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Q-conditional symmetry as a source of exact solutions of generalized Burgers equation

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В роботі проаналізовано умовну симетрію гіперболічного узагальнення рівняння Бюргерса. Використання узагальненої симетрії дозволило одержати нові точні розв'язки, що описують різноманітні хвильові структури.

In this paper the conditional symmetry to a hyperbolic generalization of Burgers equation is studied. Employment of the generalized symmetry enabled to obtain new exact solutions, describing the evolution of various wave patterns.

1. Introduction. In last few years we dealt with different methods of obtaining analytical solutions of nonlinear PDE's that are not completely integrable [1–3], paying special attention to the following generalization of Burgers equation (GBE) [4]:

$$\tau u_{tt} - \kappa u_{xx} + A u u_x + B u_t + H u_x = f(u).$$

Here and henceforth lower indices mean partial derivatives with respect to corresponding variables. The classical symmetry methods [5, 6] are very popular in obtaining exact solutions to nonlinear PDEs, but for non-zero constants classical symmetries of GBE are reduced to the generators of translations ∂_t and ∂_x, giving rise to travelling-wave solutions. So in this study we proceed further on and look for solutions which cannot be described in terms of travelling waves. To do this we employ so-called Q-conditional symmetry methods [7–11].

Let us consider equation

$$F(x, t, u, u_x, u_t, \dots) = 0. \tag{1}$$

It is of common knowledge, that within the classical Lie algorithm [5, 6] we look for the operator

$$Q = \xi_1 \partial_x + \xi_2 \partial_t + \phi \partial_u, \tag{2}$$

such that

$$\hat{Q} F|_{F(x,t,u,u_x,u_t,\dots)=0} = 0, \tag{3}$$

where $\hat{Q}$ denotes the proper prolongation of Q.

Looking for the Q-conditional symmetry, we pose the additional condition:

$$Q(u(x,t) - u) = 0 = \xi_1 \, u_x + \xi_2 \, u_t - \phi \tag{4}$$

and solve the equations:

$$\hat{Q} F|_{F(x,t,u,u_x,u_t,\dots)=0,\, Q=0,\, Q_1=0,\, Q_2=0,\dots} = 0, \tag{5}$$

where $Q = 0$, $Q_1 u = 0$, $Q_2 u = 0$, $\dots$ denote equation (4) and its differential consequences of the corresponding orders. The additional condition allows finding much wider classes of reductions to GBE.

2. Brief overview of the cases. We deal with the equation:

$$\begin{aligned} &\tau u_{tt} - \kappa u_{xx} + A u u_x + B u_t + H u_x = \\ &\quad = f(u) = \lambda_0 + \lambda_1 u + \lambda_2 u^2 + \lambda_3 u^3. \end{aligned} \tag{6}$$

To examine the conditional symmetry of (6), we consider it together with the equation (4). Here ξ_1, ξ_2, ϕ depend on the variables x, t, u. We assume that τ, κ, A, λ_3 are non-zero and examine symmetries of the system (4), (6). Let us notice, that whenever ξ_1 (or ξ_2) is non-zero, it can be scaled to 1.

Case I: $\xi_2 = 1$, $\xi_1 \neq 0$. Using (6), (4) and its differential consequences we can eliminate u_t and all the second derivatives of $u(t, x)$[1]. After computing the prolongation of Q and performing the splitting procedure we obtain four determining equations:

$$e1 = 3 f(u)(\kappa - \tau \xi_1^2) \xi_{1u} + \tau \phi^2 (2 \tau \xi_1 {\xi_{1u}}^2 + \kappa \xi_{1uu} - \tau \xi_1^2 \xi_{1uu}) -$$

[1]Let us notice that in case when $\kappa - \tau \xi_1^2 = 0$ the above procedure fails. This situation is thoroughly examined in section 3.

