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Classification of admissible transformations of differential equations

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Концепцію групової класифікації модифіковано і поширено до класифікації допустимих перетворень у класах диференціальних рівнянь. З цією метою переглянуто існуючі поняття групового аналізу. Описано недавно введені поняття (умовна група еквівалентності, нормалізований клас диференціальних рівнянь) та досліджено їх властивості.

The framework of group classification is modified and extended to classification of admissible transformations in classes of differential equations. For this purpose, existing notions of group analysis are revised. Recently introduced notions (conditional equivalence group, normalized class of differential equations) are described and their properties are investigated.

1. Introduction. The beginnings of the theory of Lie groups and Lie algebras were inseparably linked with group analysis of differential equations and, in particular, with group classification problems. Inspired by the idea of creating a universal theory of integration of ordinary differential equations similar to the Galois theory of solving algebraic equations, S. Lie developed the theory of continuous transformation groups, classified such locally non-singular groups acting on the complex and real planes, described their differential invariants and then carried out group classification of second-order ordinary differential equations.

At present there is a substantial number of papers devoted to studying important classes of differential equations of theoretical and mathematical physics, biology, financial mathematics and other sciences from the Lie symmetry point of view (see e.g. all references in this paper and citation therein). The group classification in a class of (systems of) differential equations is reduced to integration of a complicated overdetermined system of partial differential equations with respect to both coefficients of infinitesimal symmetry operators and arbitrary elements. That is why it is a considerably more complicated problem than finding the Lie symmetry group of a single differential equation. Whereas programs for solving the latter problem had been created for most existing symbolic calculations packages a significant progress in computer realization of the group classification algorithm was achieved only recently.

Classes of differential equations are usually chosen based on their importance for applications without any mathematical background, although such choice is an important step in successful and exhaustive classification. It is a well-established fact that in the presence of certain properties with respect to point transformations the implementation of group classification is simplified and final results can be formulated in a clear and complete form. As is often the case, before mathematical notions are defined in a rigorous and precise form, they can be implicitly used for a long time. This commonplace is particularly true for the notion of a normalized class of differential equations, which was introduced recently [6, 18, 19] and may become a cornerstone of the framework of classification problems of group analysis. Knowledge that a class of differential equations is normalized allows to reduce the group classification problem in this class to subgroup analysis of the corresponding equivalence group.

Development of techniques of group analysis also allows to formulate and to solve new classification problems concerning transformational properties of differential equations. In particular, the notions of conditional equivalence group and normalized class of differential equations gives a language to describe complete sets of admissible transformations for classes of differential equations.

In this paper we outline extension of the framework of group classification to classification of admissible transformations in classes of differential equations. For this purpose, existing notions of group analysis (class of differential equations, equivalence group, gauge equivalence group [15], form-preserving transformation [8–10]) are discussed and revised. Recently introduced notions (conditional equivalence group [20], normalized class of differential equations [6,19]) are described and their properties are investigated.

Note 1. All functions are assumed to be smooth (e.g. analytical) and defined on certain subsets of their variables. A point transformation in the space of the variables $z = (z_1, \dots, z_k)$ is a smooth function φ: $\tilde{z} = \varphi(z)$ which is invertible at least locally.

2. Classes of systems of differential equations. Let $\mathcal{L}_\theta$ be a system $L(x,u_{(p)},\theta(x,u_{(p)}))=0$ of l differential equations for m unknown functions $u=(u^1,\dots,u^m)$ of n independent variables $x=(x_1,\dots,x_n)$. Here $u_{(p)}$ denotes the set of all the derivatives of u with respect to x of order no greater than p, including u as the derivatives of the zero order. $L=(L^1,\dots,L^l)$ is a tuple of l fixed functions depending on x, $u_{(p)}$ and θ. θ denotes the tuple of arbitrary (parametric) functions $\theta(x,u_{(n)})=(\theta^1(x,u_{(p)}),\dots,\theta^k(x,u_{(p)}))$ running the set $\mathcal{S}$ of solutions of the auxiliary system $S(x,u_{(p)},\theta_{(q)}(x,u_{(p)}))=0$. This system consists of differential equations with respect to θ, where x and $u_{(p)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order no greater than q. Sometimes the set $\mathcal{S}$ is additionally constrained by the non-vanish condition $S'(x,u_{(p)},\theta_{(q)}(x,u_{(p)}))\neq 0$ with another tuple S' of differential functions. In what follows we call the functions θ as arbitrary elements. Denote *the class of systems $\mathcal{L}_\theta$ with the arbitrary elements θ running $\mathcal{S}$* as $\mathcal{L}|_\mathcal{S}$.

Let $\mathcal{L}^k_\theta$ denote the set of all algebraically independent differential consequences of $\mathcal{L}_\theta$, which have, as differential equations, orders no greater than k. We identify $\mathcal{L}^k_\theta$ with the manifold determined by $\mathcal{L}^k_\theta$ in the jet space $J^{(k)}$. In particular, $\mathcal{L}_\theta$ is identified with the manifold determined by $\mathcal{L}^p_\theta$ in $J^{(p)}$.

It should be noted that the above definition of a class of systems of differential equations is not complete. The problem is that correspondence $\theta\to\mathcal{L}_\theta$ between arbitrary elements and systems (treated not as formal algebraic expressions but as real systems of differential equations or manifolds in $J^{(p)}$) may be not one-to-one. Namely, the same system may correspond to different values of arbitrary elements. A reason of this indeterminancy is that different values θ and $\tilde\theta$ of arbitrary elements can result after substitution of them to L in the same expression in x and $u_{(p)}$. Moreover, it is enough for $\mathcal{L}^p_\theta$ and $\mathcal{L}^p_{\tilde\theta}$ to coincide if the associated system completed with independent differential consequences differ each from other with a nonsingular matrix being a function in the variables of $J^{(p)}$.

The values θ and $\tilde\theta$ of arbitrary elements are called *gauge-equivalent* ($\theta\overset{\mathrm{g}}{\sim}\tilde\theta$) if $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ are the same system of differential equations. For the correspondence $\theta\to\mathcal{L}_\theta$ to be one-to-one, the set $\mathcal{S}$ of arbitrary elements should be factorized with respect to the gauge equivalence relation. We formally consider $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ as different representations of the same system from $\mathcal{L}|_\mathcal{S}$. It is often possible to realize gauge informally via changing the chosen representation of the class under consideration with replacement of the number k of arbitrary elements and the differential functions L and S although then this may result in more complicated calculations.

