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Analytical approximate capillary surfaces*

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Analytical approaches to capillary (meniscus) problem in infinite horizontal channel and axisymmetric container are developed. For these cases, finding the menisci reduces to free-boundary problems for specific systems of ordinary differential equations. Their solutions describe capillary curves, resulted from intersection of menisci and (depending on the container type) either cross-section or meridional plane. Further studies on capillary waves require to know analytical approximations in the C^n, $n \geq 3$ metrics. An objective consists of constructing analytical approximate solutions. The paper focuses on limits of applicability of Taylor-polynomial and Padé approximations, which were proposed for this class of capillary problems in 1984 by Barnyak & Timokha.

Розвиваються аналітичні підходи до капілярних (меніск) проблем у нескінченому горизонтальному каналі і осесиметричному контейнері. Для цих випадків знаходження менісків зводиться до задачі з невідомою границею зі спеціальною системою звичайних диференціальних рівнянь. Їхні розв'язки описують капілярні криві, які виникають у перетині менісків та чи поперечного перерізу, чи меридіональної площини (залежно від форми контейнера). Подальші дослідження капілярних хвиль вимагають знання аналітичних наближень у метриці C^n, $n \geq 3$. Метою є побудова відповідних аналітичних наближених розв'язків. Стаття присвячена дослідженню границь застосованості аналітичних наближень Тейлора і Паде, які було запропоновано для цього класу капілярних задач у 1984 році Барняком та Тимохою.

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Introduction

Growing up interest to micro-scale technology, which were extensively developed last decade for getting smart materials and drugs, is a motivation for paying a dedicated insight into capillary meniscus problems whose mathematical formulation was given in breakthrough works of Thomas Young [9]. Various aspects of these problems were studied during the 70-80th inspired by practical interests to spacecraft applications. A detailed review of historical aspects of the capillary problems can be found in [4,7,8], at least, as they stand when these famous books were issued. A task in studying capillary phenomena could consist of examining the liquid sloshing dynamics occurring relative to the capillary meniscus. Standing capillary waves are described by a spectral boundary problem whose properties were studied by Nikolay Kopachevskiy [5,6,8] for both ideal (potential flows) and viscous incompressible liquids. The spectral boundary problem contains spectral parameter in a boundary condition on the capillary surface Σ_0. The boundary condition has surface-dependent coefficients, which are functions of the meniscus solution and its higher (up to third-order) spatial derivatives. To consider and analyse capillary-sloshing problem, one must therefore know either exact (rarely exists) or accurate analytical approximation of the static capillary meniscus surface in the C^n, $n \geq 3$-metrics.

The present paper considers two capillary surface problems for partly-filled infinite channels and axisymmetric reservoirs. Finding the capillary surface (meniscus) reduces to boundary value problems for systems of ordinary differential equations. The ODEs describe capillary lines, which are an intersection of either cross or meridional plane, respectively. We show that the capillary lines are solutions of one-parameter families of the Cauchy problem for the ODEs. Following Barnyak & Timokha [2], we construct the Taylor and Padé approximations of these solutions. Their radii of convergence are estimated. Whereas the capillary lines for channels may be effectively approximated by using both Taylor and Padé approximations, the approximations are less accurate for axisymmetric reservoirs.

1 Capillary surface in infinite channels

Consider the Oz-symmetric and, generally speaking, closed infinite channel (horizontal tube) whose rigid walls are defined by the function $y =$

$\pm f(z)$ as in fig. 1. The tube is partly filled with a liquid whose hydrostatic shape, which is affected by gravity force (parallel to Oz) and surface tension, is bounded with the capillary surface $\Sigma_0 = \{(x, y, z) : -\infty < x < \infty,\ (y, z) \in l_0\}$ and the wetted tank surface $S_0 = \{(x, y, z) : -\infty < x < \infty,\ (y, z) \in l_1\}$. In the cross-section, Σ_0 is fully determined by capillary curve l_0 but S_0 is defined by l_1.

