_i$ $(i = 1, 2, 3)$ belong to the class of operators $c(u)\partial_u$, and in accordance to the statement obtained in the process of proof of theorem 2, there is no realizations of the algebra $so(3)$ in this class of operators. Thus, there exist no nonlinear equation of form (1) whose invariance algebra is isomorphic to the algebra $so(4)$.

2. The type G_2 contains one compact real form g_2 and one noncompact form g_2'. Since $g_2 \cap g_2' \sim su(2) \oplus su(2) \sim so(4)$ and algebra $so(4)$ does not have realizations in the given class of operators, in this class of operators the algebras g_2 and g_2' also do not have realizations.

3. For the algebras $so(3, 1)$ we will use Cartan's decomposition: $so(3, 1) = \langle e_1, e_2, e_3\rangle \dot{\oplus} \langle N_1, N_2, N_3\rangle$, де $\langle e_1, e_2, e_3\rangle = so(3)$, $[e_i, N_j] = \varepsilon_{ijl} N_l$, $[N_i, N_j] = -\varepsilon_{ijl} e_l$; $i, j, l = 1, 2, 3$; ε_{ijl} is antisymmetric tensor of third rank, $\varepsilon_{123} = 1$.

It can be proved that the realization has the form:

$$
\begin{aligned}
&e_1 = \partial_x, \quad N_1 = \partial_y,\\
&e_2 = \epsilon_1(\operatorname{sh} y \cos x \partial_x - \operatorname{ch} y \sin x \partial_y) + \operatorname{sh} y \sin x \partial_u,\\
&e_3 = -\epsilon_1(\operatorname{sh} y \sin x \partial_x + \operatorname{ch} y \cos x \partial_y) + \operatorname{sh} y \cos x \partial_u,\\
&N_2 = \epsilon_1(\operatorname{ch} y \sin x \partial_x + \operatorname{sh} y \cos x \partial_y) - \operatorname{ch} y \cos x \partial_u,\\
&N_3 = \epsilon_1(\operatorname{sh} y \cos x \partial_x - \operatorname{sh} y \sin x \partial_y) + \operatorname{ch} y \sin x.
\end{aligned}
$$

The respective invariant solution having the form

$$\Delta u = \lambda e^{2\epsilon_1 u}, \quad \lambda \in \mathbb{R}, \quad \lambda \neq 0,$$

is the subcase of equation (7). □

Invariance with respect to decomposable algebras. In accordance to the above here we study existence of equations of form (1) invariant with respect to algebras of form $S \oplus N$, i.e. satisfying conditions

$$[S, S] \subset S, \quad [N, N] \subset N, \quad [S, N] = 0. \tag{8}$$

Possible realisations $so^1(3)$, $so^2(3)$, $sl^1(2, \mathbb{R})$, $sl^2(2, \mathbb{R})$ of semisimple algebras S (Levi factor) have been found in the theorems 1 and 2.

There exist six classes of nonlinear equations of form (1), whose maximal invariance algebras can be decomposed into the direct sum of the Levi factor and solvable Lie algebra:

1) $\Delta u = \operatorname{ch}^{-2} y \tilde{F}(\omega), \quad \omega = (u_x^2 + u_y^2) \operatorname{ch}^2 y : \quad so^1(3) \oplus \langle \partial_u \rangle;$
2) $\Delta u = \operatorname{ch}^{-2} y \tilde{F}(\omega), \quad \omega = (u_x \operatorname{ch} y - \operatorname{sh} y)^2 + u_y^2 \operatorname{ch}^2 y :$
   $so^2(3) \oplus \langle \partial_u \rangle;$
3) $\Delta u = y^{-2} \tilde{F}(\omega), \quad \omega = (u_x^2 + u_y^2) y^2 : \quad sl^1(2, \mathbb{R}) \oplus \langle \partial_u \rangle;$
4) $\Delta u = y^{-2} \tilde{F}(\omega), \quad \omega = 4y^2 u_y^2 + (1 - 2y u_x)^2 : \quad sl^2(2, \mathbb{R}) \oplus \langle \partial_u \rangle;$
5) $\Delta u = \lambda \operatorname{ch}^{-1} y \sqrt{u_x^2 + u_y^2}, \quad \lambda \neq 0 : \quad so^1(3) \oplus \langle \partial_u, u\partial_u \rangle;$
6) $\Delta u = \lambda y^{-1} \sqrt{u_x^2 + u_y^2}, \quad \lambda \neq 0 : \quad sl^2(2, \mathbb{R}) \oplus \langle \partial_u, u\partial_u \rangle.$

Here $\tilde{F} = \tilde{F}(\omega)$ is an arbitrary smooth function, $\tilde{F}_{\omega\omega} \neq 0$.

Invariance with respect to non-decomposable algebras. Here we will investigate an existence of equations of form (1) invariant with respect to algebras of form $S \Subset N$, i.e. satisfying conditions

$$[S, S] \subset S, \quad [N, N] \subset N, \quad [S, N] \subset N. \tag{9}$$

In this investigation we will use results of the paper [2], where classification of Lie algebras of dimension not greater then eight, and whose Levi factor coincides with the algebras $so(3)$ and $sl(2, \mathbb{R})$.

