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Adaptive Lukovsky's formulas for the resulting hydrodynamic force and moment owing to sloshing in an upright circular tank*

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Lukovsky's asymptotic formulas for the resulting hydrodynamic force and moment are derived as if they follow from the adaptive (infinite-dimensional) multimodal theory of the liquid sloshing dynamics in an upright circular base container. The result is given in a tensor form introduced in notations of the original paper by Faltinsen, Lukovsky and Timokha (2016).

Виводяться формули Луковського адаптивного асимптотичного типу для результуючих гідродинамічної сили й моменту, що пов'язуються з коливаннями рідини у вертикальному круговому баці. Результат представлено в термінах позначень із оригінальної роботи Фалтінсена, Луковського й Тимохи (2016).

Introduction

Linear and weakly-nonlinear and fully-nonlinear multimodal theories are common for studying the liquid sloshing dynamics in an upright circular cylindrical tank performing a small-amplitude three-dimensional oscillatory motion. The theories reduce the original free-boundary problem to different kinds of (modal) systems of ordinary differential equations

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with respect to the hydrodynamic generalised coordinates – the time-dependent coefficients in a Fourier–type (functional) presentation of the free surface. By getting either transient (solving the Cauchy problem) or steady-state wave (time-periodic condition) solution of the modal systems makes it possible, using the aforementioned Fourier representation, to describe the free-surface elevation as well as the velocity and pressure fields. Furthermore, substituting this modal solution (the hydrodynamic generalised coordinates) into the so-called Lukovsky's formulas [1,4,5] gives the resulting hydrodynamic force and moment due to the pressure load on the wetted tank surface.

I.A. Lukovsky derived his formulas in the most general, fully-nonlinear form. The formulas had to equip the so-called Miles-Lukovsky modal theory [1,4,5], which is fully-nonlinear and, therefore, mathematically equivalent to the original free-surface sloshing problem. However, the Miles-Lukovsky theory is of a rather abstract nature so that its usage is disputable to effectively conduct numerical simulations and/or make analytical studies. This was in many details discussed in reviews [1,6]. To facilitate analytical studies of the resonant nonlinear sloshing, the theory had to be simplified to a weakly-nonlinear form. The simplification includes an analytical (asymptotic) reduction of both the governing (modal) equations, which couple the hydrodynamic generalised coordinates, and the Lukovsky formulas for the hydrodynamic force and moment. A requirement consists of postulating a series of specific asymptotic relations between the non-dimensional forcing magnitude (implies the higher-order asymptotic scale) and the hydrodynamic generalised coordinates, which, because some of them are resonantly excited, are characterised by a lower asymptotic order. The procedure needs neglecting the higher asymptotic quantities than the non-dimensional forcing magnitude; it leads to weakly-nonlinear modal equations and Lukovsky's formulas. In the most general case, the aforementioned asymptotic relations take the so-called adaptive form.

Recently, an adaptive infinite-dimensional asymptotic modal system was derived in [2] to describe the resonant liquid sloshing in an upright circular base container. The adaptive intermodal asymptotic ordering assumes then that all the hydrodynamic generalised coordinates have the same asymptotic order $O(\epsilon^{1/3})$ where $O(\epsilon) \ll 1$ measures the non-dimensional forcing magnitude. The present paper equips [2] with the corresponding adaptive Lukovsky formulas, which were not derived in the original and forthcoming [3] papers based on the adaptive and

Narimanov-Moiseev–type multimodal asymptotic ordering.

1 Non-dimensional modal solution

We consider the liquid sloshing dynamics in an upright circular rigid tank performing a small-magnitude almost periodic three-dimensional motion with the circular frequency σ. The sloshing is analysed in the non-dimensional statement suggesting the tank radius R_0 is the characteristic size and $T = 2\pi/\sigma$ is the characteristic time.

The time-dependent liquid domain $Q(t)$ is confined by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$. The free-surface elevations are considered in the tank-fixed coordinate system $Oxyz$ whose coordinate plane Oxy coincides with the mean free surface Σ_0 and Oz is the symmetry axis. The small-magnitude tank motions are governed by the time-dependent vectors $\boldsymbol{v}_O(t) = (\dot\eta_1, \dot\eta_2, \dot\eta_3)$ and $\boldsymbol{\omega}(t) = (\dot\eta_4, \dot\eta_5, \dot\eta_6)$, which describe the non-dimensional translatory and instant angular velocities of the tank, respectively, and the generalised coordinates $\eta_i(t) = O(\epsilon) \ll 1$, $i = 1, ...6$ determine the three-dimensional body motions.