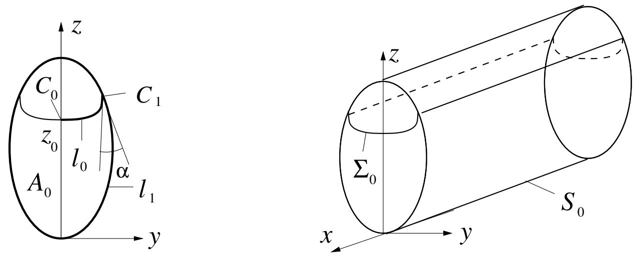

Figure 1. Capillary surface Σ_0 in an infinite closed channel (horizontal tube). Three-dimensional and cross-section views. Capillary curve l_0 is resulted from intersection of Σ_0 and the Oyz plane. Curve l_1 implies the intersection with the wetted tank surface. The present study assumes that the tank surface is defined as the single-valued presentation $y = \pm f(z)$. The gravity acceleration is parallel to the Oz axis.

1.1 Mathematical formulation

The problem on the capillary curve l_0 is furthermore considered in nondimensional statement, which appears after introducing the characteristic length r_0 of the two-dimensional cross-sectional area A_0 (confined by l_0 and l_1, fig. 1). Following [8], we assume that l_0 is defined in the normal parametric form,

$$l_0 = \{(y, z) : y = y(s), z = z(s));\ 0 \le s \le s_1\},$$

where $s = 0$ implies the starting point, $C_0 = (0, z_0)$, on the Oz-axis, but s_1 is the actual length of l_0 and implies the contact point $C_1 = (y(s_1) = f(z(s_1)), z(s_1))$ of l_0 and l_1. These two points C_0 (coordinate z_0) and C_1 are unknown *a priori*.

According to chapter 1 of [8], the capillary curve l_0 is governed by the following system of ODEs

$$y'' = -z' \left(\mathrm{Bo}\, z + c\right), \quad z'' = y' \left(\mathrm{Bo}\, z + c\right), \tag{1}$$

where Bo is the Bond number (Bo= $\rho g r_0^2/T_s$; ρ is the liquid density, g is the gravity acceleration and T_s is the surface tension) and c is an unknown nondimensional parameter. System (1) should be equipped with the initial conditions

$$y(0) = z'(0) = 0, \quad y'(0) = 1 \tag{2}$$

as well as one can suggest

$$z(0) = z_0 > 0, \tag{3}$$

which depends on the unknown value z_0 (vertical coordinate of C_0).

Remark 1.1. *Because l_0 adopts normal parametrisation by s, the system (1) has the integral*

$$y'^2(s) + z'^2(s) \equiv 1. \tag{4}$$

Accounting for the unknown parameter c implies that the Cauchy problem (1)-(3) determines the *two-parameter* family of curves

$$l_0^* = \{(y(s; z_0, c), z(s; z_0, c)) : z_0 > 0, \, s \geq 0\} \tag{5}$$

in the coordinate plane Oyz. Solving the *capillary problem* consists of finding $l_0 \in l_0^*$, which is characterised by

(a) monotonic $z(s)$ on $0 < s < s_1$, where s_1 determines the first intersection point C_1 of l_0 and l_1 ($y(s_1) = f(z(s_1))$) as shown in fig. 1,
(b) the given contact angle α between l_0 and l_1,

$$\text{atan2}(1, y'(z(s_1))) - \text{atan2}(z'(s_1), y'(s_1)) = \alpha, \tag{6}$$

(c) the constant liquid volume (cross-sectional area $|A_0|$)

$$\int_0^{z_0} f(z)\, dz + \int_{z_0}^{s_1} [f(z(s)) - y(s)]\, |z'(s)|\, ds = \tfrac{1}{2}|A_0| = \text{const}. \tag{7}$$

To the authors best knowledge, there are no theorems on solvability of the capillary surface problem (1)-(3) + (a)-(c). However, such a solution (not necessary stable) should exist from a physical point of view, at least, for positive Bo.

Remark 1.2. *When $Bo = 0$ (zero-gravity, weightless conditions), the Cauchy problem (1)-(3) has the exact analytical integral*

$$l_0^* = \{(y(s; z_0, c) = c^{-1}\sin(cs), \; z(s; z_0, c) = c^{-1}(\cos(cs) - 1) + z_0\}, \tag{8}$$

which imply, as expected, a two-parameter class of circles of the radius c^{-1} with the centre $(0, z_0 - c^{-1})$.