Definition 1. The classes $\mathcal{L}|_{\mathcal{S}}$ and $\mathcal{L}'|_{\mathcal{S}'}$ are called *similar* if $n = n'$, $m = m'$, $p = p'$, $k = k'$ and there exists a point transformation $\Psi\colon (x, u_{(p)}, \theta) \to (x', u'_{(p)}, \theta')$ which is projectible on the space of $(x, u_{(q)})$ for any $0 \le q \le p$, and $\Psi|_{(x,u_{(q)})}$ being the q-th order prolongation of $\Psi|_{(x,u)}$, $\Psi\mathcal{S} = \mathcal{S}'$ and $\mathcal{L}_{\theta'} = \Psi|_{(x,u)}\mathcal{L}_\theta$.

Hereafter the action of a such point transformation Ψ in the space of $(x, u_{(p)}, \theta)$ on arbitrary elements from $\mathcal{S}$ as pth-order differential functions is given by the formula:

$$\tilde\theta = \Psi\theta \quad \text{if} \quad \tilde\theta(x, u_{(p)}) = \Psi^\theta\Big(\Theta(x, u_{(p)}), \theta\big(\Theta(x, u_{(p)})\big)\Big),$$

where $\Theta = (\mathrm{pr}_p\Psi|_{(x,u)})^{-1}$ and pr_p denotes the operation of standard prolongations of a point transformations to the derivatives of orders not greater than p.

The set of transformations used in definition 1 can be extended via admitting different kinds of dependence on arbitrary elements in the ways as it is made for equivalence groups below.

Similar classes of systems have similar properties with the group analysis point of view.

Subclasses are singled out in the class $\mathcal{L}|_{\mathcal{S}}$ with additional auxiliary systems (or non-vanish conditions) which are attached to the main auxiliary system and the set of non-vanish conditions. Note that unions and intersections of subclasses of $\mathcal{L}|_{\mathcal{S}}$ also are subclasses of $\mathcal{L}|_{\mathcal{S}}$:

$$\mathcal{L}|_{\mathcal{S}'} \cup \mathcal{L}|_{\mathcal{S}''} = \mathcal{L}|_{\mathcal{S}'\cup\mathcal{S}''}, \quad \mathcal{L}|_{\mathcal{S}'} \cap \mathcal{L}|_{\mathcal{S}''} = \mathcal{L}|_{\mathcal{S}'\cap\mathcal{S}''}, \quad \mathcal{S}', \mathcal{S}'' \subset \mathcal{S}.$$

3. Admissible transformations. For $\theta, \tilde\theta \in \mathcal{S}$ we call the set of point transformations which maps the system $\mathcal{L}_\theta$ into the system $\mathcal{L}_{\tilde\theta}$ as the *set of admissible transformations from $\mathcal{L}_\theta$ into $\mathcal{L}_{\tilde\theta}$* and denote it by $\mathrm{T}(\theta, \tilde\theta)$. The maximal point symmetry group G_θ of the system $\mathcal{L}_\theta$ coincides with $\mathrm{T}(\theta, \theta)$. If the systems $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ are equivalent with respect to point transformations then $\mathrm{T}(\theta, \tilde\theta) = G_\theta \circ \varphi^0 = \varphi^0 \circ G_{\tilde\theta}$, where φ^0 is a fixed transformation from $\mathrm{T}(\theta, \tilde\theta)$. Otherwise, $\mathrm{T}(\theta, \tilde\theta) = \varnothing$.

The set $\mathrm{T}(\theta, \mathcal{L}|_{\mathcal{S}}) = \{\, (\tilde\theta, \varphi) \mid \tilde\theta \in \mathcal{S}, \mathrm{T}(\theta, \tilde\theta) \ne \varnothing, \varphi \in \mathrm{T}(\theta, \tilde\theta)\,\}$ is called the *set of admissible transformations of the system $\mathcal{L}_\theta$ in the class $\mathcal{L}|_{\mathcal{S}}$*.

Analogously, $\mathrm{T}(\mathcal{L}|_\mathcal{S}) = \{(\theta,\tilde\theta,\varphi) \mid \theta,\tilde\theta\in\mathcal{S},\ \mathrm{T}(\theta,\tilde\theta)\neq\varnothing,\ \varphi\in\mathrm{T}(\theta,\tilde\theta)\}$ is called the *set of admissible transformations in* $\mathcal{L}|_\mathcal{S}$.

First the set of admissible transformations was described by Kingston and Sophocleous for a class of generalised Burgers equations. These authors call transformations of such type *form-preserving ones* [8–10].

Notions and results adduced in this and the next sections can be reformulated in the infinitesimal terms by means of using the notions of vector fields, Lie algebras instead of point transformations, Lie groups etc. For instance, see [3] for the definition of "cones of tangent equivalences", which is the infinitesimal analogue of the definition of $\mathrm{T}(\theta,\mathcal{L}|_\mathcal{S})$. Ibid a non-trivial example of semi-normalized classes of differential equations (see definition 7) is investigated in the framework of the infinitesimal approach.

In the case of one dependent variable ($m=1$) we can extend above and below notions to contact transformations.

An element $(\theta,\tilde\theta,\varphi)$ from $\mathrm{T}(\mathcal{L}|_\mathcal{S})$ is called *a gauge admissible transformations in* $\mathcal{L}|_\mathcal{S}$ if $\theta \overset{\mathrm{g}}{\sim} \tilde\theta$ and φ is the identical transformation.

Proposition 1. *Similar classes have similar sets of admissible transformations. Namely, a similarity transformation* Ψ *from the class* $\mathcal{L}|_\mathcal{S}$ *into the class* $\mathcal{L}'|_{\mathcal{S}'}$ *generates a one-to-one mapping* Ψ^T *from* $\mathrm{T}(\mathcal{L}|_\mathcal{S})$ *into* $\mathrm{T}(\mathcal{L}'|_{\mathcal{S}'})$ *via the rule* $(\theta',\tilde\theta',\varphi') = \Psi^\mathrm{T}(\theta,\tilde\theta,\varphi)$ *if* $\theta'=\Psi\theta$, $\tilde\theta'=\Psi\tilde\theta$ *and* $\varphi' = \Psi|_{(x,u)}{}^{-1}\circ\varphi\circ\Psi|_{(x,u)}$. *Here* $(\theta,\tilde\theta,\varphi)\in\mathrm{T}(\mathcal{L}|_\mathcal{S})$, $(\theta',\tilde\theta',\varphi')\in\mathrm{T}(\mathcal{L}'|_{\mathcal{S}'})$.

