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On equations of Korteweg–de Vries type with highest symmetry properties

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Представлено результати групової класифікації одного класу нелінійних еволюційних рівнянь третього порядку, що допускать чотирьохвимірні алгебри Лі операторів симетрії.

We present results on group classification of one class of third-order nonlinear evolution equations admitting four-dimensional solvable Lie algebras of symmetry operators.

The standard Korteweg–de Vries equation $u_t = u_{xxx} + uu_x$ belongs to the family of evolution equations

$$u_t = u_{xxx} + F(t, x, u, u_x, u_{xx}), \tag{1}$$

where $u = u(t, x)$, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xxx} = \frac{\partial^3 u}{\partial x^3}$.

The problem of group classification of equation (1) was solved by F. Güngor, V. Lahno and R. Zhdanov [1]. But their result of group classification is not complete. They obtained all classes of nonlinear equations of the form (1) that admit one-, two-, three- and four-dimensional solvable Lie algebras.

Here we investigate the symmetry properties of nonlinear equations of the form (1) whose invariance algebras are isomorphic to solvable Lie algebras $2A_{2.2} = \langle e_1, e_2\rangle \oplus \langle e_3, e_4\rangle$ ($[e_1, e_2] = e_2, [e_3, e_4] = e_4$), $A_{2.2} \oplus 2A_1 = \langle e_1, e_2\rangle \oplus \langle e_3\rangle \oplus \langle e_4\rangle$ ($[e_1, e_2] = e_2$) and $A_{3.3} \oplus A_1 = \langle e_1, e_2, e_3\rangle \oplus \langle e_4\rangle$ ($[e_2, e_3] = e_1, [e_1, e_2] = [e_1, e_3] = 0$).

According to [1] the complete list of such equations contains following nine equations:

1) $u_t = u_{xxx} + u_x^3 - 3u_x u_{xx} + x^{-2} u_x \tilde{F}(\omega)$,

$$\omega = x(u_x^{-1}u_{xx} - u_x);$$

2) $u_t = u_{xxx} + \frac{\lambda}{3t}\omega_1 \ln|\omega_1| + \frac{\omega_1}{t}\tilde{F}(\omega), \quad \omega_1 = t^{\frac{1}{3}}u_x,$
$\omega = t^{\frac{1}{3}}u_x^{-1}u_{xx}, \quad \lambda \in \mathbb{R};$

3) $u_t = u_{xxx} - \lambda x u_x - \lambda u_x \ln|u_x| + u_x\tilde{F}(\omega),$
$\omega = u_x^{-1}u_{xx}, \quad \lambda \neq 0;$

4) $u_t = u_{xxx} - (1+\lambda^{-1})u_x + e^{-x}\tilde{F}(\omega),$
$\omega = e^x(u_x + u_{xx}), \quad \lambda \neq 0;$

5) $u_t = u_{xxx} - \gamma^{-1}(1+\gamma^3)u_x + e^{(\gamma-\beta^{-1})x-t}\tilde{F}(\omega),$
$\omega = e^{t+(\beta^{-1}-\gamma)x}(\gamma u_x - u_{xx}), \quad \gamma\beta \neq 0;$

6) $u_t = u_{xxx} - u_x + e^{-x}\tilde{F}(\omega), \quad \omega = e^x(u_x + u_{xx});$

7) $u_t = u_{xxx} + u_x\tilde{F}(\omega), \quad \omega = u_{xx}u_x^{-1};$

8) $u_t = u_{xxx} - (\lambda^3+1)\lambda^{-1}u_x + e^{-t+\lambda x}\tilde{F}(\omega),$
$\omega = e^{t-\lambda x}(\lambda u_x - u_{xx}), \quad \lambda \neq 0;$

9) $u_t = u_{xxx} + \lambda^{-1}x - \beta u_x + \tilde{F}(u_{xx}), \quad \lambda > 0, \quad \beta \in \mathbb{R}.$ (2)