1.2 The set l_0^* as an one-parameter family of curves for $\mathrm{Bo} \neq 0$

When $|\mathrm{Bo}| \neq 0$, the following substitution

$$y(s; z_0, c) = \frac{Y(|\mathrm{Bo}|^{1/2}s; \xi)}{|\mathrm{Bo}|^{1/2}}, \quad z(s; z_0, c) = \frac{Z(|\mathrm{Bo}|^{1/2}s; \xi)}{|\mathrm{Bo}|^{1/2}} - \frac{c}{\mathrm{Bo}} \tag{9}$$

redefines l_0^* as the ξ-parametric family of curves

$$L_0^* = \left\{ (Y(S;\xi), Z(S;\xi)) : \xi = z_0|\mathrm{Bo}|^{1/2} + \frac{c\,\mathrm{sgn}(\mathrm{Bo})}{|\mathrm{Bo}|^{-1/2}},\ S = |\mathrm{Bo}|^{1/2}s \geq 0 \right\}, \tag{10}$$

which is governed by the Cauchy problem

$$Y'' = -\mathsf{b}\,Z'Z, \quad Z'' = \mathsf{b}\,Y'Z, \ (\mathsf{b} = \mathrm{sgn}(\mathrm{Bo})); \tag{11a}$$

$$Y(0;\xi) = Z'(0;\xi) = 0, \ Y'(0;\xi) = 1, \ Z(0;\xi) = \xi, \tag{11b}$$

where the prime now means differentiation by S.

Remark 1.3. *The curves L_0^* by (11) are also normally parametrised and, therefore,*

$$Y'^2(S) + Z'^2(S) \equiv 1 \ \ \textit{for all} \ \ S. \tag{12}$$

Remark 1.4. *The ODEs (11a) are invariant with respect to substitution $Z := -Z$. This means that one can concentrate, without loss of generality, on non-negative $\xi \geq 0$.*

Henceforth, we concentrate on the ξ-family L_0^* with $\xi \geq 0$ pursuing an analytically-given approximation in a neighbourhood of $S = 0$. We will prove, consequently, that Y and Z are analytical functions at $S = 0$ and meromorphic function in the complex plane $S \in \mathbb{C}$ having only an infinite set of simple poles.

1.2.1 Analytical properties

As remarked in [2], the Cauchy problem (11) admits an integral constructed in terms of elliptic functions. To get this integral, one should rewrite (11a) in the following equivalent form

$$Y' = \cos\beta, \quad Z' = \sin\beta, \ \beta' = \mathsf{b}\,Z \tag{13}$$

and, furthermore, by using the substitution $\beta = -2\mathrm{i}\,\mathrm{Ln}\,\varphi(S)$ ($\mathrm{i}^2 = -1$ is the complex imaginary), the system (13) transforms to

$$Y' = \tfrac{1}{2}\left(\varphi^2 + \varphi^{-2}\right), \; Z' = -\mathrm{i}\,\tfrac{1}{2}\left(\varphi^2 - \varphi^{-2}\right), \; \varphi' = \mathrm{i}\tfrac{1}{2}\,\mathsf{b}\,\varphi Z, \tag{14}$$

which needs the initial conditions

$$Y(0;\xi) = 0, \quad Z(0;\xi) = \xi. \tag{15}$$

The second and third equations of (14) do not depend on variable Y. This makes it possible to find Z in an analytical form.

Indeed, when Bo>0, (14) has the following integral

$$Z = -\mathrm{i}\frac{\sqrt{\varphi^4 - (\xi^2+2)\varphi^2 + 1}}{\varphi} \tag{16}$$

coupling Z and φ. Substituting (16) into the last equation of (14) shows that φ is an inverse of the Christoffel-Schwartz integral mapping the half-plane onto a rectangle, i.e.,

$$S = 2\int_0^{\varphi} \frac{d\varphi}{\sqrt{\varphi^4 - (\xi^2+2)\varphi^2 + 1}} + S_0, \tag{17}$$

where S_0 is an arbitrary constant. The two integrals (16) and (17) imply the solution of the last two equations of (14).