Result. *There exist no nonlinear equations of form* (1) *with invariance algebras of such structure.*

Invariance with respect to solvable algebras of symmetry operators. For each two-dimensional and three-dimensional solvable algebras $A_{2.i} = \langle e_1, e_2\rangle$, $(i = 1, 2)$ and $A_{3.i} = \langle e_1, e_2, e_3\rangle$ $(i = 1, \dots, 9)$ below we adduce realizations and corresponding forms of the functions F.

$A_{2.1}$ $([e_1, e_2] = 0)$:

$$A^1_{2.1} = \langle \partial_x, \partial_y\rangle, \quad F = \tilde F(u, u_x, u_y);$$
$$A^2_{2.1} = \langle \partial_x, \partial_u\rangle, \quad F = \tilde F(y, u_x, u_y);$$
$$A^3_{2.1} = \langle \partial_u, g(x, y)\partial_u\rangle \ (g \neq \text{const})$$
$$F = \frac{g_x u_x + g_y u_y}{g_x^2 + g_y^2}(g_{xx} + g_{yy}) + G(x, y, \omega),$$
$$\omega = g_y u_x - g_x u_y, \quad g_x^2 + g_y^2 \neq 0.$$

$A_{2.2}$ $([e_1, e_2] = e_2)$:

$$A^2_{2.2} = \langle -x\partial_x - y\partial_y, \partial_x\rangle : \quad F = (u_x^2 + u_y^2)\tilde F(u, \omega_1, \omega_2),$$
$$\omega_1 = yu_x,\ \omega_2 = yu_y;$$
$$A^1_{2.2} = \langle \partial_x - u\partial_u, \partial_u\rangle : \quad F = e^{-x}\tilde F(y, \omega_1, \omega_2),$$
$$\omega_1 = e^x u_x, \quad \omega_2 = e^x u_y;$$
$$A^3_{2.2} = \langle -u\partial_u, \partial_u\rangle : \quad F = (u_x + u_y)\tilde F(x, y, \omega), \quad \omega = u_x u_y^{-1}.$$

Let us note that for arbitrary forms of functions $\tilde F$ the respective realisations are the maximal invariance algebras of equations.

$A_{3.1}$ $([e_j, e_l] = 0,\ j, l = 1, 2, 3)$:

$$A^1_{3.1} = \langle \partial_x, \partial_y, \partial_u\rangle, \quad F = G(u_x, u_y);$$
$$A^2_{3.1} = \langle \partial_x, \partial_u, f(y)\partial_u\rangle, \ \ f'(y) \neq 0, \quad F = \frac{f''}{f'}u_y + G(y, u_x);$$
$$A^3_{3.1} = \langle \partial_u, f(x, y)\partial_u, g(x, y)\partial_u\rangle.$$

Further analysis of the determining equations for operators $f(x, y)\partial_u$ and $g(x, y)\partial_u$ shows that either the respective invariant equation is linear or $g(x, y) = \lambda f(x, y) + \mu$, where λ, μ are constants, and then the case $A^3_{3.1}$ is reduced to the two-dimensional case.

$A_{3.2}$ $([e_1, e_2] = e_2,\ [e_1, e_3] = [e_2, e_3] = 0)$:

$$A^1_{3.2} = \langle -x\partial_x - y\partial_y, \partial_x, \partial_u\rangle, \quad F = u_x^2 G(yu_x, yu_y);$$

$$A^2_{3.2} = \langle -u\partial_u, \partial_u, \partial_x\rangle, \quad F = (u_x + u_y)G\left(y, \frac{u_x}{u_y}\right);$$

$$A^3_{3.2} = \langle \lambda\partial_y - u\partial_u, \partial_u, \partial_x\rangle, \quad \lambda \neq 0,$$
$$F = (u_x + u_y)G(e^{y/\lambda}u_x, e^{y/\lambda}u_y);$$

$$A^4_{3.2} = \langle \partial_x - u\partial_u, \partial_u, f(y)e^{-x}\partial_u\rangle, \quad f(y) \neq 0,$$
$$F = -\frac{f + f''}{f}u_x + e^{-x}G(y, e^x f' u_x + e^x f u_y).$$

$A_{3.3}$ ($[e_2, e_3] = e_1$, $[e_1, e_2] = [e_1, e_3] = 0$):

$$A^1_{3.3} = \langle \partial_u, \partial_x, \lambda\partial_y + x\partial_u\rangle, \lambda \neq 0, \quad F = G(\lambda u_x - y, u_y);$$
$$A^2_{3.3} = \langle \partial_u, (f(y) - x)\partial_u, \partial_x\rangle, \quad F = -f''u_x + G(y, u_y + f'u_x).$$