Using the standard Lie approach we prove that the maximal invariance group of equations (1) is generated by the operator

$$\mathrm{v} = \tau(t)\partial_t + \left(\tfrac{1}{3}\dot{\tau}x + \rho(t)\right)\partial_x + \eta(t,x,u)\partial_u, \tag{3}$$

where the functions τ, ρ, η and F are arbitrary solutions of a single partial differential equation

$$\begin{aligned} -3u_x\dot{\rho} - xu_x\ddot{\tau} - 9u_xu_{xx}\eta_{uu} - 3u_x^3\eta_{uuu} + 3\eta_t - 9u_{xx}\eta_{xu} - \\ - 9u_x^2\eta_{xuu} - 9u_x\eta_{xxu} - 3\eta_{xxx} + 3(\eta_u - \dot{\tau})F + (2u_{xx}\dot{\tau} - \\ - 3u_{xx}\eta_u - 3u_x^2\eta_{uu} - 6u_x\eta_{xu} - 3\eta_{xx})F_{u_{xx}} + (u_x\dot{\tau} - 3u_x\eta_u - \\ - 3\eta_x)F_{u_x} - 3\eta F_u - 3\tau F_t - (3\rho + x\dot{\tau})F_x = 0. \end{aligned} \tag{4}$$

Here the dot over a symbol stands for the time derivative.

The equations (2) contain arbitrary functions of one variable. Therefore we utilize the Lie–Ovsyannikov method [2, 3] of group classification of differential equations. We consider in more detail first and sixth equations (2).

In first equation (2)

$$F = u_x^3 - 3u_xu_{xx} + x^{-2}u_x\tilde{F}(\omega), \quad \omega = x(u_x^{-1}u_{xx} - u_x).$$

From the equation (4) we find that the functions τ, ρ, η in the operator (3) and the function $\tilde{F}$ satisfy following system of equations:

$$\begin{aligned}
&\eta_{uuu} - \eta_u = 0;\\
&x^{-1}[\eta_{uu} - \eta_u]\tilde{F}_\omega + 3x^{-1}[\eta_{uu} - \eta_u]\omega + 3[\eta_{xuu} - \eta_{xu}] = 0;\\
&x^{-1}[x^{-2}\rho\omega - 2\eta_x + 2\eta_{xu}]\tilde{F}_\omega - 2x^{-3}\rho\tilde{F} =\\
&\quad = 3x^{-1}(\eta_x - \eta_{xu})\omega + 3(\eta_{xx} - \eta_{xxu}) - \tfrac{1}{3}x\ddot{\tau} - \dot{\rho};\\
&[x^{-2}\eta_x\omega - x^{-1}\eta_{xx}]\tilde{F}_\omega - x^{-2}\eta_x\tilde{F} = \eta_{xxx} - \eta_t. \qquad (5)
\end{aligned}$$

If $\tilde{F}$ is arbitrary function, then from (5) we obtain that corresponding operator v has following form:

$$\mathrm{v} = (C_1 t + C_2)\partial_t + \tfrac{1}{3}C_1 x\partial_x + C_3 e^u\partial_u + C_4\partial_u,$$

where $C_1, C_2, C_3, C_4 \in \mathbb{R}$.

The corresponding invariance algebra is isomorphic to solvable Lie algebra $2A_{2.2}$: $e_1 = -t\partial_t \frac{1}{3}x\partial_x$, $e_2 = \partial_t$, $e_3 = \partial_u$, $e_4 = e^u\partial_u$.