Proceeding in similar way for Bo<0 derives the integrals

$$Z = \mathrm{i}\frac{\sqrt{\varphi^4 + (\xi^2-2)\varphi^2 + 1}}{\varphi}, \tag{18}$$

$$S = 2\mathrm{i}\int_0^{\varphi} \frac{d\varphi}{\sqrt{\varphi^4 + (\xi^2-2)\varphi^2 + 1}} + S_0, \tag{19}$$

which are an analogy for (16) and (17), respectively. When $\xi^2 > 4$, the Christoffel-Schwartz integral (18) maps the upper half-plane onto a rectangle. If $\xi^2 < 4$, it transforms the unit circle to the rectangle, but $\xi^2 = 4$ implies

$$\varphi = \mathrm{i}\frac{1 + Ce^S}{1 - Ce^S}, \quad C = \frac{1-\mathrm{i}}{1+\mathrm{i}}. \tag{20}$$

Because $\varphi(S)$ has, according to the Schwartz principle, only simple poles and zeros, one can prove the following theorem.

Theorem 1.1. *The Cauchy problem (11) determines the meromorphic functions Y and Z by variable $S \in \mathbb{C}$, which are characterised by an infinite set of simple poles located as specified in fig. 2 (a) for either* $\mathrm{Bo} > 0$ *or* $\mathrm{Bo} < 0,\ \xi^2 > 4$*, but the case* $\mathrm{Bo} < 0,\ \xi^2 < 4$ *implies simple poles, which are located as in fig. 2 (b). When* $\mathrm{Bo} < 0,\ \xi^2 = 4$*, the simple poles are located at* $S = \pm\pi \mathrm{i}(\frac{1}{2} + 2k),\ k \in \mathbb{Z}$.

Remark 1.5. *Even though one can find integrals (16)-(20) of (11a), they are difficult to use in practical computations. A simpler way could be rewriting the first equation of (11) in the form*

$$Y'' = -\tfrac{1}{2}\mathsf{b}\,(Z^2)' \Rightarrow Y' = -\mathsf{b}\,Z^2 + [1 + \tfrac{1}{2}\mathsf{b}\,\xi^2].$$

Substituting the last expression into the second equation of (11a) derives the Cauchy problem

$$Z'' - \mathsf{b}\,Z = \tfrac{1}{2}\xi^2 Z(1 - Z^2);\ \ Z(0) = \xi,\ \ Z'(0) = 0 \tag{21}$$

whose solution can be constructed in terms of elliptic functions. Alternatively, (21) may be solved numerically.

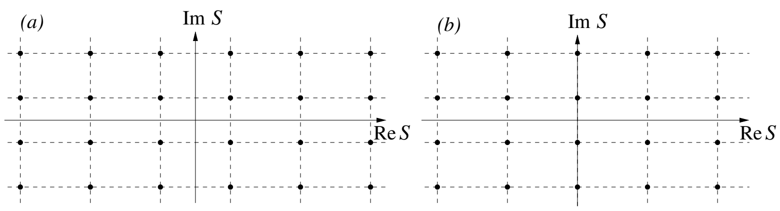

Figure 2. Location of simple poles for $Y(S;\xi),\ Z(S;\xi),\ S \in \mathbb{C}$, which are determined by (11) for different ξ as it follows from Theorem 1.1. The case (a) corresponds to $\mathrm{Bo} > 0$ or $\mathrm{Bo} < 0,\ \xi^2 > 4$ but (b) – $\mathrm{Bo} < 0,\ \xi^2 < 4$. When $\mathrm{Bo} < 0,\ \xi^2 = 4$, the simple poles are located at $S = \pm\pi \mathrm{i}(\frac{1}{2} + 2k),\ k \in \mathbb{Z}$.

Remark 1.6. *From physical point of view, $Z(S;\xi)$ should be a monotonic function by $S \in \mathbb{R}$ until it reaches the contact point C_1. One should remember that the contact point is located somewhere on the interval* $0 < S_1 \le S_2$*, where S_2 is the lowest root of $Z'(S_2) = 0$, if the root exists.*

Theorem 1.1 states that Y and Z are analytical functions for any $S \in \mathbb{C}$ except at the specified points where they have simple poles. The latter means that one can attempt to construct a Taylor-polynomial approximation of Y and Z in a neighbourhood of $S = 0$. Apart from the Taylor approximation, one can test the Padé approximant, which has to handle a finite set of simple poles in $\mathbb{C}$.