The adaptive asymptotic modal method introduces the modal (Fourier) representation of the free surface

$$\zeta(r,\theta,t) = \sum_{Mi} \mathcal{R}_{Mi}(r)\cos(M\theta)\, p_{Mi}(t) + \sum_{mi} \mathcal{R}_{mi}(r)\sin(m\theta)\, r_{mi}(t), \quad (1)$$

where $\mathcal{R}_{Mi}(r) = \alpha_{Mi} J_M(k_{Mi} r)$ with the Bessel function of the first kind, k_{Mi}, $i \geq 1$ are the roots of $J'_M(k_{Mi}) = 0$, and

$$p_{Mi}(t) \sim r_{Mi}(t) = O(\epsilon^{1/3})$$

are the *hydrodynamic generalised coordinates*. The normalising multipliers α_{Mi} are required to provide the identity

$$\int_0^1 r\mathcal{R}^2_{Mi}(r)\, dr = 1 \quad \Rightarrow \quad \alpha^2_{Mi} = \frac{2k_{Mi}}{J^2_M(k_{Mi})(k^2_{Mi} - M^2)}. \quad (2)$$

Here and thereafter, the *large summation indices imply summation from zero to infinity but small indices mean change from one to infinity.*

The adaptive asymptotic modal theory neglects the $o(\epsilon)$-terms in the governing equations and all other hydrodynamic characteristics. A result in [2] is a weakly-nonlinear infinite-dimensional system of ordinary differential equations with respect to p_{Mi} and r_{Mi} whose right-hand site is a linear vector-function with respect to η_i and their time derivatives.

The forthcoming derivations adopt notations from [2]. This includes tensors in Appendix A and several shortcuts, e.g.,

$$\kappa_{Lk} = k_{Lk}\tanh(k_{Lk}h), \quad \mathcal{Z}_{Lk}(z) = \frac{\cosh(k_{Lk}(z+h))}{k_{Lk}\sinh(k_{Lk}h)} \tag{3}$$

where h is the non-dimensional liquid depth.

2 Hydrodynamic force

Adopting the asymptotic modal solution (1) and the original Lukovsky formula for the dimensional hydrodynamic force $\boldsymbol{F}(t)$ from, e.g., [1, 4], one can write down

$$\begin{aligned}\boldsymbol{F}(t) = &\left[M_l R_0 \sigma^2\right]\big(\boldsymbol{g} - \dot{\boldsymbol{v}}_O - \boldsymbol{\omega}\times\boldsymbol{v}_O - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}_{lC}) - \dot{\boldsymbol{\omega}}\times\boldsymbol{r}_{lC} - \ddot{\boldsymbol{r}}_{lC} \\ &- 2\boldsymbol{\omega}\times\dot{\boldsymbol{r}}_{lC}\big) = \left[M_l R_0\sigma^2\right]\big(\boldsymbol{g} - \dot{\boldsymbol{v}}_O - \dot{\boldsymbol{\omega}}\times\boldsymbol{r}_{lC_0} - \ddot{\boldsymbol{r}}_{lC} + o(\epsilon)\big) \\ &= \left[M_l R_0\sigma^2\right]\big(F_1(t)\,\boldsymbol{e}_1(t) + F_2(t)\,\boldsymbol{e}_2(t) + F_3(t)\,\boldsymbol{e}_3(t) + o(\epsilon)\big),\end{aligned} \tag{4}$$

where the dot means the time derivative in the $Oxyz$ coordinates,

$$\dot{\boldsymbol{a}}(t) = \dot{a}_1(t)\,\boldsymbol{e}_1(t) + \dot{a}_2(t)\,\boldsymbol{e}_2(t) + \dot{a}_3(t)\,\boldsymbol{e}_3(t)$$

