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No-go theorem on reduction operators of linear second-order parabolic equations

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Доведено, що проблема опису всіх операторів редукції, тобто операторів некласичної (умовної) симетрії, довільного лінійного рівняння з частинними похідними другого порядку з двома незалежними змінними еквівалентна в певному сенсі розв'язанню вихідного рівняння.

The problem on description of the reduction operators, i.e. the operators of nonclassical (conditional) symmetry, of an arbitrary second-order linear parabolic partial differential equation in two independent variables proves to be equivalent, in some sense, to solving the initial equation.

1. Introduction. The notion of nonclassical symmetry (called also Q-conditional or, simply, conditional symmetry) was introduced in [1] by the example of the one-dimensional linear heat equation and a partial class of operators. A precise and rigorous definition was suggested later (see e.g. [4,5,14]). In contrast to classical Lie symmetry, the system of determining equations on the coefficients of conditional symmetry operators of the heat equation was found to be nonlinear and less overdetermined. First this system was investigated in [12] in detail, where the system was partially linearized and its Lie symmetries were found. The problem on conditional symmetries of the heat equation was completely solved in [3]. Namely, in the both arising cases the maximal Lie invariance algebras of the determining equations were calculated and the determining equations were reduced to the initial equation with nonlocal transformations. Results of [3] were in [2,6,7] extended to a class of linear transfer equations which generalize the heat equation. Thus, for these equations the "no-go" theorems on reduction of determining equations for coefficients of conditional symmetry operators to the initial equations were proved in detail and wide multi-parametric families of exact solutions were constructed with non-Lie reductions. It was observed in [13] that the proof of the theorem from [3] on linearization of determining equations in case of conditional symmetry operators with vanishing coefficients of ∂_t are extended to the class of one-dimensional evolution equations. This theorem was also generalized to multi-dimensional evolution equations [8] and even systems of such equations [11].

In this paper we investigate the class of second-order linear parabolic partial differential equation in two independent variables, which have the general form

$$Lu = u_t - a^2(t,x)u_{xx} - a^1(t,x)u_x - a^0(t,x)u = 0, \tag{1}$$

where the coefficients a^i, $i = 1, 2, 3$, are (real or complex) analytic functions of t and x, $a^2 \neq 0$. The “no-go” theorems on reduction of determining equations for coefficients of conditional symmetry operators to the initial equations are proved for class (1). All possible reductions to ordinary differential equations are described.

Conditional invariance of a differential equation with respect to a vector field is equivalent to that any ansatz associated with the vector field reduces the equation to a differential equation with the lesser by 1 number of independent variables [14]. That is why, below we use the shorter and more natural term “reduction operator” instead of “operator of conditional symmetry” and say that an operator reduces a differential equation in case the equation is reduced by the associated ansatz.

2. Determining equations for reduction operators. Preliminary description of reduction operators of equations (1) is given by the following theorem.

Theorem 1. *Any reduction operator of equation* (1) *is equivalent to either an operator $\partial_t + g^1(t,x)\partial_x + (g^2(t,x)u + g^3(t,x))\partial_u$ with the coefficients g^1, g^2 and g^3 satisfying the system*

$$\begin{aligned}
&\tilde{L}g^1 + Ha^1 + a^1_x g^1 + 2a^2 g^2_x + a^1_t = 0,\\
&\tilde{L}g^2 - Ha^0 - a^0_x g^1 - a^0_t = 0,\\
&\tilde{L}g^3 - a^0 g^3 = 0,
\end{aligned} \tag{2}$$

where $\tilde{L} = \partial_t - a^2\partial_{xx} - a^1\partial_x + H$ and $H = 2g^1_x - (a^2_x g^1 + a^2_t)/a^2$, or an operator $\partial_x + \eta(t,x,u)\partial_u$, where the function η is a solution of the *equation*

$$\eta_t = a^2(\eta_{xx} + 2\eta\eta_{xu} + \eta^2\eta_{uu}) + a^2_x(\eta_x + \eta\eta_u) \\ + (a^1\eta)_x + a^0(\eta - u\eta_u) + a^0_x u. \tag{3}$$

Proof. In the case of two independent variables t and x, reduction operators are written as $Q = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$, where $(\tau,\xi) \neq (0,0)$. The conditional invariance criterion for equation (1) and the operator Q has the form [4]

$$Q_{(2)}Lu \,\Big|_{\,Lu=0,\;\; Q[u]=0,\;\; D_tQ[u]=0,\;\; D_xQ[u]=0} = 0, \tag{4}$$

where $Q_{(2)}$ is the standard second prolongation of Q, $Q[u] = \eta - \tau u_t - \xi u_x$ is its characteristic, D_t and D_x denote the total differentiation operators with respect to t and x. All equalities hold true as algebraic relations in the second-order jet space $J^{(2)}$ over the space of the independent variables (t,x) and the dependent variable u.