1.2.2 The Taylor approximation

M. Barnyak [1] was most probably the first one who proposed to adopt the Taylor approximation for solving the capillary meniscus problem. Postulating this approximation

$$Y = \sum_{k=1}^{N} a_k S^{2k-1}, \quad Z = \sum_{k=1}^{N} b_k S^{2k-2}, \quad N \to \infty \tag{22}$$

and substituting it into (11), derives, by gathering similar quantities S^m, the recurrence formulas

$$\begin{aligned} &a_1 = 1, \; b_1 = \xi, \\ &b_{j+1} = \frac{\mathsf{b}}{2j(2j-1)} \sum_{k=1}^{j} b_k a_{j-k+1}(2(j-k)+1), \\ &a_{j+1} = -\frac{\mathsf{b}}{j(2j+1)} \sum_{k=1}^{j} b_k b_{j-k+2}(j-k+1), \quad j \geq 1. \end{aligned} \tag{23}$$

According to Theorem 1.1 radius of convergence (R_T) of (22) is finite and, in the limit $N \to \infty$, it coincides with distance to the nearest simple pole in the complex plane. The radius is a function of $\xi \geq 0$ and $\mathsf{b} = \pm 1$.

When N is finite, the Taylor approximation (22) is applicable for $|S| \leq S_T(N, \xi, \epsilon) < R_T$, where ϵ is a given accuracy. An estimate of the radius $S_T(N, \xi, \epsilon)$ follows from the condition $S_T = \max S_*$ such that

$$|{Z'}^2(S) + {Y'}^2(S) - 1| \leq \epsilon, \; 0 < S < S_*(N, \xi, \epsilon). \tag{24}$$

Here, we used Remark 1.3.

1.2.3 The Padé approximant

The meromorphic solution $Y(S; \xi)$ and $Z(S; \xi)$ are characterised by simple poles in the complex plane $S \in \mathbb{C}$ as shown in fig. 2. This means that using the Taylor solution (22) deduces the Pagé approximant

$$Y = S\frac{1+\sum_{i=1}^{L} p_i^{(Y)} S^2}{1+\sum_{i=1}^{M} q_i^{(Y)} S^2},\ Z = \frac{\xi+\sum_{i=1}^{L} p_i^{(Z)} S^2}{1+\sum_{i=1}^{M} q_i^{(Z)} S^2},\ M+L=N-1. \quad (25)$$

For given N, M (the number of simple poles accounted for), ξ and the accuracy ϵ, one can define radius of convergence of (25) as $S_P = \max S_*$, where S_* is defined by (24) with Y and Z by (25). Because the Padé approximant should account for the nearest simple poles, we expect to improve accuracy and increase radius of convergence with respect to the Taylor polynomials, i.e. $S_T < S_P$.

1.2.4 Limits of applicability

Could the Taylor and Padé approximations provide a solution of the capillary meniscus problem? The answer depends on how large are radii of convergence S_T and S_P to guarantee that (22) and (25) make it possible to reach the point C_1 for any ξ. As stated in Remark 1.6, sufficient condition for that with the given accuracy ϵ is that the radii exceed S_2, i.e. $S_T \geq S_2$ and/or $S_P \geq S_2$, respectively, where S_2 is the lowest root of $Z'(S_2;\xi) = 0$. To compute S_2 as a function of $\xi > 0$, one can use the Runge-Kutta method for the Cauchy problem (11) rewritten in the normal form ($y_1 = Y, y_2 = Y', y_3 = Z, y_4 = Z'$)

$$\begin{aligned} &y_1' = y_2,\ y_2' = -\mathsf{b}\, z_2 z_1,\ z_1' = z_2,\ z_2' = \mathsf{b}\, y_2 z_1; \\ &y_1(0) = z_2(0) = 0,\ y_2(0) = 1,\ z_1(0) = \xi. \end{aligned} \quad (26)$$

A double precision (digits=16) FORTRAN-code was used to evaluate $S_T(\xi), S_P(\xi)$ and $S_2(\xi)$ as functions of ξ for the fixed dimension $N = 40$ and the accuracy $\epsilon = 10^{-7}$, $M = 4$ (the eight nearest simple poles are accounted for by the Padé approximant) in fig. 3. The results on $S_T(\xi)$ are marked by the dashed lines but $S_P(\xi)$ is denoted by the dots. The graph for $S_2(\xi)$ is drawn by the solid lines.