($\boldsymbol{e}_i(t)$, $i = 1,2,3$ are the coordinate units of the body-fixed coordinate system $Oxyz$), M_l is the liquid mass, $\boldsymbol{g}$ is the non-dimensional gravity vector in the body-fixed coordinate system whose linear ($= O(\epsilon)$) components take the form

$$\boldsymbol{g} = g\eta_5\boldsymbol{e}_1 - g\eta_4\boldsymbol{e}_2 - g\boldsymbol{e}_3, \quad \boldsymbol{g}_0 = -g\boldsymbol{e}_3 = O(1); \tag{5}$$

and

$$\begin{aligned}\boldsymbol{r}_{lC} &= \frac{1}{\pi h}\left[\boldsymbol{e}_1\int_{Q(t)} x dQ + \boldsymbol{e}_2\int_{Q(t)} y dQ + \boldsymbol{e}_3\int_{Q(t)} z dQ\right] \\ &= \boldsymbol{r}_{lC_0} + \boldsymbol{r}_{lC_s} = \boldsymbol{e}_1\left(\frac{1}{h}\sum_i P_i p_{1i}(t)\right) + \boldsymbol{e}_2\left(\frac{1}{h}\sum_i P_i r_{1i}(t)\right) \\ &\qquad + \boldsymbol{e}_3\left(-\frac{h}{2} + \frac{1}{h}\sum_i p_{0i}^2 + \frac{1}{2h}\sum_{m,i}(p_{mi}^2 + r_{mi}^2)\right),\end{aligned} \tag{6a}$$

$$\boldsymbol{r}_{lC_0} = -\tfrac{1}{2}h\boldsymbol{e}_3, \quad \boldsymbol{r}_{lC_s} = r_{1lC_s}\boldsymbol{e}_1 + r_{2lC_s}\boldsymbol{e}_2 + r_{3lC_s}\boldsymbol{e}_3, \tag{6b}$$

$$\left(P_i = \int_0^1 r^2 \mathcal{R}_{1i}(r)\, dr = \alpha_{1n} \frac{J_1(k_{1n})}{k_{1n}^2} \right) \tag{6c}$$

is the non-dimensional liquid mass centre $\boldsymbol{r}_{lC}$, $\boldsymbol{r}_{lC_0}$ defines its hydrostatic position, but $\boldsymbol{r}_{lC_s}$ determines the sloshing-related mass centre motions.

Substituting (5), (6a) and (6b) into (4) derives the three scalar force components $F_i(t)$, $i = 1, 2, 3$:

$$F_1(t) = g\eta_5 - \ddot{\eta}_1 + \tfrac{1}{2}h\ddot{\eta}_5 - h^{-1}\sum_i P_i\, \ddot{p}_{1i}, \tag{7a}$$

$$F_2(t) = -g\eta_4 - \ddot{\eta}_2 - \tfrac{1}{2}h\ddot{\eta}_4 - h^{-1}\sum_i P_i\, \ddot{r}_{1i}, \tag{7b}$$

$$\begin{aligned} F_3(t) = -g - \ddot{\eta}_3 - 2h^{-1}\sum_i \left(\ddot{p}_{0i}p_{01} + \dot{p}_{0i}^2\right) \\ - h^{-1}\sum_{m,i}\left(\ddot{p}_{mi}p_{mi} + \dot{p}_{mi}^2 + \ddot{r}_{mi}r_{mi} + \dot{r}_{mi}^2\right). \end{aligned} \tag{7c}$$

3 The Stokes-Joukowski potentials

The Lukovsky formula for the resulting hydrodynamic moment relative to the origin O needs to know the free-surface depending Stokes-Joukowski potentials. The non-dimensional scalar Stokes-Joukowski potentials, $\Omega_i(x, y, z; \{p_{Mi}, r_{mi}\})$, $i = 1, 2, 3$, are components of the vector-function $\boldsymbol{\Omega} = \Omega_1\boldsymbol{e}_1 + \Omega_2\boldsymbol{e}_2 + \Omega_3\boldsymbol{e}_3$, which is a solution of the Neumann boundary value problem

$$\nabla^2\boldsymbol{\Omega} = \boldsymbol{0} \;\text{ in }\; Q(t), \quad \partial_n\boldsymbol{\Omega} = \boldsymbol{r}\times\boldsymbol{n} \;\text{ on }\; S(t) + \Sigma(t), \tag{8}$$

where $\boldsymbol{r} = x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + z\boldsymbol{e}_3$ and $\boldsymbol{n}$ is the outer normal unit vector. Finding the scalar Stokes-Joukowski potentials is the same as getting the harmonic functions satisfying the Neumann boundary conditions

$$\partial_n\Omega_1 = yn_z - zn_y,\; \partial\Omega_2 = zn_x - xn_z,\; \partial_n\Omega_3 = xn_y - yn_x \text{ on } S(t) + \Sigma(t), \tag{9}$$

where n_x, n_y and n_z are components of the outer normal vector in the Cartesian coordinate system $Oxyz$.