$$- 2\kappa\tau\xi_{1u}\phi_t + 2\tau^2\xi_1^2\xi_{1u}\phi_t + B\kappa\xi_{1t} - 2H\tau\xi_1\xi_{1t} - 2Au\tau\xi_1(\xi_{1t} +$$
$$+ B\tau\xi_1^2\xi_{1t} + 2\kappa\tau\phi_u\xi_{1t} + 2\tau^2\xi_1^2\phi_u\xi_{1t} + 2\tau^2\xi_1{\xi_{1t}}^2 + 2\kappa\tau\xi_1\phi_{tu} -$$
$$- 2\tau^2\xi_1^3\phi_{tu} + \phi(-2\tau^2\xi_1^3\phi_{uu} + 2\tau\xi_1(\kappa\phi_{uu} - \xi_{1u}(H + Au -$$
$$- 2\tau\xi_{1t})) - \kappa(A + 2B\xi_{1u} - 2\tau\xi_{1tu}) + \tau\xi_1^2(A + 4(B + \tau\phi_u)\xi_{1u} -$$
$$- 2\tau\xi_{1tu})) + \kappa\tau\xi_{1tt} - \tau^2\xi_1^2\xi_{1tt} - H\kappa\xi_{1x} - Au\kappa\xi_{1x} + 2B\kappa\xi_1\xi_{1x} -$$
$$- H\tau\xi_1^2\xi_{1x} - Au\tau\xi_1^2\xi_{1x} + 4\kappa\tau\xi_1\phi_u\xi_{1x} - 2\kappa\tau\xi_1\xi_{1x}^2 + 2\kappa^2\phi_{xu} -$$
$$- 2\kappa\tau\xi_1^2\phi_{xu} - \kappa^2\xi_{1xx} + \kappa\tau\xi_1^2\xi_{1xx} = 0,$$
$$e2 = -\tau\phi^2(2\xi_1(B + \tau\phi_{1u})\xi_{1u} + \kappa\phi_{1uu} - \tau\xi_1^2\phi_{1uu}) - B\kappa\phi_t +$$
$$+ B\tau\xi_1^2\phi_t - 2\tau^2\xi_1\phi_t(\xi_{1t} - \kappa\tau(\phi_{tt} + \tau^2\xi_1^2(\phi_{tt} - H\kappa\phi_x -$$
$$- Au\kappa\phi_x + H\tau\xi_1^2\phi_x + Au\tau\xi_1^2\phi_x + 2\kappa\tau(\xi_{1t}\phi_x - 2\kappa\tau\phi_t\xi_{1x} +$$
$$+ 2\kappa\tau\xi_1\phi_x\xi_{1x} + f(u)((-\kappa + \tau\xi_1^2)(\phi_u + 2(\tau\xi_1(\xi_{1t} + \kappa\xi_{1x})) +$$
$$+ \phi((\kappa - \tau\xi_1^2)f'(u) - 2(-(\tau f(u)\xi_1\xi_{1u}) + \tau\xi_1(\tau\xi_{1u}\phi_t + (B +$$
$$+ \tau\phi_u\xi_{1t} - \tau^2\xi_1^2\phi_{xu} + \kappa(\tau\phi_{xu} - \tau\xi_{1u}\phi_x + (B + \tau\phi_u\xi_{1x}))) +$$
$$+ \kappa^2\phi_{xx} - \kappa\tau\xi_1^2\phi_{xx} = 0,$$
$$e3 = -4\tau^2\phi\xi_1^2\xi_{1u}^2 - 2\xi_{1u}((H + Au)\kappa + \tau\xi_1^3(B + \tau\phi_u -$$
$$- \tau\xi_1^2(H + Au - 2\tau\xi_{1t} - \kappa\xi_1(B + \tau(\phi_u - 2\tau\xi_{1x})) + (\kappa -$$
$$- \tau\xi_1^2)((\kappa - \tau\xi_1^2)\phi_{uu} - 2(\tau\phi\xi_1\xi_{1uu} + \tau\xi_1(\xi_{1xt} + \kappa\xi_{1xt})) = 0,$$
$$e4 = 2\tau\xi_1\xi_{1u}^2 + \kappa\xi_{1uu} - \tau\xi_1^2\xi_{1uu} = 0.$$

Since the first three equations are very complicated, we start our analysis from the last and the simplest one.