The extension of symmetry properties of first equation (2) takes place in two cases:

$$\begin{aligned}
(1)\quad &\tilde{F} = \lambda\omega^2 \ \left(\lambda \neq 0, -\tfrac{3}{2}\right): \text{ here}\\
&\tau = C_1 t + C_2, \quad \rho = C_3, \quad \eta = C_4 e^u + C_5,\\
&C_i \in \mathbb{R} \ \ (i = 1, 2, \ldots, 5);\\
(2)\quad &\tilde{F} = -\tfrac{3}{2}\omega^2: \text{ here}\\
&\tau = C_1 t + C_2, \quad \rho = C_3, \quad \eta = C_4 e^u + C_5 e^{-u} + C_6,\\
&C_i \in \mathbb{R} \ \ (i = 1, 2, \ldots, 6).
\end{aligned}$$

In sixth equation (2) $F = -u_x + e^{-x}\tilde{F}(\omega)$, $\omega = e^x(u_x + u_{xx})$, $\tilde{F}_{\omega\omega} \neq 0$, and from the equation (4) we find that the functions τ, ρ, η, $\tilde{F}$ satisfy following system of equations:

$$\begin{aligned}
&\eta_{uuu} = 0; \quad \eta_{uu}(1 - \tilde{F}_\omega) - \eta_{xuu} = 0;\\
&(\dot{\tau} + 6\eta_{xu})\tilde{F}_\omega = -9e^{-x}\omega\eta_{uu} - 3\dot{\rho} - x\ddot{\tau} + 9\eta_{xu} - 9\eta_{xxu};\\
&[e^x(2\dot{\tau} - 3\eta_u - 3\rho - x\dot{\tau})\omega - 3\eta_x - 3\eta_{xx}]\tilde{F}_\omega +\\
&\quad + e^{-x}(3\eta_u - 3\dot{\tau} + 3\rho + x\dot{\tau})\tilde{F} - 9e^{-x}\omega\eta_{xu} +\\
&\quad + 3\eta_t - 3\eta_x - 3\eta_{xxx} = 0. \qquad (6)
\end{aligned}$$

From second equation (6) we obtain the condition

$$\eta_{uu}\tilde{F}_{\omega\omega} = 0,$$

consequently $\eta_{uu} = 0$.

From third equation (6) we obtain the condition

$$(\dot{\tau} + 6\eta_{xu})\tilde{F}_{\omega\omega} = 0,$$

consequently

$$\dot{\tau} + 6\eta_{xu} = 0, \quad -3\dot{\rho} - x\ddot{\tau} + 2\dot{\tau} + 9\eta_{xu} - 9\eta_{xxu} = 0.$$

From obtained relations we obtain following values of the functions τ, ρ, η :

$$\tau = C_1 t + C_2, \quad \rho = \tfrac{1}{6}C_1 t + C_3,$$
$$\eta = \left[-\tfrac{1}{6}C_1 x + \gamma(t)\right] u + \beta(t,x), \quad C_1, C_2, C_3 \in \mathbb{R}.$$

Fourth equation (6) transforms into following system:

$$\begin{aligned}
&\tfrac{1}{2}C_1\tilde{F}_\omega = -3\dot{\gamma} + \tfrac{1}{2}C_1,\\
&\left[e^{-x}\left(2C_1 - \tfrac{1}{2}xC_1 - 3\gamma - \tfrac{1}{2}C_1 t - 3C_3\right)\omega - 3\beta_x - 3\beta_{xx}\right]\tilde{F}_\omega +\\
&\quad + e^{-x}\left(-3C_1 + \tfrac{1}{2}xC_1 + \tfrac{1}{2}C_1 t + 3C_3 + 3\gamma\right)\tilde{F} =\\
&\quad = -\tfrac{3}{2}e^{-x}C_1\omega - 3\beta_t - 3\beta_x + 3\beta_{xxx}.
\end{aligned} \tag{7}$$