Proposition 2. $\mathrm{T}(\mathcal{L}|_{\mathcal{S}'})\subset\mathrm{T}(\mathcal{L}|_\mathcal{S})$ *for any subclass* $\mathcal{L}|_{\mathcal{S}'}$ *of the class* $\mathcal{L}|_\mathcal{S}$. *If* $\mathcal{L}|_{\mathcal{S}''}$ *is another subclass of* $\mathcal{L}|_\mathcal{S}$ *then* $\mathrm{T}(\mathcal{L}|_{\mathcal{S}'})\cap\mathrm{T}(\mathcal{L}|_{\mathcal{S}''}) = \mathrm{T}(\mathcal{L}|_{\mathcal{S}'\cap\mathcal{S}''})$.

A number of notions connected with admissible transformations in classes of systems of differential equations can be reformulated in terms of the category theory [23].

4. Equivalence groups. The usual equivalence group of the class $\mathcal{L}|_\mathcal{S}$ is defined in a rigorous way via the notion of admissible transformations. Namely, any element Φ from the *usual equivalence group* $G^\sim = G^\sim(\mathcal{L}|_\mathcal{S})$ of the class $\mathcal{L}|_\mathcal{S}$ is a point transformation in the space of $(x,u_{(p)},\theta)$, which is projectible on the space of $(x,u_{(p')})$ for any $0\le p'\le p$, and $\Phi|_{(x,u_{(p')})}$ being the p'-th order prolongation of $\Phi|_{(x,u)}$, and $\forall\theta\in\mathcal{S}$: $\Phi\theta\in\mathcal{S}$ and $\Phi|_{(x,u)}\in\mathrm{T}(\theta,\Phi\theta)$.

Let us remind that the point transformation φ: $\tilde z=\varphi(z)$ in the space of the variables $z=(z_1,\dots,z_k)$ is called projectible on the space of the variables $z'=(z_{i_1},\dots,z_{i_{k'}})$, where $1\le i_1<\dots<i_{k'}\le k$, if the

expressions for $\tilde{z}'$ depend only on z'. We denote the restriction of φ on the space of z' as $\varphi|_{z'}$: $\tilde{z}' = \varphi|_{z'}(z')$.

If the arbitrary elements θ explicitly depend on x and u only (one always can do it formally, assuming derivatives as new dependent variables), we can admit dependence of transformations of (x, u) on θ and consider the *generalized equivalence group* $G^{\sim}_{\rm gen} = G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$ [16]. Any element Φ from $G^{\sim}_{\rm gen}$ is a point transformation in the space of (x, u, θ) such that $\forall\theta \in \mathcal{S}$: $\Phi\theta \in \mathcal{S}$ and $\Phi(\cdot,\cdot,\theta(\cdot,\cdot))|_{(x,u)} \in \mathrm{T}(\theta, \Phi\theta)$.

The action of $\Phi \in G^{\sim}_{\rm gen}$ on arbitrary elements as functions of (x, u) is given by the formula: $\tilde{\theta} = \Phi\theta$ if $\tilde{\theta}(x, u) = \Phi^{\theta}(\Theta(x, u), \theta(\Theta(x, u)))$, where $\Theta = (\Phi(\cdot,\cdot,\theta(\cdot,\cdot))|_{(x,u)})^{-1}$.

Roughly speaking, $G^{\sim}$ is the set of admissible transformations which can be applied to any $\theta \in \mathcal{S}$ and $G^{\sim}_{\rm gen}$ is formed by the admissible transformations which can be separated to classes parameterized with θ running whole $\mathcal{S}$.

It is possible to consider other generalizations of equivalence groups, e.g. groups with transformations which are point with respect to independent and dependent variables and include nonlocal expressions with arbitrary elements [7,24]. Let us give definitions of some generalizations.

Definition 2. The *extended equivalence group* $\bar{G}^{\sim} = \bar{G}^{\sim}(\mathcal{L}|_{\mathcal{S}})$ of the class $\mathcal{L}|_{\mathcal{S}}$ is formed by the transformations each of which are compositions $\Phi^1 \circ \Phi^2$, where $\forall\theta \in \mathcal{S}$: $(\Phi^1 \circ \Phi^2)\theta \in \mathcal{S}$ and $\Phi^1|_{(x,u)} \in \mathrm{T}(\theta, (\Phi^1 \circ \Phi^2)\theta)$. Here Φ^1 is a point transformation in the space of $(x, u_{(p)}, \theta)$, which is projectible on the space of $(x, u_{(p')})$ for any $0 \le p' \le p$, and $\Phi^1|_{(x,u_{(p')})}$ being the p'-th order prolongation of $\Phi^1|_{(x,u)}$. Φ^2 is an invertible transformation in the space of arbitrary elements assumed as functions of $(x, u_{(p)})$, and Φ^2 having certain special properties.

Definition 3. A transformation Φ is called to belong to the *extended generalized equivalence group* $\bar{G}^{\sim}_{\rm gen} = \bar{G}^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$ of the class $\mathcal{L}|_{\mathcal{S}}$ iff $\forall\theta \in \mathcal{S}$: $\Phi\theta \in \mathcal{S}$ and, after fixing θ, Φ becomes a point transformation from $\mathrm{T}(\theta, \Phi\theta)$.

The classes of chosen transformations with respect to arbitrary elements should be specified depending on the investigated classes of systems of differential equations. We do not point out a fixed kind of equivalence group where it is possible, implying any of the above kind.

Similar classes of systems of differential equations have similar equivalence groups.

The equivalence group generates an equivalence relations on the set of admissible transformations. Namely, the admissible transformations $(\theta^1, \tilde\theta^1, \varphi^1)$ and $(\theta^2, \tilde\theta^2, \varphi^2)$ from $\mathrm{T}(\mathcal{L}|_\mathcal{S})$ are called $G^\sim$*-equivalent* if there exist $\Phi \in G^\sim$ such that $\theta^2 = \Phi\theta^1$, $\tilde\theta^2 = \Phi\tilde\theta^1$ and $\varphi^2 = \Theta^{-1} \circ \varphi^1 \circ \Theta$, where $\Theta = \Phi|_{(x,u)}$ (or $\Theta = \Phi(\cdot,\cdot,\theta(\cdot,\cdot))|_{(x,u)}$ in case of $G^\sim_{\rm gen}$).

5. Group classification problems. Let $G^\cap = G^\cap(\mathcal{L}|_\mathcal{S}) = \bigcap_{\theta\in\mathcal{S}} G_\theta$ be the common part of G_θ, $\theta \in \mathcal{S}$, which is called the *kernel of the maximal point symmetry groups* of systems from the class $\mathcal{L}|_\mathcal{S}$. Note that $G^\cap$ can be naturally embedded into $G^\sim$ via trivial (identical) prolongation of the kernel transformations to the arbitrary elements. The associated subgroup of $G^\sim$ is normal.