Case Ii: $\xi_{1u} \neq 0$. Introducing the new function $\xi_{1u} = \Psi(\xi_1)$ and consequently $\xi_{1uu} = \Psi'(\xi_1)\Psi(\xi_1)$, we obtain the integrable equation:

$$2\tau\xi_1\Psi(\xi_1)^2 + \kappa\Psi'(\xi_1)\Psi(\xi_1) - \tau\xi_1^2\Psi'(\xi_1)\Psi(\xi_1) = 0. \tag{7}$$

Equation (7) is satisfied by the following function:

$$\xi_1 = \sqrt{\frac{\kappa}{\tau}}\tanh\left(\sqrt{\kappa\tau}c_1(x,t)(u + c_2(x,t))\right). \tag{8}$$

Function ϕ can be calculated from e3=0. Unfortunately e1=0 gives us either $\tau = 0$ or $\kappa = 0$ or $\lambda_3 = \lambda_2 = A = 0$. So all the possibilities are in contradiction with our assumptions.

Case Iii: $\xi_{1u} = 0$. To solve $e4 = 0$ we can also put $\xi_{1u} = 0$. Then the third determining equation ($e3 = 0$) takes the form: $(\kappa - \tau\xi_1^2)\phi_{uu} = 0$. Since $(\kappa - \tau\xi_1^2) \neq 0$, then $\phi_{uu} = 0$. In other words, $\phi = a(x,t)u + b(x,t)$ and the remaining determining equations are as follows:

$$
\begin{aligned}
e1 = {} & -(A\kappa(ua+b)) + A\tau(ua+b)\xi_1^2 + 2\kappa\tau\xi_1 a_t - 2\tau^2\xi_1^3 a_t + \\
& + B\kappa\xi_{1t} + 2\kappa\tau a\xi_{1t} - 2H\tau\xi_1(\xi_{1t} - 2Au\tau\xi_1\xi_{1t} + B\tau\xi_1^2\xi_{1t} + \\
& + 2\tau^2 a\xi_1^2\xi_{1t} + 2\tau^2\xi_1\xi_{1t}^2 + \kappa\tau\xi_{1tt} - \tau^2\xi_1^2\xi_{1tt} + 2\kappa^2 a_x - \\
& - 2\kappa\tau\xi_1^2 a_x - H\kappa\xi_{1x} - Au\kappa\xi_{1x} + 2B\kappa\xi_1\xi_{1x} + 4\kappa\tau a\xi_1\xi_{1x} - \\
& - H\tau\xi_1^2\xi_{1x} - Au\tau\xi_1^2\xi_{1x} - 2\kappa\tau\xi_1\xi_{1x}^2 - \kappa^2\xi_{1xx} + \kappa\tau\xi_1^2\xi_{1xx} = 0, \\
e2 = {} & -\kappa a f(u) + \tau a f(u)\xi_1^2 + \kappa(ua+b)f'(u) - \tau(ua+b)\xi_1^2 f'(u) - \\
& - 2\kappa\tau(ua+b)a_t + 2\tau^2(ua+b){\xi_1}^2 a_t - B\kappa(ua_t + b_t) + \\
& + B\tau\xi_1^2(ua_t + b_t) - 2B\tau(ua+b)\xi_1\xi_{1t} - 2\tau^2 a(ua+b)\xi_1\xi_{1t} + \\
& + 2\tau f(u)\xi_1\xi_{1t} - 2\tau^2\xi_1(ua_t + b_t)\xi_{1t} - \kappa\tau(ua_{tt} + b_{tt}) + \\
& + \tau^2\xi_1^2(ua_{tt} + b_{tt}) - H\kappa(ua_x + b_x) - Au\kappa(ua_x + b_x) + \\
& + H\tau\xi_1^2(ua_x + b_x) + Au\tau\xi_1^2(ua_x + b_x) + 2\kappa\tau(\xi_{1t}(ua_x + b_x) - \\
& - 2B\kappa(ua+b)\xi_{1x} - 2\kappa\tau a(ua+b)\xi_{1x} + 2\kappa f(u)\xi_{1x} - \\
& - 2\kappa\tau(ua_t + b_t)\xi_{1x} + 2\kappa\tau\xi_1(ua_x + b_x)\xi_{1x} + \kappa^2(ua_{xx} + b_{xx}) - \\
& - \kappa\tau\xi_1^2(ua_{xx} + b_{xx}).
\end{aligned}
$$