Fig. 3 shows that usage of the Padé approximant is more preferable – the radius S_P is larger than S_T, sometimes twice. One can see that $S_T, S_P \to +\infty$ as $\xi \to 0$ and $S_T, S_P \to 0$ as $\xi \to +\infty$. However, comparing $S_2(\xi)$ with $S_T(\xi)$ and $S_P(\xi)$ shows the Taylor polynomials and Padé approximant have different limits of applicability depending on ξ and Bo. When Bo>0 both the Taylor and Padé approximantions are

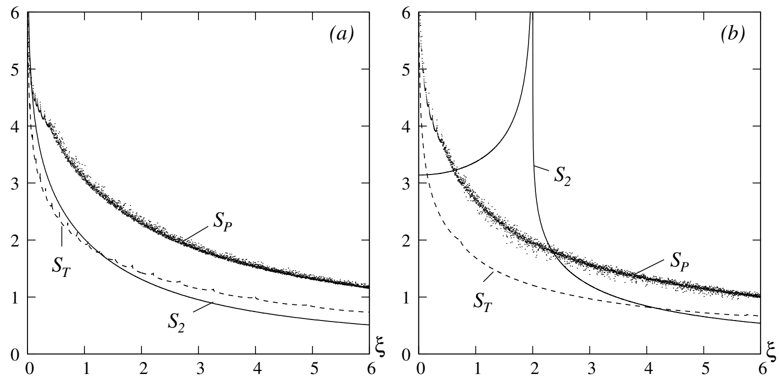

Figure 3. The radii of convergence for the Taylor $S_T(\xi)$ and Padé $S_P(\xi)$ approximations with $N = 40, \epsilon = 10^{-7}$ and $M = 4$ as well as the upper bound value $S_2(\xi)$. The case (a) – Bo > 0 and (b) – Bo < 0. The constructed approximations provide the capillary meniscus solution when $S_T \geq S_2$ (for the Taylor polynomials) and/or $S_P \geq S_2$ (for the Padé approximant).

well applicable slightly away from $\xi = 0$ so that the sufficient condition $S_T \geq S_2$ is satisfied for, approximately, $\xi \geq 1.25$ but $S_P \geq S_2$ as $\xi \geq 0.067$. The latter means that the Padé approximant may uniformly be applied to the capillary problem for positive Bo. Specifically, when Bo < 0, Y and Z become non-periodic functions by $S > 0$ as $\xi = 2$ and, therefore, $S_2 \to \infty$ for $\xi \to 2$. However, both S_T and S_P are finite at $\xi = 2$. As a consequence, the constructed analytical solutions (22) and (25) become inapplicable for a wide interval about $\xi = 2$.

2 Axisymmetric capillary surface

2.1 Mathematical formulations

Cavities of revolution may provide either axisymmetric or exotic (non-symmetric) capillary surface. The exotic surface was theoretically predicted and, later on, validated in the Space experiments [3]. In the present paper, we exclusively focus on studying the axisymmetric capillary meniscus whose mathematical formulation reduces, as in the previous section,

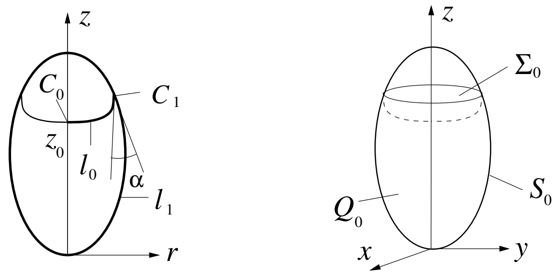

Figure 4. The same as in fig. 1 but for containers of revolution and three-dimensional axisymmetric capillary surfaces.

to the system of ODEs [8]

$$r'' = -z'\left(\mathrm{Bo}\, z - \frac{z'}{r} + c\right), \quad z'' = r'\left(\mathrm{Bo}\, z - \frac{z'}{r} + c\right), \tag{27}$$

which determines the capillary curve l_0 resulted from intersection of capillary surface and meridional plane as shown in fig. 4. The capillary curve l_0 is represented in the normal parametric form

$$l_0 = \{(r, z) : r = r(s), z = z(s)); \ 0 \le s \le s_1\},$$

where $s = 0$ implies the starting point, $C_0 = (0, z_0)$, on the Oz-axis, s_1 is the curve length and $C_1 = (f(z(s_1)), z(s_1))$ is the contact point with the wetted tank surface.