The group classification problem for the class $\mathcal{L}|_\mathcal{S}$ is to describe all $G^\sim$-inequivalent values of $\theta \in \mathcal{S}$ together with the corresponding groups G_θ, for which $G_\theta \neq G^\cap$. The solution of the group classification problem is the list of pairs $(\mathcal{S}_\gamma, \{G_\theta, \theta \in \mathcal{S}_\gamma\})$, $\gamma \in \Gamma$. Here $\{\mathcal{S}_\gamma, \gamma \in \Gamma\}$ is a family of subsets of $\mathcal{S}$, $\bigcup_{\gamma\in\Gamma} \mathcal{S}_\gamma$ contains only $G^\sim$-inequivalent values of θ with $G_\theta \neq G^\cap$, and for any $\theta \in \mathcal{S}$ with $G_\theta \neq G^\cap$ there exists $\gamma \in \Gamma$ such that $\theta \in \mathcal{S}_\gamma \bmod G^\sim$. Structures of G_θ are similar for different values of $\theta \in \mathcal{S}_\gamma$ under fixed γ. In particular, G_θ, $\theta \in \mathcal{S}_\gamma$, have the same arbitrariness of group parameters.

Group classification problems in the above formulation are very complicated and, in the general case, are impossible to be solved since they leads to systems of functional differential equations. That is why, one usually considers only the connected component $G^{\rm p}_\theta$ of unity for each θ instead of the whole group G_θ. $G^{\rm p}_\theta$ is called the *principal (symmetry) group* of the system $\mathcal{L}_\theta$. The generators of one-parametric subgroups of $G^{\rm p}_\theta$ form a Lie algebra A_θ of vector fields in the space of (x,u), which is called the *maximal Lie invariance (or principal) algebra* of infinitesimal symmetry operators of $\mathcal{L}_\theta$. The kernel of principal groups of the class $\mathcal{L}|_\mathcal{S}$ is the group $G^{\cap\rm p} = G^{\cap\rm p}(\mathcal{L}|_\mathcal{S}) = \bigcap_{\theta\in\mathcal{S}} G^{\rm p}_\theta$ for which the Lie algebra is $A^\cap = A^\cap(\mathcal{L}|_\mathcal{S}) = \bigcap_{\theta\in\mathcal{S}} A_\theta$.

Knowing A_θ, one can reconstruct G_θ. Then the problem of group classification is reformulated in finding all possible inequivalent cases of extensions for A_θ, i.e. in listing all $G^\sim$-inequivalent values of the arbitrary parameters θ together with A_θ satisfying the condition $A_\theta \neq A^\cap$ [1, 17].

6. Gauge equivalence groups. The equivalence group $G^\sim$ of the class $\mathcal{L}|_\mathcal{S}$ can contain transformations which act only on arbitrary elements and do not really change systems, i.e. which generate gauge admissible transformations. In general, transformations of such type can be considered as trivial [15] (gauge) equivalence transformations and form the *gauge* subgroup $G^{\mathrm{g}\sim} = \{\Phi \in G^\sim \mid \Phi x = x,\ \Phi u = u,\ \Phi\theta \overset{g}{\sim} \theta\}$ of the equivalence group $G^\sim$. Moreover, $G^{\mathrm{g}\sim}$ is a normal subgroup of $G^\sim$.

Application of gauge equivalence transformations is equivalent to rewriting systems in another form. In spite of regular equivalence transformations, their role in group classification comes not to choice of representatives in equivalence classes but to choice of form of these representatives. It is quite common that the gauge equivalence relation on the set of arbitrary elements of a class of differential equations is generated by its gauge equivalence group.

We use the name "gauge equivalence transformation" since there exist really trivial equivalence transformations which do not transform even arbitrary elements. Such transformations arise if the auxiliary system implies functional dependence of arbitrary elements. They form normal subgroups in the corresponding equivalence groups and in the corresponding gauge equivalence groups. We will neglect these transformations and assume that equivalence groups coincide if they have the same factor group with respect to the trivial equivalence subgroups.

7. Conditional equivalence groups. The concept of *conditional equivalence* arises as an extension of the notion of conditional symmetry transformations of a single system of differential equations [4] to equivalence transformations in classes of systems. It is even more natural than the concept of conditional symmetry since description of any class includes, as a necessary element, an auxiliary system (a *condition*) for arbitrary elements. Imposing additional constraints on arbitrary elements, we may single out a subclass in the class under consideration, the equivalence group of which is not contained in the equivalence group of the whole class.

Let $\mathcal{L}|_{\mathcal{S}\cap\mathcal{S}'}$ denote the subclass of the class $\mathcal{L}|_{\mathcal{S}}$, which is singled out with the additional constrained system $S'(x, u_{(p)}, \theta_{(q)}(x, u_{(p)})) = 0$. Here $\mathcal{S}\cap\mathcal{S}'$ is the set of solutions of the united system $S = 0$, $S' = 0$. We assume that the united system is compatible for the subclass to be nonempty.

Definition 4. The equivalence group $G^\sim(\mathcal{L}|_{\mathcal{S}\cap\mathcal{S}'})$ of the subclass $\mathcal{L}|_{\mathcal{S}\cap\mathcal{S}'}$ is called a *conditional equivalence group* of the whole class $\mathcal{L}|_{\mathcal{S}}$ under the condition $S' = 0$. The conditional equivalence group is called *nontrivial* iff it is not a subgroup of the equivalence group $G^\sim(\mathcal{L}|_{\mathcal{S}})$.

The equivalence group $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ generates an equivalence relation on the set of pairs of additional auxiliary conditions and the corresponding conditional equivalence groups. Namely, if a transformation from $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ transforms the system $S'=0$ to the system $S''=0$ then the conditional equivalence groups $G^{\sim}(\mathcal{L}|_{\mathcal{S}\cap\mathcal{S}'})$ and $G^{\sim}(\mathcal{L}|_{\mathcal{S}\cap\mathcal{S}''})$ are similar with respect to this transformation and will be called *$G^{\sim}$-equivalent.*

Basing on the concept of conditional equivalence, we can formulate the problem of description of $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ similarly to the group classification problem. Nontrivial additional auxiliary conditions for arbitrary elements naturally arise under studying $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$. Steps of investigation could be the following:

1. Construction of $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ (or $G^{\sim}_{\rm gen}(\mathcal{L}|_{\mathcal{S}})$ etc).
2. Description of conditional equivalence transformations in $\mathcal{L}|_{\mathcal{S}}$, i.e. searching for a complete family of $G^{\sim}$-inequivalent additional auxiliary conditions S_γ, $\gamma\in\Gamma$, such that any S_γ determines the set $\mathcal{S}_\gamma$ of arbitrary elements, for which $G^{\sim}(\mathcal{L}|_{\mathcal{S}\cap\mathcal{S}_\gamma})\not\subset G^{\sim}(\mathcal{L}|_{\mathcal{S}})$.
3. Finding admissible transformations which belong to no conditional equivalence groups.