Taking into account that $f(u) = \lambda_3 u^3 + \lambda_2 u^2 + \lambda_1 u + \lambda_0$, we collect the terms at different powers of u. Nullifying them, we obtain (among others) the following equations:

$$
\begin{aligned}
& a(x,t)(\kappa - \tau\xi_1)^2 + 2\tau\xi_1\xi_{1t} + \kappa\xi_{1x} + \tau\xi_1^2\xi_{1x} = 0, \\
& a(x,t)(\kappa - \tau{\xi_1}^2) + \tau\xi_1\xi_{1t} + \kappa\xi_{1x} = 0.
\end{aligned}
$$

The above system can be presented in the following equivalent form:

$$
a(x,t) = -\xi_{1x}, \quad \xi_1\xi_{1x} = -\xi_{1t}.
$$

The second equation is a model equation of gas dynamics. Its general solution is as follows [12]:

$$
x = t\xi_1 + \Phi(\xi_1),
$$

where $\Phi(\cdot)$ is an arbitrary function.

For $\Phi(\xi_1) = 0$

$$\xi_1(x,t) = x/t, \quad a(x,t) = -1/t$$

and remaining equations of this group are satisfied providing the following conditions are fulfilled:

$$b(x,t) = \frac{-\lambda_2}{3t\lambda_3}, \quad B = 0, \quad \lambda_0 = \frac{\lambda_2^3}{27\lambda_3^2},$$
$$A = \frac{-3H\lambda_3}{\lambda_2}, \quad \lambda_1 = \frac{\lambda_2^2}{3\lambda_3}.$$

Using the ansatz

$$u(x,t) = \frac{-\lambda_2 x + 3\lambda_3 R(\frac{t}{x})}{3\lambda_3 x},$$

we get in this case the reduced equation

$$\lambda_2(\kappa\xi_1^2 - \tau)R''(\xi_1) + 3H\lambda_3\xi_1 R'(\xi_1)R(\xi_1) + \lambda_3\lambda_2 R(\xi_1)^3 + \\ + 4\kappa\lambda_2\xi_1 R'(\xi_1) + 3H\lambda_3 R(\xi_1)^2 + 2\kappa\lambda_2 R(\xi_1) = 0. \quad (9)$$

Equation (9) is a nonlinear non-autonomous ODE which is, generally speaking, nonintegrable. A particular solution can be obtained using the ansatz-based method [1], slightly modified for the purposes of non-autonomous case. Thus, we represent the solution in the form

$$R(\xi_1) = \frac{p}{1 + q(\xi_1)\exp(\alpha(\xi_1))}. \quad (10)$$

The numerator of equation (9) contains then different powers of $\mathrm{Exp}\,(\alpha(\xi_1))$. Splitting the equation we can calculate the constants:

$$p = \pm\sqrt{2\kappa/\lambda_3}, \quad H = \mp 2/3\sqrt{2\kappa/\lambda_3}\lambda_2,$$

and the function $q(\xi_1)$. The simplified solutions are as follows

$$R(\xi_1) = \pm\frac{\sqrt{2\kappa}F(\xi_1)}{\sqrt{\lambda_3}(F(\xi_1) + \sqrt{\kappa\xi_1^2 - \tau})},$$
$$R(\xi_1) = \pm\frac{\sqrt{2\kappa}}{\sqrt{\lambda_3}(1 + F(\xi_1)\sqrt{\kappa\xi_1^2 - \tau})},$$

where $F(\xi_1) = \mathrm{arctanh}\,(\sqrt{\kappa/\tau}\xi_1)$.