The Neumann boundary conditions (9) have a specific form for the upright circular base tank when considering them, separately, on the bottom, wetted walls, and the free surface. On the flat bottom, $n_x = n_y = 0, n_z = -1$ and, therefore, (9) transforms to

$$\partial_z\Omega_1 = y = r\sin\theta,\ \partial_z\Omega_2 = -x = -r\cos\theta,\ \partial_z\Omega_3 = 0 \text{ at } z = -h; \quad (10)$$

on the vertical wall, $n_x = \cos\theta, n_y = \sin\theta, n_z = 0$ and, therefore,

$$\partial_r\Omega_1 = -z\sin\theta,\ \partial_r\Omega_2 = z\cos\theta,\ \partial_r\Omega_3 = 0 \text{ at } r = 1. \quad (11)$$

The free surface $\Sigma(t)$ is defined by $z = \zeta(r, \theta) = 0$ or, in other words, by the hydrodynamic generalised coordinates in (1). To write down the corresponding Neumann boundary conditions, one should first introduce the outer normal vector

$$\boldsymbol{n} = \nabla_{xyz}[z-\zeta]/||\nabla_{xyz}[z-\zeta]||$$

where, according to (53) in Appendix B,

$$\begin{aligned} n_x||\nabla_{xyz}(z-\zeta)|| &= -\cos\theta\,\partial_r\zeta + r^{-1}\sin\theta\,\partial_\theta\zeta, \\ n_y||\nabla_{xyz}(z-\zeta)|| &= -\sin\theta\,\partial_r\zeta - r^{-1}\cos\theta\,\partial_\theta\zeta, \\ n_z||\nabla_{xyz}(z-\zeta)|| &= 1. \end{aligned} \quad (12)$$

Using these expressions, the Neumann boundary conditions in (9) on $\Sigma(t)$ take the following form

$$\nabla\Omega_1 \cdot \nabla(z-\zeta) = r\sin\theta + \zeta\left(\sin\theta\,\partial_r\zeta + r^{-1}\cos\theta\,\partial_\theta\zeta\right), \quad (13a)$$

$$\nabla\Omega_2 \cdot \nabla(z-\zeta) = -r\cos\theta + \zeta\left(-\cos\theta\,\partial_r\zeta + r^{-1}\sin\theta\,\partial_\theta\zeta\right), \quad (13b)$$

$$\nabla\Omega_3 \cdot \nabla(z-\zeta) = -\partial_\theta\zeta, \quad (13c)$$

where

$$\nabla\Omega_i \cdot \nabla(z-\zeta) = \partial_z\Omega_i - \partial_r\Omega_i\,\partial_r\zeta - r^{-2}\partial_\theta\Omega_i\,\partial_\theta\zeta. \quad (14)$$

The forthcoming goal consists, after substituting (1) into the Neumann boundary conditions (13), of getting an asymptotic approximation of the scalar Stokes-Joukowski potentials in terms of the hydrodynamic generalised coordinates $p_{Mi}, r_{Mi} = O(\epsilon^{1/3})$,

$$\Omega_{ni} = \Omega_{0i} + \Omega_{1i} + \Omega_{2i} + O(\epsilon),\quad \Omega_{ni} = O(\epsilon^{n/3});\quad n = 1, 2, 3, \quad (15)$$

starting with the zero-order approximation, which suggests the generalised coordinates p_{Mi} and r_{Mi} are zero.

3.1 Zero-order approximation

The zero-order approximation, Ω_{0i}, is an attribute of the linear sloshing problem. There is an analytical solution of the corresponding Neumann problem in the unperturbed liquid domain (the flat free surface with $p_{Mi} = r_{Mi} = 0$ in (13), (14)). This solution can be found, e.g., in the chapter 5 of [1]. For the adopted normalisation, this analytical solution was derived into [3]. It takes the form

$$\Omega_{01} = -\mathcal{F}(r,z)\sin\theta, \quad \Omega_{02} = \mathcal{F}(r,z)\cos\theta, \quad \Omega_{03} = 0, \tag{16}$$

where

$$\mathcal{F}(r,z) = rz - 2\sum_n \frac{P_n}{k_{1n}}\mathcal{R}_{1n}(r)\frac{\sinh(k_{1n}(z+\frac{1}{2}h))}{\cosh(\frac{1}{2}k_{1n}h)} \tag{17}$$

and P_n is defined in (6c).