The system (27) is equipped with the initial conditions

$$r(0) = z'(0) = 0, \quad r'(0) = 1, \ z(0) = z_0 > 0, \tag{28}$$

where z_0 is unknown *a priori*. Because l_0 is based on the normal parametrisation, the system has the integral

$$r'^2(s) + z'^2(s) \equiv 1. \tag{29}$$

The Cauchy problem (27)-(28) determines the *two-parameter* family of curves

$$l_0^* = \{(r(s; z_0, c), z(s; z_0, c)) : \ z_0 > 0, \ s \ge 0\}. \tag{30}$$

To find z_0 and c and, therefore, $l_0 \in l_0^*$, one should satisfy, for the monotonic function $z(s)$ on $0 < s < s_1$, the contact angle condition

$$\mathrm{atan2}(1, r'(z(s_1))) - \mathrm{atan2}(z'(s_1), r'(s_1)) = \alpha, \tag{31}$$

in $C_1 = (r(s_1), z(s_1)) = (f(z(s_1)), z(s_1))$ as well as the liquid volume (mass) conservation condition

$$\int_0^{z_0} f^2(z)\,dz + \int_{z_0}^{s_1} \left[f^2(z(s)) - r^2(s)\right] |z'(s)|\,ds = |V_0|/\pi = \text{const}. \quad (32)$$

When Bo $\neq 0$, the substitution

$$r(s; z_0, c) = \frac{R(|\text{Bo}|^{1/2}s; \xi)}{|\text{Bo}|^{1/2}}, \quad z(s; z_0, c) = \frac{Z(|\text{Bo}|^{1/2}s; \xi)}{|\text{Bo}|^{1/2}} - \frac{c}{\text{Bo}} \quad (33)$$

redefines the set l_0^* as the ξ-parametric family of curves

$$L_0^* = \left\{(R(S;\xi), Z(S;\xi)) : \xi = z_0|\text{Bo}|^{1/2} + \frac{c\,\mathsf{b}}{|\text{Bo}|^{-1/2}},\ S = |\text{Bo}|^{1/2}s \geq 0\right\}, \quad (34)$$

coming from the Cauchy problem

$$R'' = -Z'\left(\mathsf{b}\,Z - \frac{Z'}{R}\right), \quad Z'' = R'\left(\mathsf{b}\,Z - \frac{Z'}{R}\right), \quad (35a)$$

$$Y(0;\xi) = Z'(0;\xi) = 0,\ Y'(0;\xi) = 1,\ Z(0;\xi) = \xi. \quad (35b)$$

The identity

$$R'^2(S) + Z'^2(S) \equiv 1 \ \text{ for all } \ S \quad (36)$$

remains invariant. We see that the ODEs (35a) are invariant with respect to the substitution $Z := -Z$ and therefore, the forthcoming analysis may concentrate on the case $\xi \geq 0$.

2.2 The Taylor and Padé approximations of L_0^*

The solution R, Z can also be suggested as analytical functions of $S \in \mathbb{C}$ at $S = 0$. In the contrast to the previous section, we cannot prove that R and Z are meromorphic functions but only show, following [2], that R and Z may have the simple pole singularity. This means that using the Taylor and Padé approximations of the Cauchy problem (35) has no rigorous mathematical argumentation but could be considered as a numerical experiment.

Adopting the Taylor polynomials

$$R = \sum_{k=1}^{N} a_k S^{2k-1}, \quad Z = \sum_{k=1}^{N} b_k S^{2k-2}, \quad N \to \infty, \quad (37)$$

leads to the recurrence formulas

$$
\begin{aligned}
&a_1 = 1, \;\; b_1 = \xi, \\
&b_{j+1} = \frac{1}{(2j)^2}\left[\mathsf{b}\sum_{m=1}^{j} a_{j-m+1}[2(j-m)+1]\sum_{l=1}^{m} a_l b_{m-l+1}\right. \\
&\qquad \left. -4j\sum_{m=1}^{j-1} m\, a_{j-m+1} b_{m+1}\right], \\
&a_{j=1} = -\frac{1}{j(2j+1)}\left[\sum_{m=1}^{j-1} a_{m+1}a_{j-m+1}\, m\,(2m+1)\right. \\
&\quad \left. +\sum_{m=1}^{j} b_{j-m+2}(j-m+1)\left(\mathsf{b}\sum_{l=1}^{m} b_l a_{m-l+1} - 2m b_{m+1}\right)\right], \; j > 1.
\end{aligned} \tag{38}
$$