Case II: $\xi_2 = 1$, $\xi_1 = 0$. With these assumptions determining equations are as follows:

$$
\begin{aligned}
&e1 = \phi_{uu} = 0,\\
&e2 = f(u)\phi_u + B\phi_t - \phi(f'(u) - 2\tau\phi_{tu}) + \tau\phi_{tt} +\\
&\qquad + H\phi_x + Au\phi_x - \kappa\phi_{xx} = 0,\\
&e3 = A\phi - 2\kappa\phi_{xu} = 0.
\end{aligned}
$$

From the firs equation we get the representation for u:

$$\phi(x,t,u) = a(x,t)u + b(x,t).$$

On virtue of this, the last equation becomes as follows:

$$A[b(x,t) + a(x,t)u] - 2\kappa a_x(x,t) = 0.$$

Since $a(x,t)$ and $b(x,t)$ are independent of u and $A \neq 0$ we must put $a(x,t) = 0$ and as a consequence $b(x,t) = 0$. The condition (4) gives $u_t(x,t) = 0$ and then $u = u(x)$.

Case III: $\xi_2 = 0$, $\xi_1 = 1$. We get in this case three determining equations:

$$
\begin{aligned}
&e1 = \phi_{uu} = 0,\\
&e2 = \phi_{ut} = 0,\\
&e3 = A\phi^2 + f(u)\phi_u + B\phi_t + \tau\phi_{tt} + H\phi_x + Au\phi_x -\\
&\qquad - \phi(f'(u) + 2\kappa\phi_{xu}) - \kappa\phi_{xx} = 0.
\end{aligned}
$$

From the first two equations we obtain that

$$\phi(x,t,u) = a(x)u + b(x,t).$$

The last equation then takes the form:

$$
\begin{aligned}
&af(u) + A(b+au)^2 - 2\kappa a_x(b+au) + Bb_t + H(a_x u + b_x) + \\
&\quad + Au(a_x u + b_x) - (b+au)f'(u) + \tau b_{tt} - \kappa(a_{xx}u + b_{xx}) = 0.
\end{aligned}
\tag{11}
$$

Expression (11) can be splitted by powers of u. The coefficient of u^3 is $-2\lambda_3 a(x)$ while that of u^2 is $-\lambda_2 a + Aa^2 - 3\lambda_3 b + Aa'$. If we put $a(x) = 0$, then this implies $b(x,t) = 0$ and we again encounter the situation where

u is function of one independent variable. So we rather put $\lambda_3 = 0$, $\lambda_2 \neq 0$. Then $a(x)$ can be easily calculated:

$$a(x) = \frac{\lambda_2 e^{\frac{x\lambda_2}{A}}}{A e^{\frac{x\lambda_2}{A}} - e^{\lambda_2 c_1}}. \tag{12}$$

Nullifying the coefficient of u in (11), we obtain a first order linear equation, which gives us the form of $b(x,t)$:

$$b(x,t) = \frac{e^{\frac{x\lambda_2}{A}}(\lambda_2 e^{\lambda_2 c_1}(-AH + \kappa\lambda_2) + A^2 e^{\frac{x\lambda_2}{A}} c_2(t))}{A^2(e^{\lambda_2 c_1} - A e^{\frac{x\lambda_2}{A}})^2}. \tag{13}$$

Substituting (12), (13) into (11) and equating to zero the coefficient of u^0 (the free term), we obtain a very complicated expression which determines an unknown function $c_2(t)$. To our luck, this expression can be splitted as well. This time we equate to zero coefficients of $e^{3c1\lambda_2}$, $e^{3x\lambda_2/A}$, $e^{2c1\lambda_2 + x\lambda_2/A}$ and $e^{(c1+2x/A)\lambda_2}$ (equations obtained this way we denote as (eqn1)–(eqn4)).