Actually, the proposed procedure is wide of optimality. We return to discussion of it after presentation of a more developed technique.

8. Normalized classes of differential equations. Solving group classification problems is essentially simpler if the class $\mathcal{L}|_{\mathcal{S}}$ of system differential equations under consideration has an additional property of normalization with respect to point transformations. The procedure of investigation of $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ can also be additionally enhanced with consideration of conditional equivalence groups for subclasses possessing this property.

Definition 5. The class $\mathcal{L}|_{\mathcal{S}}$ is called *normalized* if $\forall(\theta,\tilde\theta,\varphi)\in\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ $\exists\Phi\in G^{\sim}$: $\tilde\theta=\Phi\theta$ and $\varphi=\Phi|_{(x,u)}$.

The class $\mathcal{L}|_{\mathcal{S}}$ is called *normalized in generalized sense* if $\forall(\theta,\tilde\theta,\varphi)\in\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ $\exists\Phi\in G^{\sim}_{\rm gen}$: $\tilde\theta=\Phi\theta$ and $\varphi=\Phi(\cdot,\cdot,\theta(\cdot,\cdot))|_{(x,u)}$.

Proposition 3. *If the class $\mathcal{L}|_{\mathcal{S}}$ is normalized (in usual or generalized sense) then for any $\theta^0\in\mathcal{S}$ the point symmetry group G_{θ^0} coincides with restriction, on the space of (x,u), of the subgroup of $G^{\sim}$ (or $G^{\sim}_{\rm gen}$) preserving the value $\theta=\theta^0(x,u_{(p)})$.*

Definition 6. The class $\mathcal{L}|_\mathcal{S}$ is called *strongly normalized* if it is normalized and $G^\sim|_{(x,u)} = \prod_{\theta\in\mathcal{S}} G_\theta$.

The class $\mathcal{L}|_\mathcal{S}$ is called *strongly normalized in generalized sense* if it is normalized in generalized sense and $\forall\theta^0\in\mathcal{S}$: $G^\sim_{\rm gen}|^{\theta=\theta^0}_{(x,u)} = \prod_{\theta\in\mathcal{S}_{\theta^0}} G_\theta$, where $\mathcal{S}_{\theta^0} = \{\theta'\in\mathcal{S} \mid G^\sim_{\rm gen}|^{\theta=\theta'}_{(x,u)} = G^\sim_{\rm gen}|^{\theta=\theta^0}_{(x,u)}\}$.

Definition 7. The class $\mathcal{L}|_\mathcal{S}$ is called *semi-normalized* if $\forall(\theta,\tilde\theta,\varphi)\in \mathrm{T}(\mathcal{L}|_\mathcal{S})$ $\exists\tilde\varphi\in G_\theta$, $\exists\Phi\in G^\sim$: $\varphi = \tilde\varphi\circ\Phi|_{(x,u)}$, i.e.

$$\mathrm{T}(\mathcal{L}|_\mathcal{S}) = \{(\theta,\Phi\theta,\tilde\varphi\circ\Phi|_{(x,u)}) \mid \theta\in\mathcal{S},\ \tilde\varphi\in G_\theta,\ \Phi\in G^\sim\}.$$

($\mathrm{T}(\mathcal{L}|_\mathcal{S}) = \{(\theta^0,\Phi\theta^0,\tilde\varphi\circ\Phi|^{\theta=\theta^0}_{(x,u)}) \mid \theta^0\in\mathcal{S},\ \tilde\varphi\in G_\theta,\ \Phi\in G^\sim_{\rm gen}\}$ if $\mathcal{L}|_\mathcal{S}$ is *semi-normalized in generalized sense.*)

Roughly speaking, the class $\mathcal{L}|_\mathcal{S}$ is normalized if any admissible transformation in this class belongs to the equivalence group $G^\sim$ and is strongly normalized if additionally $G^\sim|_{(x,u)}$ is generated by elements from G_θ, $\theta\in\mathcal{S}$. The set of admissible transformations of a semi-normalized class is generated by the transformations from the equivalence group of the whole class and the transformations from the Lie symmetry groups of equations of this class.

Intersection of normalized subclasses of the class $\mathcal{L}|_\mathcal{S}$ with the same equivalence group $G^\sim_0$ is a normalized subclass possessing $G^\sim_0$ as a subgroup of the equivalence group, which generates the whole corresponding set of admissible transformations. Indeed, let $\mathcal{L}|_{\mathcal{S}'}$ and $\mathcal{L}|_{\mathcal{S}''}$ be normalized subclasses of the class $\mathcal{L}|_\mathcal{S}$ and $G^\sim(\mathcal{L}|_{\mathcal{S}'}) = G^\sim(\mathcal{L}|_{\mathcal{S}''}) = G^\sim_0$. If $\Phi\in G^\sim_0$ then $(\theta,\Phi\theta,\Phi|_{(x,u)})\in \mathrm{T}(\mathcal{L}|_{\mathcal{S}'\cap\mathcal{S}''})$ for any $\theta\in\mathcal{S}'\cap\mathcal{S}''$, i.e. $\Phi\in G^\sim(\mathcal{L}|_{\mathcal{S}'\cap\mathcal{S}''})$. In view of normalization of $\mathcal{L}|_{\mathcal{S}'}$ or $\mathcal{L}|_{\mathcal{S}''}$, for any $(\theta,\tilde\theta,\varphi)\in \mathrm{T}(\mathcal{L}|_{\mathcal{S}'\cap\mathcal{S}''})$ there exist $\Phi\in G^\sim_0$ such that $\tilde\theta=\Phi\theta$ and $\varphi = \Phi|_{(x,u)}$. Therefore, $\mathcal{L}|_{\mathcal{S}'\cap\mathcal{S}''}$ is a normalized subclass. The proof in case of normalization in generalized sense is analogous.

9. Examples of normalized classes. There exist a number of obvious examples of normalized classes. Thus, it is intuitively understandable that the extreme cases of classes formed by either a single system of differential equations or all systems having a fixed number of independent variables, unknown functions and differential equations with or without restriction of order are normalized. Let us demonstrate it within the framework of the above formal approach.