One can see that Ω_3 has no the zero-order component but Ω_1 and Ω_2 and similar by the coordinates r, z and differ only by the azimuthal coordinate θ.

3.2 First- and second-order approximate Ω_1

The first-order approximation of the Stokes-Joukowski potential Ω_1, Ω_{11} from (15), is a harmonic function, which satisfies the zero-Neumann condition on the wetted tank surface (at $z = -h$ and $r = 1$) but the Neumann boundary condition on the unperturbed free surface is non-zero but derivable from (13a) by keeping the $O(\epsilon^{1/3})$-order quantities. It takes the form

$$\partial_z\Omega_{11} = -\partial_z^2\Omega_{01}\zeta + \partial_r\Omega_{01}\partial_r\zeta + r^{-2}\partial_\theta\Omega_{01}\partial_\theta\zeta \quad \text{at} \quad z = 0. \tag{18}$$

Substituting (16), (17) and (1) into (18) yields

$$\begin{aligned}\partial_z\Omega_{11} = 2\sum_{Mi} p_{Mi}\sum_a \tilde{P}_a\big[&\sin\theta\cos(M\theta)(\mathcal{R}'_{Mi}\mathcal{R}'_{1a} - k_{1a}^2\mathcal{R}_{Mi}\mathcal{R}_{1a}) \\ &- M\cos\theta\sin(M\theta)r^{-2}\mathcal{R}_{Mi}\mathcal{R}_{1a}\big] \\ + 2\sum_{mi} r_{mi}\sum_a \tilde{P}_a\big[&\sin\theta\sin(m\theta)(\mathcal{R}'_{mi}\mathcal{R}'_{1a} - k_{1a}^2\mathcal{R}_{mi}\mathcal{R}_{1a}) \\ &+ m\cos\theta\cos(m\theta)r^{-2}\mathcal{R}_{mi}\mathcal{R}_{1a}\big],\end{aligned} \tag{19}$$

where

$$\tilde{P}_a = P_a k_{1a}^{-1} \tanh(\tfrac{1}{2} k_{1a} h). \tag{20}$$

Using the standard projective scheme derives the following solution

$$\Omega_{11} = \sum_{Lk} \mathcal{R}_{Lk}(r) \mathcal{Z}_{Lk}(z) \cos(L\theta) \sum_{mi} O^{1,r}_{(Lk),(mi)} r_{mi}(t) + \sum_{lk} \mathcal{R}_{lk}(r) \mathcal{Z}_{lk}(z) \sin(l\theta) \sum_{Mi} O^{1,p}_{(lk),(Mi)} p_{Mi}(t), \tag{21}$$

where

$$O^{1,r}_{(Lk),(mi)} = \frac{2}{\Lambda_{LL}} \sum_a \tilde{P}_a \Big\{ \Lambda_{L,m1} \big(\lambda'_{(mi)(1a),(Lk)} - k_{1a}^2 \lambda_{(mi)(1a)(Lk)} \big) + m \Lambda_{mL1,} \bar{\lambda}_{(mi)(1a)(Lk)} \Big\}, \tag{22a}$$

$$O^{1,p}_{(lk),(Mi)} = \frac{2}{\Lambda_{ll}} \sum_a \tilde{P}_a \Big\{ \Lambda_{M,1l} \big(\lambda'_{(Mi)(1a),(lk)} - k_{1a}^2 \lambda_{(Mi)(1a)(lk)} \big) - M \Lambda_{1,Ml} \bar{\lambda}_{(Mi)(1a)(lk)} \Big\} \tag{22b}$$

within notations from (3) and Appendix B .

The second-order approximation Ω_{21} is also a harmonic function satisfying the zero-Neumann boundary condition at $z = -h$ and $r = 1$. The only non-zero Neumann boundary condition at $z = 0$ can be obtained by the Taylor expansion in ζ (and its derivatives) applied to (13a). The result is

$$\partial_z \Omega_{21} = \zeta \left(\sin\theta \partial_r \zeta + r^{-1} \cos\theta \partial_\theta \zeta \right) + \zeta \left(\partial^2_{zr} \Omega_{01} \partial_r \zeta + r^{-2} \partial^2_{z\theta} \Omega_{01} \partial_\theta \zeta \right) - \tfrac{1}{2} \partial_z^3 \Omega_{01} \zeta^2 - \partial_z^2 \Omega_{11} \zeta + \partial_r \Omega_{11} \partial_r \zeta + r^{-2} \partial_\theta \Omega_{11} \partial_\theta \zeta \quad \text{at} \quad z = 0. \tag{23}$$