Solving (eqn1) we get

$$\lambda_0 = \frac{1}{A^4}(AH - \kappa\lambda_2)(A^2\lambda_1 - AH\lambda_2 + \kappa\lambda_2^2). \tag{14}$$

Adding (eqn2) to (eqn3), we obtain the expression

$$A\left[H\lambda_2 - c_2(t)\right]\left[\lambda_2\left(AH - \kappa\lambda_2\right) - Ac_2(t)\right] = 0.$$

Thus, $c_2(t)$ must be constant. Under this assumption we finally solve equations (eqn3)–(eqn4) and conclude that

$$c_2 = \lambda_2(H - \kappa\lambda_2/A).$$

Now equation (4) takes on the form

$$u_x = \frac{\lambda_2 e^{\frac{x\lambda_2}{A}}}{A e^{\frac{x\lambda_2}{A}} - e^{\lambda_2 c_1}} u + \frac{e^{\frac{x\lambda_2}{A}}\lambda_2(-AH + \kappa\lambda_2)}{A^2(e^{\lambda_2 c_1} - A e^{\frac{x\lambda_2}{A}})}. \tag{15}$$

Solving (15), we obtain:

$$u = \frac{1}{A^2}\left[-AH + k\lambda_2 + A^2\left(e^{c_1\lambda_2} - A e^{\frac{x\lambda_2}{A}}\right) c_3(t)\right]. \tag{16}$$

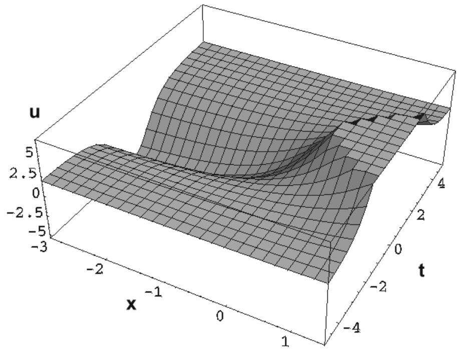

Fig. 1. The example solution of the form (16), $A = \lambda_2 = c = \tau = \kappa = 1$, $H = -1$, $c_4 = c_5 = 0$.

Finally, inserting (16) into (6), we obtain determining equation for unknown function $c_3(t)$:

$$A^2(\tau c_3''(t) + Bc_3'(t)) - A^2 c\lambda_2 c_3(t)^2 - \\ - (A^2\lambda_1 - 2AH\lambda_2 + 2\kappa\lambda_2^2)c_3(t) = 0, \tag{17}$$

where $c = e^{c_1\lambda_2}$. Some special solutions to this equation can be easily found. Let us introduce the new variable $g(c_3) = c_3'(t)$. Then $c_3''(t) = g'g$ and (17) becomes as follows:

$$A^2(\tau g'(c_3)g(c_3) + Bg(c_3)) - A^2 c\lambda_2 c_3^2 - \\ - (A^2\lambda_1 - 2AH\lambda_2 + 2\kappa\lambda_2^2)c_3 = 0.$$

For $B = 0$ it can be solved just by two integrations:

$$\int \tau g'(c_3)g(c_3)dg - A^2 c\lambda_2 c_3^3 - (A^2\lambda_1 - 2AH\lambda_2 + 2\kappa\lambda_2^2)c_3 dc_3 = \\ = \tfrac{1}{2}g(c_3)^2 - \tfrac{1}{3}A^2 c\lambda_2 c_3^2 - \tfrac{1}{2}(A^2\lambda_1 - 2AH\lambda_2 + 2\kappa\lambda_2^2)c_3^2 = c_4. \tag{18}$$