$A_{3.4}$ ($[e_1, e_3] = e_1$, $[e_2, e_3] = e_1 + e_2$, $[e_1, e_2] = 0$):

$$A^1_{3.4} = \langle \partial_u, \partial_x, x\partial_x + y\partial_y + (u + x)\partial_u\rangle, \quad F = e^{-u_x}G(u_y, ye^{-u_x});$$
$$A^2_{3.4} = \langle \partial_u, (f(y) - x)\partial_u, \partial_x + u\partial_u\rangle,$$
$$F = \frac{f'u_y - u_x}{1 + (f')^2}f'' + e^x G(y, f'e^{-x}u_x + e^{-x}u_y).$$

$A_{3.5}$ ($[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_2] = 0$):

$$A^1_{3.5} = \langle \partial_x, \partial_y, x\partial_x + y\partial_y\rangle, \quad F = (u_x + u_y)^2 G\left(u, \frac{u_x}{u_y}\right);$$
$$A^2_{3.5} = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + u\partial_u\rangle, \quad F = y^{-1}G(u_x, u_y);$$
$$A^3_{3.5} = \langle \partial_u, f(y)\partial_u, \partial_x + u\partial_u\rangle, \quad f' \neq 0,$$
$$F = \frac{u_y}{f'}f'' + e^x G(y, f'e^{-x}u_x);$$
$$A^4_{3.5} = \langle \partial_u, f(x, y)\partial_u.u\partial_u\rangle,$$

The equation that is invariant with respect to $A^4_{3.5}$ is linear.

$A_{3.6}$ ($[e_1, e_3] = e_1$, $[e_2, e_3] = -e_2$, $[e_1, e_2] = 0$):

$$A^1_{3.6} = \langle \partial_x, \partial_u, x\partial_x + y\partial_y - u\partial_u\rangle, \quad F = y^{-3}G(y^2u_x, y^2u_y);$$
$$A^2_{3.6} = \langle \partial_u, e^{2x}f(y)\partial_u, \partial_x + u\partial_u\rangle, \quad f(y) \neq 0,$$
$$F = \frac{2fu_x + f'u_y}{4f^2 + (f')^2}(f'' + 4f) + e^x G(y, f'e^{-x}u_x - 2fe^{-x}u_y);$$

$A_{3.7}$ ($[e_1, e_3] = e_1$, $[e_2, e_3] = qe_2$ $(0 < |q| < 1)$, $[e_1, e_2] = 0$):

$$A^1_{3.7} = \langle \partial_x, \partial_u, x\partial_x + y\partial_y + qu\partial_u \rangle, \quad F = y^{q-2} G(y^{1-q}u_x, y^{1-q}u_y);$$

$$A^2_{3.7} = \langle \partial_u, \partial_x, qx\partial_x + qy\partial_y + u\partial_u \rangle,$$

$$F = y^{(1-2q)/q} G(y^{(q-1)/q}u_x, y^{(q-1)/q}u_y);$$

$$A^3_{3.7} = \langle \partial_u, e^{(1-q)x} f(y)\partial_u, \partial_x + u\partial_u \rangle,$$

$$F = \frac{(1-q)fu_x + f'u_y}{(1-q)^2 f^2 + (f')^2}(f'' + f(1-q)^2) + \\ + e^x G(y, f'e^{-x}u_x + (q-1)fe^{-x}u_y).$$

$A_{3.8}$ ($[e_1, e_3] = -e_2$, $[e_2, e_3] = e_1$, $[e_1, e_2] = 0$):

$$A^1_{3.8} = \langle \partial_x, \partial_y, y\partial_x - x\partial_y \rangle, \ F = G(u, u_x^2 + u_y^2);$$

$$A^2_{3.8} = \langle \partial_u, \operatorname{tg}(f(y) - x)\partial_u, \partial_x - u \operatorname{tg}(f(y) - x)\partial_u \rangle,$$

$$F = \frac{f'u_y - u_x}{(f')^2 + 1} f'' + 2(f'u_y - u_x) \operatorname{tg}(f - x) + \\ + (f'u_x + u_y) G(y, \cos(f - x)(f'u_x + u_y)).$$

$A_{3.9}$ ($[e_1, e_3] = qe_1 - e_2$, $[e_2, e_3] = e_1 + qe_2$ $(q > 0)$, $[e_1, e_2] = 0$):

$$A^1_{3.9} = \langle \partial_x, \partial_y, (qx + y)\partial_x + (qy - x)\partial_y \rangle,$$

$$F = (u_x^2 + u_y^2) G\left(u, \ln(u_x^2 + u_y^2) + 2q \arctan \frac{u_x}{u_y}\right);$$

$$A^2_{3.9} = \langle \partial_u, \operatorname{tg}(f(y) - x)\partial_u, \partial_x + (q - \operatorname{tg}(f(y) - x))u\partial_u \rangle,$$

$$F = \frac{f'u_y - u_x}{(f')^2 + 1} f'' + 2(f'u_y - u_x) \operatorname{tg}(f - x) + \\ + (f'u_x + u_y) G(y, \cos(f - x)(f'u_x + u_y)e^{-qx}).$$

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