Consider a system $L(x, u_{(p)}) = 0$ of l differential equations for m unknown functions u of n independent variables x, which admits the maximal point symmetry group G. We assume that the tuple θ consists of a single arbitrary element denoted also as θ and L depends on θ constantly. The auxiliary system $\mathcal{S}$ for the arbitrary element θ is possible to be chosen in different ways. Here we discuss two possibilities.

The first one is to constrain θ with a single (algebraic or differential) equation, for example, $\theta = 0$. Hence, $\mathcal{S}$ is a one-element set consisting of the function identically vanishing on $J^{(p)}$, $\mathrm{T}(\mathcal{L}|_\mathcal{S}) = \{\,(0,0,\varphi) \mid \varphi \in G\,\}$ and $G^\sim = \{\,(\tilde{x},\tilde{u}) = \varphi(x,u),\ \tilde{\theta} = F(x,u_{(p)},\theta)\theta \mid \varphi \in G,\ F(\cdot,\cdot,0) \neq 0\,\}$, i.e. in view of definition 1 the class $\mathcal{L}|_\mathcal{S}$ is normalized. It possesses the nonempty trivial equivalence group $G^\sim_{\rm triv} = \{\,(\tilde{x},\tilde{u}) = (x,u),\ \tilde{\theta} = F(x,u_{(p)},\theta)\theta \mid F(\cdot,\cdot,0) \neq 0\}$ which should be neglected, and $G^\sim/G^\sim_{\rm triv} = \{\,(\tilde{x},\tilde{u}) = \varphi(x,u),\ \tilde{\theta} = \theta \mid \varphi \in G\,\}$.

The second possibility is to demand no constraints on θ, so $\mathcal{S}$ is the whole set of p-th order differential functions of (x,u), $\mathrm{T}(\mathcal{L}|_\mathcal{S}) = \{\,(\theta,\tilde{\theta},\varphi) \mid \theta,\tilde{\theta} \in \mathcal{S},\ \varphi \in G\}$ and $G^\sim = \{\,(\tilde{x},\tilde{u}_{(p)}) = \mathrm{pr}_p\,\varphi(x,u_{(p)}),\ \tilde{\theta} = F(x,u_{(p)},\theta) \mid \varphi \in G,\ \partial F/\partial\theta \neq 0\,\}$. Therefore, $\mathcal{L}|_\mathcal{S}$ is normalized. This class gives an example of classes without one-to-one correspondence between arbitrary elements and systems of differential equations.

The class of all systems of l differential equations for m unknown functions of n independent variables, which have order no greater than p, (here l, m, n and p are fixed integers) can be included within the framework of the formal approach after putting the left part of equations themselves as arbitrary elements and taking the empty auxiliary system S, i.e. $k = l$, $L \equiv \theta$ and $\mathcal{S}$ is the whole set of l-tuples of functionally independent p-th order differential functions of (x,u). Then $\mathrm{T}(\mathcal{L}|_\mathcal{S}) = \{\,(\theta,\tilde{\theta},\varphi) \mid \theta \in \mathcal{S},\ \tilde{\theta} = F(x,u_{(p)},\mathrm{pr}_p\,\varphi)\circ\theta,\ |\partial\varphi/\partial(x,u)| \neq 0,\ \partial F/\partial\theta|_{\theta=0} \neq 0\,\}$ and $G^\sim = \{\,\Phi = (\varphi(x,u), F(x,u_{(p)},\theta)) \mid |\partial\varphi/\partial(x,u)| \neq 0,\ \partial F/\partial\theta|_{\theta=0} \neq 0\,\}$ that obviously shows normalization of this class.

Normalization property has been proved in some ways for a number of different classes of differential equations being important for application. For example, generalized Burgers equations [8], eikonal equations of space dimensions 1, 2 and 3 [3], quasi-linear one-dimensional evolutions equations [2, 25], different multi-dimensional quasi-linear parabolic equations [23], $(1+1)$-dimensional generalized nonlinear wave equations [14], different kinds of $(1+1)$-dimensional nonlinear Schrödingher equations [5, 6, 19, 21, 22, 26], multi-dimensional generalized nonlinear Schrödingher equations [11].

10. Normalized classes and group classification problems. The notion of normalized classes was implicitly used in solving the group classification problems for many classes of system of differential equations. The most known classical group classification problems such as the Lie's classifications of second-order ordinary differential equations [13] and of second-order two-dimensional linear partial differential equations [12] were solved with essential usage of strong normalization of the above classes. Similar classification technique implicitly based on the properties of normalized classes was recently applied in solving group classification problems by a number of authors (see e.g. [2,3,5,14,21,25, 26]).

Proposition 4. *Let the class* $\mathcal{L}|_{\mathcal{S}}$ *be normalized and* G^i, $i=1,2$, *be local groups of point transformations in the space of* (x,u), *for which* $\mathcal{S}^i=\{\theta\in\mathcal{S}\,|\,G^{\rm p}_\theta=G^i\}\neq\varnothing$. *Then* $\mathcal{S}^1\sim\mathcal{S}^2 \bmod G^\sim$ *iff* $G^1\sim G^2 \bmod G^\sim$.

Proposition 5. *Two systems from a semi-normalized class are transformed each to other by a point transformation iff they are equivalent with respect to the equivalence group of this class.*

Proposition 6. *Any normalized class of systems of differential equations is semi-normalized.*

Proposition 7. *Let the class* $\mathcal{L}|_{\mathcal{S}}$ *be normalized and a subset* $\mathcal{S}'$ *of* $\mathcal{S}$ *determine a subclass* $\mathcal{L}|_{\mathcal{S}'}$ *which is invariant under action of* $G^\sim(\mathcal{L}|_{\mathcal{S}})$. *Then the subclass* $\mathcal{L}|_{\mathcal{S}'}$ *is normalized (in the same sense).* $G^\sim(\mathcal{L}|_{\mathcal{S}})$ *is a subgroup of* $G^\sim(\mathcal{L}|_{\mathcal{S}'})$, *which generates* $\mathrm{T}(\mathcal{L}|_{\mathcal{S}'})$ *and, if* $\mathcal{L}|_{\mathcal{S}}$ *is normalized in usual sense, coincides with* $G^\sim(\mathcal{L}|_{\mathcal{S}'})$ *up to gauge equivalence transformations in* $\mathcal{L}|_{\mathcal{S}'}$.