Because $\partial_z \Omega_{01} = r \sin\theta$ at $z = 0$, two first summands in (23) are equal. Furthermore, using the obvious Fourier expansions following from the Parseval identity

$$r = \sum_a P_a \mathcal{R}_{1a}(r) \quad \text{and} \quad 1 = \sum_a P_a \mathcal{R}'_{1a}(r),$$

the boundary condition (23) itransforms to the following form

$$\partial_z\Omega_{21} = \sum_{MiNj} p_{Mi}p_{Nj}\Big\{ \sum_a P_a\big[\sin\theta\cos(M\theta)\cos(M\theta)\big(2\mathcal{R}'_{Nj}\mathcal{R}'_{1a}\mathcal{R}_{Mi}$$
$$- k_{1a}^2\mathcal{R}_{Nj}\mathcal{R}_{1a}\mathcal{R}_{Mi}\big) - 2N\cos\theta\cos(M\theta)\sin(N\theta)r^{-2}\mathcal{R}_{Nj}\mathcal{R}_{1a}\mathcal{R}_{Mi}\big]$$
$$+ \sum_{Ab} O^{1,p}_{(Ab),(Mi)}\kappa^{-1}_{Ab}\big[\sin(A\theta)\cos(N\theta)\big(\mathcal{R}'_{Ab}\mathcal{R}'_{Nj} - k^2_{Ab}\mathcal{R}_{Ab}\mathcal{R}_{Nj}\big)$$
$$- AN\cos(A\theta)\sin(N\theta)r^{-2}\mathcal{R}_{Ab}\mathcal{R}_{Nj}\big]\Big\}$$
$$+ \sum_{minj} r_{mi}r_{nj}\Big\{ \sum_a P_a\big[\sin\theta\sin(m\theta)\sin(n\theta)\big(2\mathcal{R}'_{nj}\mathcal{R}'_{1a}\mathcal{R}_{mi}$$
$$- k^2_{1a}\mathcal{R}_{mi}\mathcal{R}_{1a}\mathcal{R}_{nj}\big) + 2n\cos\theta\sin(m\theta)\cos(n\theta)r^{-2}\mathcal{R}_{nj}\mathcal{R}_{1a}\mathcal{R}_{mi}\big]$$
$$+ \sum_{Ab} O^{1,r}_{(Ab),(mi)}\kappa^{-1}_{Ab}\big[\cos(A\theta)\sin(n\theta)\big(\mathcal{R}'_{Ab}\mathcal{R}'_{nj} - k^2_{Ab}\mathcal{R}_{Ab}\mathcal{R}_{nj}\big)$$
$$- An\sin(A\theta)\cos(n\theta)r^{-2}\mathcal{R}_{Ab}\mathcal{R}_{nj}\big]\Big\}$$
$$+\sum_{Minj} p_{Mi}r_{nj}\Big\{2\sum_a P_a\big[\sin\theta\cos(M\theta)\sin(n\theta)\big(\mathcal{R}'_{nj}\mathcal{R}'_{1a}\mathcal{R}_{Mi} + \mathcal{R}'_{Mi}\mathcal{R}'_{1a}\mathcal{R}_{nj}$$
$$-k^2_{1a}\mathcal{R}_{nj}\mathcal{R}_{1a}\mathcal{R}_{Mi}\big) + \big(n\cos\theta\cos(M\theta)\cos(n\theta) - M\cos\theta\sin(n\theta)\sin(M\theta)\big)$$
$$\times r^{-2}\,\mathcal{R}_{Mi}\mathcal{R}_{1a}\mathcal{R}_{nj}\big] + \sum_{Ab}\kappa^{-1}_{Ab}\Big[O^{1,p}_{(Ab),(Mi)}\big(\sin(A\theta)\sin(n\theta)$$
$$\times\big[\mathcal{R}'_{Ab}\mathcal{R}'_{nj} - k^2_{Ab}\mathcal{R}_{Ab}\mathcal{R}_{nj}\big] + An\cos(A\theta)\cos(n\theta)r^{-2}\mathcal{R}_{Ab}\mathcal{R}_{nj}\big)$$
$$+ O^{1,r}_{(Ab),(nj)}\big(\cos(A\theta)\cos(M\theta)\big[\mathcal{R}'_{Ab}\mathcal{R}'_{Mi} - k^2_{Ab}\mathcal{R}_{Ab}\mathcal{R}_{Mi}\big]$$
$$+ AM\sin(A\theta)\sin(M\theta)r^{-2}\mathcal{R}_{Ab}\mathcal{R}_{Mi}\big)\Big] \quad \text{at} \quad z=0. \qquad (24)$$