Returning back to the variable $c_3(t)$ we obtain:

$$c_3'(t) = \pm\sqrt{a_3 c_3^3 + a_2 c_3^2 + a_0}, \tag{19}$$

where $a_3 = 2c\lambda_2/(3\tau), a_2 = (A^2\lambda_1 - 2AH\lambda_2 + 2\kappa\lambda_2^2))/(A^2\tau)$, $a_0 = 2c_4/(A^2\tau)$. After integration the function $c_3(t)$ can be written in terms of elliptic functions. In case when $c_4 = 0$ we get:

$$t + c_5 = \frac{-2}{\sqrt{a_2}} \operatorname{arctanh} \sqrt{\frac{a_2 + a_3 c_3(t)}{a_2}}.$$

Under additional conditions $A = \lambda_2 = c = \tau = \kappa = 1$, $H = -1$, $c_5 = 0$ the solution of the GBE takes the form:

$$u(x,t) = 2 + (\exp(x) - 1)\left(\tfrac{15}{2}\operatorname{sech}^2\left(\tfrac{\sqrt{5}}{2}t\right)\right), \tag{20}$$

and it is illustrated in Fig. 1.

3. Solutions of the case $\xi_2 = 1$, $\kappa - \tau\xi_1^2 = 0$. Let us consider system (6), (4) in the case then $\xi_2 = 1$, $\xi_1 = \epsilon\sqrt{\kappa/\tau}$, $\epsilon = \pm 1$. After using the condition (4), together with the proper differential consequences, equation (6) becomes as follows:

$$\begin{aligned}&[H - B\epsilon\sqrt{\kappa/\tau} + Au - 2\epsilon\sqrt{\kappa\tau}\phi_u]u_x + \\ &\quad + (B + \tau\phi_u)\phi + \tau - \epsilon\sqrt{\kappa\tau}\phi_x - f(u) = 0.\end{aligned} \tag{21}$$

Next we denote the coefficient of u_x by $k1$ and the rest of (21) by $k2$. Solving the system $k1 = k2 = 0$ we get:

$$\phi = \phi(u) = \frac{1}{2\epsilon\sqrt{\kappa\tau}}(A/2\sqrt{\tau}u^2 + u(H\sqrt{\tau} - \epsilon B\sqrt{\kappa})) + c$$

with an extra conditions

$$\lambda_0 = 1/2c(B + \epsilon H\sqrt{\kappa/\tau}), \quad \lambda_1 = \frac{-B^2\kappa + H^2\tau + 2Ac\epsilon\sqrt{\kappa}\tau^{3/2}}{4\kappa\tau},$$
$$\lambda_2 = \frac{A(3H - \epsilon B\sqrt{\kappa/\tau})}{8\kappa}, \qquad \lambda_3 = \frac{A^2}{8\kappa}.$$

To calculate the solution $u(x,t)$, let us return to equation (4):

$$u_t + \sqrt{\kappa/\tau}u_x = \phi = a_0 + a_1 u + a_2 u^2,$$

where $a_0 = c$, $a_1 = (H\sqrt{\tau} - \epsilon B\sqrt{\kappa})/(2\epsilon\sqrt{\kappa\tau})$, $a_2 = A/(4\epsilon\sqrt{\kappa\tau})$. Writing it in the characteristic form

$$\frac{dx}{\sqrt{\kappa/\tau}} = \frac{dt}{1} = \frac{du}{a_0 + a_1 u + a_2 u^2},$$

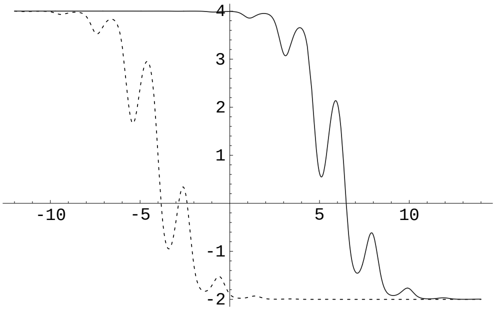

Fig. 2. Temporal evolution of solution described by formula (25) in case when $\Gamma(\omega) = \sin[2.25\omega]$.