Since (1) is an evolution equation, there are two principally different cases of finding Q: $\tau \neq 0$ and $\tau = 0$.

If $\tau \neq 0$ we can assume $\tau = 1$ up to the usual equivalence of reduction operators. There is only one unconstrained variable in (4). We choose the derivative u_x as such variable and express the other derivatives being in $Q_{(2)}Lu$ via (t,x,u) and u_x on the constrained set of $J^{(2)}$:

$$u_t = \eta - \xi u_x, \quad u_{xx} = \frac{\eta - \xi u_x - a^1 u_x - a^0}{a^2}.$$

Splitting in the obtained equation with respect to u_x results in the determining equations for coefficients ξ and η which imply $\xi_u = 0$, $\eta_{uu} = 0$, i.e. $\xi = g^1(t,x)$, $\eta = g^2(t,x)u + g^3(t,x)$. Further splitting with respect to u leads to system (2).

The condition $\tau = 0$ gives $\xi \neq 0$ since $(\tau,\xi) \neq (0,0)$. Therefore, without loss of generality we can put $\xi = 1$ in view of the usual equivalence of reduction operators. All derivatives being in $Q_{(2)}Lu$ are expressed on the constrained set of $J^{(2)}$ via the variables (t,x,u):

$$u_x = \eta, \quad u_{xx} = \eta_x + \eta\eta_u, \quad u_t = a^2(\eta_x + \eta\eta_u) + a^1\eta + a^0 u.$$

After substituting these expressions to the equation $Q_{(2)}Lu = 0$, we obtain equation (3). □

Note 1. We can essentially simplify and order investigation of reduction operators, additionally taking into account Lie symmetry transformations in case of a single equation [9] and transformations from the equivalence group or the whole set of admissible transformations in case of a class of equations [10]. Up to the equivalence relation generated by the equivalence group of class (1) on the set of pairs "(an equation of form (1), its reduction operator)", it is enough to investigate only the subclass of equations (1) with $a_2 = 1$, $a_1 = 0$.

3. No-go theorems. There is a connection of system (2) and equation (3) with initial equation (1) via non-point transformations.

Theorem 2. *Nonlinear coupled system* (2) *is reduced by the transformation*

$$g^1 = -a^2\frac{v^1v^2_{xx} - v^1_{xx}v^2}{v^1v^2_x - v^1_xv^2} - a^1, \quad g^2 = -a^2\frac{v^1_xv^2_{xx} - v^1_{xx}v^2_x}{v^1v^2_x - v^1_xv^2} + a^0,$$
$$g^3 = \frac{a^2}{v^1v^2_x - v^1_xv^2}\begin{vmatrix} v^1 & v^1_x & v^1_{xx} \\ v^2 & v^2_x & v^2_{xx} \\ v^3 & v^3_x & v^3_{xx} \end{vmatrix} \tag{5}$$

to the uncouple linear system of three copies $v^i_t - a^2v^i_{xx} - a^1v^i_x - a^0v^i = 0$ *of equation* (1) *for the functions* $v^i = v^i(t,x)$*, and the functions* v^1 *and* v^2 *being linearly independent. Hereafter* $i = 1, 2, 3$*.*

Note 2. Let $W(\varphi^1,\dots,\varphi^n) = \det(\partial^{l-1}\varphi^k/\partial x^{l-1})^n_{k,l=1}$ be the Wronskian of the functions $\varphi^k = \varphi^k(t,x)$, $k = \overline{1,n}$, with respect to the variable x. Then transformation (5) can be rewritten as

$$g^1 = -a^2\frac{(W(v^1,v^2))_x}{W(v^1,v^2)} - a^1, \quad g^2 = -a^2\frac{W(v^1_x,v^2_x)}{W(v^1,v^2)} + a^0,$$
$$g^3 = a^2\frac{W(v^1,v^2,v^3)}{W(v^1,v^2)}.$$