The standard projective scheme leads then to the following solution

$$\Omega_{21} = \sum_{Lk}\mathcal{R}_{Lk}(r)\mathcal{Z}_{Lk}(z)\cos(L\theta)\sum_{Mnij} O^{1,pr}_{(Lk),(Mi),(nj)}p_{Mi}r_{nj}$$
$$+ \sum_{lk}\mathcal{R}_{lk}(r)\mathcal{Z}_{lk}(z)\sin(l\theta)\Bigg[\sum_{MiNj} O^{1,pp}_{(lk),(Mi),(Nj)}p_{Mi}p_{Nj}$$
$$+ \sum_{minj} O^{1,rr}_{(lk),(mi),(nj)}r_{mi}r_{nj}\Bigg], \qquad (25)$$

where

$$O^{1,pr}_{(Lk),(Mi),(nj)} = \frac{2}{\Lambda_{LL}} \sum_a P_a \Big[\Lambda_{LM,1n} \big(\lambda'_{(nj)(1a),(Mi)(Lk)} + \lambda'_{(Mi)(1a),(nj)(Lk)}$$
$$- k^2_{1a} \lambda_{(nj)(1a)(Mi)(Lk)} \big) + (n\Lambda_{LMn1,} - M\Lambda_{L1,Mn}) \bar{\lambda}_{(nj)(1a)(Mi)(Lk)} \Big]$$
$$+ \sum_{Ab} \kappa^{-1}_{Ab} \Big[O^{1,p}_{(Ab),(Mi)} \big(\Lambda_{L,An} \big[\lambda'_{(Ab)(nj),(Lk)} - k^2_{Ab} \lambda_{(Ab)(nj)(Lk)} \big]$$
$$+ An\Lambda_{AnL,} \bar{\lambda}_{(Ab)(nj)(Lk)} \big) + O^{1,r}_{(Ab),(nj)} \big(\Lambda_{AML,} \big[\lambda'_{(Ab)(Mi),(Lk)}$$
$$- k^2_{Ab} \lambda_{(Ab)(Mi)(Lk)} \big] + AM\Lambda_{L,Am,} \bar{\lambda}_{(Ab)(Mi)(Lk)} \big) \Big], \quad \text{(26a)}$$

$$O^{1,pp}_{(lk),(Mi),(Nj)} = \frac{2}{\Lambda_{ll}} \sum_a P_a \Big[\Lambda_{MN,l1} \big(\lambda'_{(Nj(1a),(Mi)(lk)}$$
$$- \tfrac{1}{2} k^2_{1a} \lambda_{(Mi)(1a)(Nj)(lk)} \big) - N\Lambda_{1M,lN} \bar{\lambda}_{(mi)(1a)(Nj)(lk)} \Big]$$
$$+ \sum_{Ab} O^{1,p}_{(Ab),(Mi)} \kappa^{-1}_{Ab} \Big[\Lambda_{N,Al} \big(\lambda'_{(Ab)(Nj),(lk)} - k^2_{Ab} \lambda_{(Ab)(Nj)(lk)} \big)$$
$$- AN\Lambda_{A,Nl} \bar{\lambda}_{(Ab)(Nj)(lk)} \Big], \quad \text{(26b)}$$

$$O^{1,rr}_{(lk),(mi),(nj)} = \frac{2}{\Lambda_{ll}} \sum_a P_a \Big[\Lambda_{,1nml} \big(\lambda'_{(nj)(1a),(mi)(lk)}$$
$$- \tfrac{1}{2} k^2_{1a} \lambda_{(mi)(1a)(nj)(lk)} \big) + n\Lambda_{1n,ml} \bar{\lambda}_{(mi)(1a)(nj)(lk)} \Big]$$
$$+ \sum_{Ab} O^{1,r}_{(Ab),(mi)} \kappa^{-1}_{Ab} \Big[\Lambda_{A,nl} \big(\lambda'_{(Ab)(nj),(lk)} - k^2_{Ab} \lambda_{(Ab)(nj)(lk)} \big)$$
$$- An\Lambda_{n,Al} \bar{\lambda}_{(Ab)(nj)(lk)} \Big]. \quad \text{(26c)}$$