From first equation (7) we obtain condition

$$C_1\tilde{F}_{\omega\omega} = 0,$$

consequently $C_1 = 0$, $\gamma = C_4$, $C_4 \in \mathbb{R}$. Second equation (7) reduces to equation

$$[(C_3 + C_4)\omega + \beta_x + \beta_{xx}]\tilde{F}_\omega - (C_3 + C_4)\tilde{F} = e^x(\beta_t + \beta_x - \beta_{xxx}),$$

from which we obtain condition

$$[(C_3 + C_4)\omega + \beta_x + \beta_{xx}]\tilde{F}_{\omega\omega} = 0.$$

Consequently, $C_3 + C_4 = 0$, $\beta_x + \beta_{xx} = 0$, $\beta_t + \beta_x - \beta_{xxx} = 0$, and the operator v (3) has following form:

$$\mathrm{v} = C_2\partial_t + C_3\partial_x + (-C_3 u + C_5 + e^{-x}C_6)\partial_u,$$

$C_2, C_3, C_4, C_5 \in \mathbb{R}$. The corresponding invariance algebra is isomorphic to solvable Lie algebra $A_{2.2} \oplus 2A_1$: $e_1 = \partial_x - u\partial_u$, $e_2 = \partial_u$, $e_3 = \partial_t$, $e_4 = e^{-x}\partial_u$.

So we have obtained that sixth equation (2) does not suppose the extension of symmetry properties. Analogous results we have obtained for 3, 4, 5 and 8 equations (2). The rest equations (2) suppose the extensions of symmetry properties. We give these equations with the corresponding invariance algebras:

1) $u_t = u_{xxx} + u_x^3 - 3u_x u_{xx} + \lambda u_x (u_x^{-1} u_{xx} - u_x)^2, \lambda \neq 0, -\frac{3}{2}$:
$\langle t\partial_t + \frac{1}{3}x\partial_x, \partial_t, \partial_x, e^u\partial_u, \partial_u\rangle$;

2) $u_t = u_{xxx} - \frac{3}{2}u_x^{-1}u_{xx}^2 - \frac{1}{2}u_x^3$:
$\langle t\partial_t + \frac{1}{3}x\partial_x, \partial_t, \partial_x, e^u\partial_u, e^{-u}\partial_u, \partial_u\rangle$;

3) $u_t = u_{xxx} + \lambda^{-1}x + m\ln|u_{xx}| - \beta u_x, \lambda \cdot m \neq 0, \beta \in \mathbb{R}$:
$\langle t\partial_t + \left(\frac{1}{3}x + \frac{2}{3}\beta t\right)\partial_x + \left[u + \frac{1}{3}t\left(\lambda^{-1}x + \frac{1}{2}\beta\lambda^{-1}t + m\right)\right]\partial_u,$
$\partial_x + \lambda^{-1}t\partial_u, (x - \beta t)\partial_u, \partial_t, \partial_u\rangle$;

4) $u_t = u_{xxx} + \lambda^{-1}x - \beta u_x + m|u_{xx}|^p, \lambda m \neq 0, p \neq 0, 1, \beta \in \mathbb{R}$:
$\langle t\partial_t + \left(\frac{1}{3}x + \frac{2}{3}\beta t\right)\partial_x + \left[\frac{2p-3}{3(p-1)}u + \frac{2p-1}{3\lambda(p-1)}tx + \right.$
$\left. + \frac{\beta}{6\lambda(1-p)}t^2\right]\partial_u, \quad \partial_x + \lambda^{-1}t\partial_u, (x - \beta t)\partial_u, \partial_t, \partial_u\rangle$;

5) $u_t = u_{xxx} + \lambda^{-1}x - \beta u_x + me^{nu_{xx}}, \lambda mn \neq 0, \beta \in \mathbb{R}$:
$\langle t\partial_t + \left(\frac{1}{3}x + \frac{2}{3}\beta t\right)\partial_x + \left[\frac{2}{3}u + \frac{1}{6n}x^2 + \left(\frac{2}{3\lambda} - \frac{\beta}{3n}\right)tx + \right.$
$\left. + \frac{\beta^2}{6n}t^2\right]\partial_u, \quad \partial_x + \lambda^{-1}t\partial_u, (x - \beta t)\partial_u, \partial_t, \partial_u\rangle$.

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