Proof. For any tuple of functions $g^i = g^i(t,x)$ there exists functions v^i determined by (5). Really, relations (5) can be rewritten in the form

$$\widehat{Q}v^1 = 0, \quad \widehat{Q}v^2 = 0, \quad \widehat{Q}v^3 = g^3, \tag{6}$$

where $\widehat{Q} = \bar{Q} - L = a^2\partial_{xx} + (a^1+g^1)\partial_x + a^0 - g^2$. (Here $\bar{Q} = \partial_t + g^1\partial_x - g^2$ and $L = \partial_t - a^2(t,x)\partial_{xx} - a^1(t,x)\partial_x - a^0(t,x)$ are linear differential operators acting in the space of functions of (t,x). $\bar{Q}$ is associated with the operator Q and L is taken from equation (1).) System (6) with respect to v^i is a system of three uncoupled second-order linear ordinary differential equations with the independent variable x, and t being treated as a parameter. We can take any fundamental tuple of solutions of the equation $\widehat{Q}v=0$ as (v^1,v^2) and any partial solution of the equation $\widehat{Q}v=g^3$ as v^3. The functions v^i are determined by (6) ambiguously, namely up to the transformations

$$\tilde{v}^p=\varphi^{pq}(t)v^q,\quad \tilde{v}^3=v^3+\psi^q(t)v^q \qquad \text{or}$$
$$v^p=\tilde{\varphi}^{pq}(t)\tilde{v}^q,\quad v^3=\tilde{v}^3+\tilde{\psi}^q(t)\tilde{v}^q,$$

where φ^{pq} and ψ^q are arbitrary functions of t, $|\varphi^{pq}|\neq 0$, $(\tilde{\varphi}^{pq})=(\varphi^{pq})^{-1}$, $\tilde{\psi}^q=-\psi^p\tilde{\varphi}^{pq}$. Hereafter $p,q=1,2$ and the summation over the repeated indices is implied.

Let (v^1,v^2,v^2) be a fixed solution of (6), where the parameter-functions $g^i=g^i(t,x)$ satisfy system (2). We will show that the functions φ^{pq} and ψ^q (or $\tilde{\varphi}^{pq}$ and $\tilde{\psi}^q$) can be chosen in such way that the functions $\tilde{v}^i=0$ will satisfy equation (1), i.e. $L\tilde{v}^i=0$.

The left parts of equations (2) can be rewritten with representation (5) as

$$R^1=a^2\frac{v^2}{v^1}\bar{R}^1+a^2\frac{v^1}{v^2}\bar{R}^2,\ \ R^2=a^2\frac{v^2_x}{v^1}\bar{R}^1+a^2\frac{v^1_x}{v^2}\bar{R}^2,\ \ R^3=\widehat{Q}Lv^3,$$

where

$$\bar{R}^1=\left(\frac{W(v^1,Lv^1)}{W(v^1,v^2)}\right)_x=\left(\frac{(Lv^1/v^1)_x}{(v^2/v^1)_x}\right)_x,$$
$$\bar{R}^2=\left(\frac{W(v^2,Lv^2)}{W(v^2,v^1)}\right)_x=\left(\frac{(Lv^2/v^2)_x}{(v^1/v^2)_x}\right)_x.$$

Two first equations $R^1=R^2=0$ of (2) is a linear system of algebraic equations with respect to the values $\bar{R}^1$ and $\bar{R}^2$ with the non-vanishing determinant $(a^2)^2W(v^1,v^2)/(v^1v^2)$. Therefore, its unique solution is $\bar{R}^1=\bar{R}^2=0$. Integration the latter equations and the equation $R^3=\widehat{Q}Lv^3=0$ with respect to x leads to the conclusion that $Lv^i=\zeta^{ip}(t)v^p$, where ζ^{ip} are functions of t. If the functions $\tilde{\varphi}^{pq}$ and $\tilde{\psi}^q$ being in the "ambiguity" transformations satisfy the system of ODEs $\tilde{\varphi}^{pq}_t=\zeta^{pq'}\tilde{\varphi}^{q'q}$, $\tilde{\psi}^q_t=\zeta^{3q'}\tilde{\varphi}^{q'q}$ then $L\tilde{v}^i=0$.