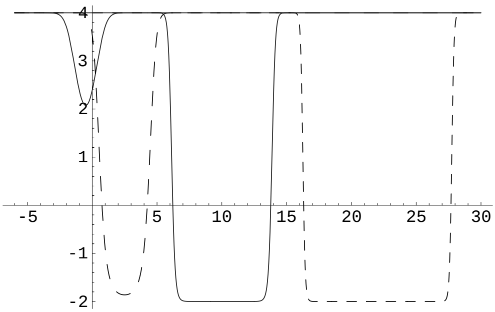

Fig. 3. Temporal evolution of solution described by formula (25) in case when $\Gamma(\omega) = -\omega^2$.

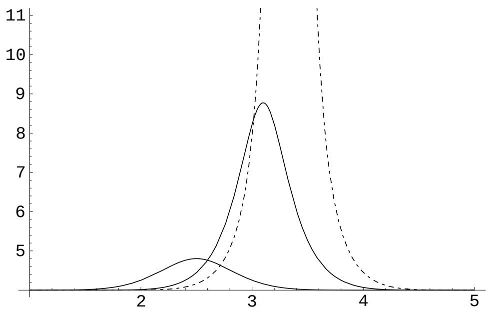

Fig. 4. A blow-up regime described by the formula (26) in case when $\Gamma(\omega) = -3.75\omega^2 + 5$.

we easily get the general solution

$$x + \Psi(\sqrt{\kappa/\tau}t - x) = \sqrt{\frac{\kappa}{\tau}} \int \frac{du}{a_0 + a_1 u + a_2 u^2}, \tag{22}$$

where $\Psi(\cdot)$ is and arbitrary function. Below we present some special solutions of (22).

For $A = \tau = \kappa = \epsilon = B = 1$, $H = 0$ equation (4) takes the form

$$u_t + u_x = \frac{1}{4}\left(u^2 - 2u + 4c\right). \tag{23}$$

General solution of this equation depends on whether $\Delta = 1 - 4c$ is positive or not. For $c < 1/4$ solution of (23) is as follows:

$$u(t,x) = \frac{u_2 G(\omega) e^{x\frac{\sqrt{\Delta}}{2}} - u_1}{G(\omega) e^{x\frac{\sqrt{\Delta}}{2}} - 1}, \tag{24}$$

where $u_1 = 1 + \sqrt{\Delta}$, $u_2 = 1 - \sqrt{\Delta}$, $G(\cdot)$ is an arbitrary function of $\omega = x - t$. Putting $c = -2$ and $G(\omega) = -e^{\Gamma(\omega)}$ we obtain the formula

$$u(x,t) = 2\frac{2 - \exp[3x/2 + \Gamma(\omega)]}{1 + \exp[3x/2 + \Gamma(\omega)]}. \tag{25}$$

For $c = -2$ and $G(\omega) = e^{\Gamma(\omega)}$ we get the solution

$$u(x,t) = 2\frac{\exp[3x/2 + \Gamma(\omega)] + 2}{1 - \exp[3x/2 + \Gamma(\omega)]}. \tag{26}$$

If $c > 1/4$ then solution to (23) is as follows:

$$u(x,t) = 1 + \beta \text{arctg}\left[\frac{\beta x}{4} + G(\omega)\right], \tag{27}$$

where $\beta = \sqrt{|1 - 4c|}$. This solution is always singular.

If $c = 1/4$ then solution to (23) is as follows:

$$u(x,t) = 1 + \frac{1}{G(\omega) - \frac{x}{4}}. \tag{28}$$

This solution is also singular.

Let us give examples of solutions corresponding to formulae (25) and (26). Thus, inserting $\Gamma(\omega) = \sin[2.25\omega]$ into equation (26), we obtain an oscillating kink-like solution, shown in Fig. 2. For $\Gamma(\omega) = -\omega^2$ this solution produces a "dark" soliton with growing "support" (Fig. 3).

In contrast to (25), solution (26) is always singular. For $\Gamma(\omega) = -3.75\omega^2 + 5$ its evolution is shown in Fig. 4, in which we see how an initial localized wave pack grows in amplitude and in a finite time gives rise to a blow-up regime.

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