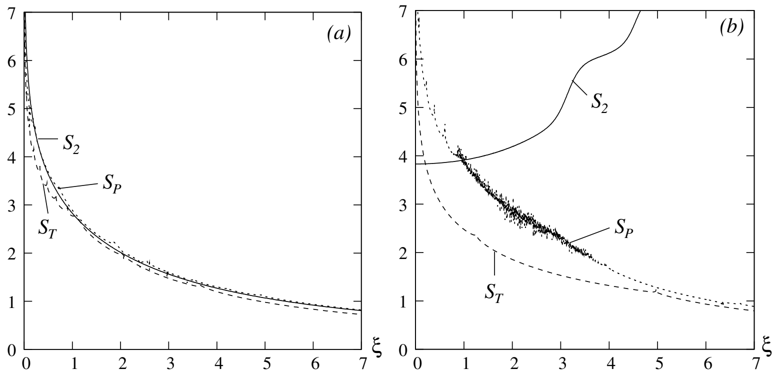

Figure 5. The same as in fig. 3 but for axisymmetric capillary surfaces; $N = 40, \epsilon = 10^{-7}$ and $M = 6$. The case (a) – Bo>0 and (b) – Bo<0. The constructed approximations provide an accurate approximation when $S_T \geq S_2$ (for the Taylor polynomials) and/or $S_P \geq S_2$ (for the Padé approximants).

Based on (37), one can get the Padé approximant

$$
R = S\frac{1+\sum_{i=1}^{L} p_i^{(R)} S^2}{1+\sum_{i=1}^{M} q_i^{(R)} S^2}, \; Z = \frac{\xi+\sum_{i=1}^{L} p_i^{(Z)} S^2}{1+\sum_{i=1}^{M} q_i^{(Z)} S^2}, \; M+L = N-1. \tag{39}
$$

One can introduce the radii of convergence $S_T \max S_*$ and $S_P \max S_*$

for (37) and (39), respectively, by using the integral (36), i.e.

$$|Z'^2(S) + R'^2(S) - 1| \leq \epsilon, \ 0 < S < S_*(N, \xi, \epsilon) \qquad (40)$$

for given N, M and ϵ.

Fig. 5 represents results of numerical experiments on the radii of convergence $S_T(\xi)$, $S_P(\xi)$ as well as $S_2(\xi)$ as functions of ξ for $N = 40, M = 6$ and $\epsilon = 10^{-7}$. The numerical results dramatically differ from those in fig. 3 When Bo$>$ 0 (the panel a), the Padé approximant slightly improves the accuracy so that $S_P \geq S_T$. However, this improvement is not as strong as for channels. Most likely, there are either other types of singularities in the complex plane (not only simple poles) or many of the simple poles are located relatively close to $S = 0$. Practically, usage of the Padé approximant guarantees rather accurate solution for the positive Bond number except, perhaps, for small ξ.

Numerical estimates of $S_T(\xi)$, $S_P(\xi)$ and $S_2(\xi)$ for negative Bond numbers (Bo$<$ 0) are presented in fig. 5 (b). Here, we see that switching from Taylor to Padé approximation may significantly increase the radius of convergence as $\xi \lesssim 5$. However, this does not help. Neither Taylor nor Padé approximations are practically applicable for solving the capillary problem with the negative Bond number. The reason is that $S_2 \to \infty$ with increasing ξ, namely, the solution becomes non-periodic in the limiting case.

3 Conclusion

The present paper tests Taylor and Padé approximations of the capillary meniscus problem in infinite channels and axisymmetric containers. This continues the study by Barnyak & Timokha [2] who suggested that rational approximation may significantly improve the numerical accuracy. We showed that, for the positive Bond number (Bo$>$ 0), using the Padé approximants may indeed provide an accurate solution of the capillary meniscus problem, except, perhaps, for large Bond numbers, when the capillary curve rapidly changes its behaviour at the contact line. In the contrast, neither Taylor polynomials nor rational approximations can guarantee getting an accurate analytical approximate solution for negative values of Bo. Our approach reduces the problem to an one-parameter set of the Cauchy problems and, as long as Bo$<$ 0, there are critical values of this parameter when one must find the solution on large interval that is impossible by using our two analytical approximate methods.

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