And vice versa, if the functions $v^i = v^i(t,x)$ satisfy the equation $Lv = 0$, and the functions v^1 and v^2 being linearly independent, then the corresponding expressions R^1, R^2 and R^3 vanish identically. It means that the functions $g^i = g^i(t,x)$ determined by (5) give a solution of system (2). □

Theorem 3. *Nonlinear equations* (3) *is reduced by composition of the nonlocal substitution* $\eta = -\Phi_x/\Phi_u$*, where* Φ *is a function of* (t,x,u)*, and the hodograph transformation*

$$\begin{aligned} &\textit{the new independent variables:} \quad \tilde t = t, \quad \tilde x = x, \quad \varkappa = \Phi, \\ &\textit{the new dependent variable:} \quad \tilde u = u \end{aligned} \tag{7}$$

to the equation $L\tilde u = \tilde u_{\tilde t} - a^2(\tilde t,\tilde x)\tilde u_{\tilde x\tilde x} - a^1(\tilde t,\tilde x)\tilde u_{\tilde x} - a^0(\tilde t,\tilde x)\tilde u = 0$*, where* $\varkappa$ *plays the role of a parameter.*

We do not adduced the proof of theorem 3 since it has already proved for both some equations [3] or subclasses [6] from class (1) and much more general classes of evolution equations [8, 11, 13].

4. Ansatzes and solutions. Inverting of theorem 3 gives the following true statement [8, 11].

Theorem 4. *For any one-parametric family of solutions of equation* (1) *there exists an operator* $\partial_x + \eta(t,x,u)\partial_u$ *which reduces equation* (1) *and with respect to which the family of solutions is invariant.*

An ansatz associated with the operator $\partial_x + \eta\partial_u$, where $\eta = -\Phi_x/\Phi_u$ and the function $\Phi = \Phi(t,x,u)$ is obtained from the one-parametric solution $u = f(t,x,\varkappa)$ of equation (1) with the inverse hodograph transformation to (7), has the form $u = f(t,x,\varphi(\omega))$, $\omega = t$. Here ω and φ are the new ("invariant") independent and dependent variables correspondingly. Substitution of the ansatz to equation (1) implies the reduced equation $\varphi_\omega = 0$, i.e. $\varphi = C = \mathrm{const}$. As a result, we obtain the obvious one-parametric solution $u = f(t,x,C)$.

Theorems 3 and 4 can be united to the single statement.

Theorem 5. *For any equation of form* (1)*, there exists one-to-one correspondence between one-parametric families of its solutions and reduction operators with zero coefficients of* ∂_t*. Namely, each operator of such kind corresponds to the family of solutions which are invariant* *with respect to this operator. The problems of construction of all one-parametric families of solutions of equation* (1) *and complete description of its reduction operators with zero coefficients of* ∂_t *are completely equivalent.*

An ansatz associated with the operator $\partial_t + g^1\partial_x + (g^2u + g^3)\partial_u$, has the form $u = v^1\varphi(\omega) + v^3$, $\omega = v^2/v^1$. Here ω and φ again denote the new ("invariant") independent and dependent variables. The functions $v^i = v^i(t, x)$ are solutions of equation (1), which are connected with the reduction operator coefficients $g^i = g^i(t, x)$ via transformation (5), and v^1 and v^2 being linearly independent. Integration of the corresponding reduced equation $\varphi_{\omega\omega} = 0$ gives $\varphi = C_2\omega + C_1$, where C_1 and C_2 are arbitrary constants. After substituting the expression for φ to the ansatz, we obtain the two-parametric solution

$$u = C_1v^1 + C_2v^2 + v^3. \tag{8}$$

And vice versa, given a two-parametric solution of form (8), the coefficients g^1, g^2 and g^3 are unambiguously repaired by formula (5).

Supposed triviality of the above ansatzes and reduced equations is connected with usage of the special representations for the solutions of the determining equations. Under this approach difficulties in construction of ansatzes and integration of reduced equations are replaced by difficulties in obtaining of the representations for coefficients of reduction operators.

4. Conclusion. The "no-go" results of this paper can be extended with investigation of Lie symmetries and Lie reductions of determining equations (2) and (3). Indeed, the maximal Lie invariance algebras of (2) and (3) are isomorphic to the maximal Lie invariance algebras of equation (1). This result is well known for the linear heat equation [3].

Let us note also that the term "no-go" has to be treated only as impossibility of exhaustive solving of the problem. At the same time, imposing additional constraints on the coefficients, one can construct a number of particular examples of reduction operators and then apply them to finding exact solutions of the initial equation. Since the determining equation has more independent variables and, therefore, more freedom degrees, it is more convenient often to guess a simple solution or a simple ansatz for the determining equation, which can give a parametric set of complicated solutions of the initial equation. (It is similar to situation with Lie symmetries of first-order ordinary differential equations.) This approach was applied to interesting subclass of equations (1), which arises under symmetry reduction of the Navier–Stokes equations [2,6,7], and allowed to construct series of multi-parametric solutions.

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