Proof. $G^\sim(\mathcal{L}|_{\mathcal{S}'})\supset G^\sim(\mathcal{L}|_{\mathcal{S}})$, since for any $\Phi\in G^\sim(\mathcal{L}|_{\mathcal{S}})$ and for any $\theta\in\mathcal{S}'$ we have $\Phi\theta\in\mathcal{S}'$, i.e. $(\theta,\Phi\theta,\Phi|_{(x,u)})\in\mathrm{T}(\mathcal{L}|_{\mathcal{S}'})$ that implies $\Phi\in G^\sim(\mathcal{L}|_{\mathcal{S}'})$. Since $\mathrm{T}(\mathcal{L}|_{\mathcal{S}'})\subset\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$, for any $(\theta,\tilde\theta,\varphi)\in\mathrm{T}(\mathcal{L}|_{\mathcal{S}'})$ there exists $\Phi\in G^\sim(\mathcal{L}|_{\mathcal{S}})$ such that $\tilde\theta=\Phi\theta$ and $\varphi=\Phi|_{(x,u)}$, i.e. the subclass $\mathcal{L}|_{\mathcal{S}'}$ is normalized. The above part of the proof is simply extended to the generalized case.

Any $\Psi\in G^\sim(\mathcal{L}|_{\mathcal{S}'})$ and any $\theta\in\mathcal{S}'$ give the admissible transformation $(\theta,\Psi\theta,\Psi|_{(x,u)})\in\mathrm{T}(\mathcal{L}|_{\mathcal{S}'})$. Therefore, there exists $\Phi\in G^\sim(\mathcal{L}|_{\mathcal{S}})$ such that $\Psi|_{(x,u)}=\Phi|_{(x,u)}$ and $\Psi\theta=\Phi\theta$. □

Note that under the above supposition the subclass $\mathcal{L}|_{\mathcal{S}\setminus\mathcal{S}'}$ has similar properties.

Given the class $\mathcal{L}|_\mathcal{S}$ and a local (connected) group G of point transformations of (x,u) such that $G=G^{\rm p}_\theta$ for some $\theta\in\mathcal{S}$, consider the subsets of $\mathcal{S}$

$$\mathcal{S}_G=\{\,\theta\in\mathcal{S}\mid G^{\rm p}_\theta\supset G\},\quad \bar{\mathcal{S}}_G=\{\,\theta\in\mathcal{S}\mid G^{\rm p}_\theta\supset G \mod G^\sim\},$$
$$\mathcal{S}'_G=\{\,\theta\in\mathcal{S}\mid G^{\rm p}_\theta=G\},\quad \bar{\mathcal{S}}'_G=\{\,\theta\in\mathcal{S}\mid G^{\rm p}_\theta=G \mod G^\sim\}.$$

Corollary 1. *Let the class $\mathcal{L}|_\mathcal{S}$ be normalized. Then $\mathcal{L}|_{\bar{\mathcal{S}}_G}$ and $\mathcal{L}|_{\bar{\mathcal{S}}'_G}$ are normalized subclasses of $\mathcal{L}|_\mathcal{S}$. $G^\sim(\mathcal{L}|_\mathcal{S})$ is a subgroup of $G^\sim(\mathcal{L}|_{\bar{\mathcal{S}}_G})$ and $G^\sim(\mathcal{L}|_{\bar{\mathcal{S}}'_G})$ and generates $\mathrm{T}(\mathcal{L}|_{\bar{\mathcal{S}}_G})$ and $\mathrm{T}(\mathcal{L}|_{\bar{\mathcal{S}}'_G})$.*

Proposition 8. *The subclass $\mathcal{L}|_{\mathcal{S}_0}$ is invariant with respect to $G^\sim(\mathcal{L}|_\mathcal{S})$, where $\mathcal{S}_0=\mathcal{S}'_{G^\cap}$, $G^\cap=G^{\cap{\rm p}}(\mathcal{L}|_\mathcal{S})$.*

Proof. Let us fix any $\Phi\in G^\sim(\mathcal{L}|_\mathcal{S})$ and any $\theta\in\mathcal{S}_0$. It is necessary to show that $\Phi\theta\in\mathcal{S}_0$. $G^{\rm p}_{\Phi\theta}=\mathrm{Ad}_\Phi G^{\rm p}_\theta=\mathrm{Ad}_\Phi G^\cap$, where Ad_Φ is the action of Φ on transformation groups: $G\ni\psi\to\varphi^{-1}\circ\psi\circ\varphi\in\mathrm{Ad}_\Phi G$, $\varphi:=\Psi|_{(x,u)}$. Since $\Phi\theta\in\mathcal{L}|_\mathcal{S}$, $G^{\rm p}_{\Phi\theta}\supset G^\cap$. If $G^{\rm p}_{\Phi\theta}=\mathrm{Ad}_\Phi G^\cap\neq G^\cap$ then $G^\cap\neq\mathrm{Ad}_{\Phi^{-1}}G^\cap\subset G^\cap$. But $\mathrm{Ad}_{\Phi^{-1}}G^\cap=G^{\rm p}_{\Phi^{-1}\theta}$, $\Phi^{-1}\theta\in\mathcal{L}|_\mathcal{S}$ and, therefore, $\mathrm{Ad}_{\Phi^{-1}}G^\cap\supset G^\cap$ that implies a contradiction. That is why, $G^{\rm p}_{\Phi\theta}=G^\cap$, i.e. $\Phi\theta\in\mathcal{S}_0$. □

Proposition 9. *$\mathcal{L}|_{\mathcal{S}'_G}$ is normalized in usual sense if $\mathcal{L}|_\mathcal{S}$ is normalized in usual sense. $\mathrm{T}(\mathcal{L}|_{\mathcal{S}'_G})$ is generated by the group $G^\sim(\mathcal{L}|_{\mathcal{S}'_G})\cap G^\sim(\mathcal{L}|_\mathcal{S})$ the projection of which in (x,u) is the normalizer of G in $G^\sim(\mathcal{L}|_\mathcal{S})|_{(x,u)}$.*

Proof. Let us fix arbitrary $(\theta,\tilde\theta,\varphi)\in\mathrm{T}(\mathcal{L}|_{\mathcal{S}'_G})$. Since $\mathrm{T}(\mathcal{L}|_{\mathcal{S}'_G})\subset\mathrm{T}(\mathcal{L}|_\mathcal{S})$, there exists $\Phi\in G^\sim(\mathcal{L}|_\mathcal{S})$ such that $\tilde\theta=\Phi\theta$ and $\varphi=\Phi|_{(x,u)}$, $\theta,\tilde\theta\in\mathcal{S}'_G$, hence $G=G^{\rm p}_{\tilde\theta}=\varphi^{-1}\circ G^{\rm p}_\theta\circ\varphi=\varphi^{-1}\circ G\circ\varphi$, i.e. $\varphi=\Phi|_{(x,u)}$ belongs to the normalizer of G in $G^\sim(\mathcal{L}|_\mathcal{S})|_{(x,u)}$.

Consider any $\Phi\in G^\sim(\mathcal{L}|_\mathcal{S})$ such that $\varphi=\Phi|_{(x,u)}$ belongs to the normalizer of G in $G^\sim(\mathcal{L}|_\mathcal{S})|_{(x,u)}$. Then $(\theta,\Phi\theta,\varphi)\in\mathrm{T}(\mathcal{L}|_{\mathcal{S}'_G})$ for arbitrary $\theta\in\mathcal{S}'_G$ since $\Phi\theta\in\mathcal{S}'_G$. Indeed, $\Phi\theta\in\mathcal{S}$ and $G=G^{\rm p}_{\Phi\theta}=\varphi^{-1}\circ G^{\rm p}_\theta\circ\varphi=\varphi^{-1}\circ G\circ\varphi=G$. Therefore, $\Phi\in G^\sim(\mathcal{L}|_{\mathcal{S}'_G})$. □

Proposition 10. *$G^{\cap{\rm p}}(\mathcal{L}|_{\mathcal{S}_G})=G$. $G^\sim(\mathcal{L}|_{\mathcal{S}_G})\subset G^\sim(\mathcal{L}|_{\mathcal{S}'_G})$. If $\mathcal{L}|_\mathcal{S}$ is normalized in usual sense, projections of these groups in (x,u) coincide.*

Proof. The first statement trivially follows from the definition of $\mathcal{L}|_{\mathcal{S}_G}$. Then in view of proposition 8 $\mathcal{L}|_{\mathcal{S}'_G}$ is invariant with respect to $G^\sim(\mathcal{L}|_{\mathcal{S}_G})$, i.e. $G^\sim(\mathcal{L}|_{\mathcal{S}_G})\subset G^\sim(\mathcal{L}|_{\mathcal{S}'_G})$. Proposition 9 implies the latter statement. In particular, $G^\sim(\mathcal{L}|_{\mathcal{S}_G})\cap G^\sim(\mathcal{L}|_\mathcal{S})=G^\sim(\mathcal{L}|_{\mathcal{S}'_G})\cap G^\sim(\mathcal{L}|_\mathcal{S})$. □

Note 2. In general, the class $\mathcal{L}|_{\mathcal{S}_G}$ is not normalized.

In view of the above propositions, the group classification problem in any normalized class of differential equations is reduced to subgroup analysis of the corresponding equivalence group. The property of strong normalization allows us to hope that essential part of subgroups will be Lie symmetry groups of systems from the class under consideration. Moreover, under classification a hierarchy of normalized classes corresponding to symmetry extension cases are naturally obtained.

11. Normalized subclasses and admissible transformations. Investigation of normalization of the class $\mathcal{L}|_{\mathcal{S}}$ or its subclasses is necessary for description of $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ and can be included as a step in studying $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$. The problem of classification of admissible transformations can be assumed solved, for example, in the following cases.

In view of the definition of normalized classes, the set of admissible transformations is known if the class proves to be normalized and its equivalence group are calculated. Then

$$\mathrm{T}(\mathcal{L}|_{\mathcal{S}}) = \{\, (\theta, \Phi\theta, \Phi|_{(x,u)}) \mid \theta \in \mathcal{S}, \Phi \in G^{\sim} \}.$$

Suppose that the class $\mathcal{L}|_{\mathcal{S}}$ is presented as a union of disjoint normalized subclasses, and there are no admissible transformations between systems from different subclasses. That is, $\mathcal{S} = \bigcup_{\gamma\in\Gamma} \mathcal{S}_\gamma$, $\mathcal{L}|_{\mathcal{S}_\gamma}$ is normalized for any $\gamma \in \Gamma$, $\mathcal{S}_\gamma \cap \mathcal{S}_{\gamma'} = \varnothing$ and $\mathrm{T}(\theta, \theta') = \varnothing$, where $\theta \in \mathcal{S}_\gamma$, $\theta' \in \mathcal{S}_{\gamma'}$, $\gamma \neq \gamma'$. Then obviously $G^{\sim}(\mathcal{L}|_{\mathcal{S}_\gamma}) \supset G^{\sim}(\mathcal{L}|_{\mathcal{S}})$ for any $\gamma \in \Gamma$ and $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ is the union of the simply constructed sets $\mathrm{T}(\mathcal{L}|_{\mathcal{S}_\gamma})$ of admissible transformations in the subclasses:

$$\mathrm{T}(\mathcal{L}|_{\mathcal{S}}) = \bigcup_{\gamma\in\Gamma} \{\, (\theta, \Phi\theta, \Phi|_{(x,u)}) \mid \theta \in \mathcal{S}_\gamma, \Phi \in G^{\sim}(\mathcal{L}|_{\mathcal{S}_\gamma}) \,\}.$$

The class of nonlinear Schrödinger equations with potentials and general modular nonlinearities has the set of admissible transformations of the above structure for all space dimensions [11, 18, 19].

A more nontrivial situation is when normalized subclasses intersect each other. Let $\mathcal{S}', \mathcal{S}'' \subset \mathcal{S}$, $\mathcal{S}' \cap \mathcal{S}'' \neq \varnothing$, the subclasses $\mathcal{L}|_{\mathcal{S}'}$ and $\mathcal{L}|_{\mathcal{S}''}$ are normalized, $\mathcal{S}' = G^{\sim}(\mathcal{L}|_{\mathcal{S}'})\ \mathcal{S}' \cap \mathcal{S}''$ and $\mathcal{S}'' = G^{\sim}(\mathcal{L}|_{\mathcal{S}''})\ \mathcal{S}' \cap \mathcal{S}''$. Then any admissible transformation $(\theta', \theta'', \varphi)$ with $\theta' \in \mathcal{S}'$ and $\theta'' \in \mathcal{S}''$, can be presented in the form $(\theta', \Phi^2(\Phi^1\theta'), (\Phi^1 \circ \Phi^2)|_{(x,u)})$, where $\Phi^1 \in G^{\sim}(\mathcal{L}|_{\mathcal{S}'})$, $\Phi^2 \in G^{\sim}(\mathcal{L}|_{\mathcal{S}''})$ and $\Phi^1\theta' \in \mathcal{S}' \cap \mathcal{S}''$.

A set of admissible transformations of such structure arises under investigation of a class of variable coefficient diffusion–reaction equations [24].

12. Conclusion. Consideration in this paper is quite informal. The aim was to give a description of major tools of modern group analysis and to present a new treatment of group classification problems. Most of adduced definitions and statements are flexible and can be made rigorous after fixing a class of system of differential equations under investigation.

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