3.3 First- and second-order approximate Ω_2

Proceeding in similar way with the zero-order approximation (16), (17) and the Neumann boundary condition (13b) on the free surface derives the first- and second-order approximations, Ω_{12} and Ω_{22}, as follows

$$\Omega_{12} = \sum_{Lk} \mathcal{R}_{Lk}(r) \mathcal{Z}_{Lk}(z) \cos(L\theta) \sum_{Mi} O^{2,p}_{(Lk),(Mi)} p_{Mi}(t)$$
$$+ \sum_{lk} \mathcal{R}_{lk}(r) \mathcal{Z}_{lk}(z) \sin(l\theta) \sum_{mi} O^{2,r}_{(lk),(mi)} r_{mi}(t), \quad (27)$$

where

$$O^{2,p}_{(Lk),(Mi)} = \frac{2}{\Lambda_{LL}} \sum_a \tilde{P}_a \Big[\Lambda_{1ML,} \big(-\lambda'_{(Mi)(1a),(Lk)} + k^2_{1a} \lambda_{(Mi)(Lk)(1a)} \big) - M\Lambda_{L,1M} \bar{\lambda}_{(Mi)(1a)(Lk)} \Big], \tag{28a}$$

$$O^{2,r}_{(lk),(mi)} = \frac{2}{\Lambda_{ll}} \sum_a \tilde{P}_a \Big[\Lambda_{1,lm} \big(-\lambda'_{(mi)(1a),(lk)} + k^2_{1a} \lambda_{(mi)(lk)(1a)} \big) + m\Lambda_{M,1l} \bar{\lambda}_{(mi)(1a)(lk)} \Big] \tag{28b}$$

and

$$\Omega_{22} = \sum_{Lk} \mathcal{R}_{Lk}(r) \mathcal{Z}_{Lk}(z) \cos(L\theta) \Bigg[\sum_{MiNj} O^{2,pp}_{(Lk),(Mi),(Nj)} p_{Mi} p_{Nj} + \sum_{minj} O^{2,rr}_{(Lk),(mi),(nj)} r_{mi} r_{nj} \Bigg] + \sum_{lk} \mathcal{R}_{lk}(r) \mathcal{Z}_{lk}(z) \sin(l\theta) \sum_{Mnij} O^{2,pr}_{(lk),(Mi),(nj)} p_{Mi} r_{nj} \tag{29}$$

where

$$O^{2,pp}_{(Lk),(Mi),(Nj)} = \frac{2}{\Lambda_{LL}} \sum_a \Big[\Lambda_{1MNL,} \big(\tfrac{1}{2} \lambda_{(Mi)(Nj)(1a)(Lk)} - \lambda'_{(Nj)(1a),(Mi)(Lk)} \big) - N\Lambda_{ML,N1} \bar{\lambda}_{(Nj)(1a)(Mi)(Lk)} \Big] + \sum_{Ab} O^{2,p}_{(Ab),(Mi)} \kappa^{-1}_{Ab} \Big[\Lambda_{ANL,} \big(\lambda'_{(Ab)(Nj),(Lk)} - k^2_{Ab} \lambda_{(Ab)(Nj)(Lk)} \big) + AN\Lambda_{L,AN} \bar{\lambda}_{(Ab)(Nj)(Lk)} \Big], \tag{30a}$$

$$O^{2,rr}_{(Lk),(mi),(nj)} = \frac{2}{\Lambda_{LL}} \sum_a \Big[\Lambda_{1L,mn} \big(\tfrac{1}{2} \lambda_{(mi)(nj)(1a)(Lk)} - \lambda'_{(nj)(1a),(mi)(Lk)} \big) + n\Lambda_{nl,1m} \bar{\lambda}_{(nj)(1a)(mi)(Lk)} \Big]$$

$$+\sum_{Ab} O^{2,r}_{(Ab),(mi)}\kappa^{-1}_{Ab}\Big[\Lambda_{L,An}\big(\lambda'_{(Ab)(nj),(Lk)} - k^2_{Ab}\lambda_{(Ab)(nj)(Lk)}\big) + an\,\Lambda_{AnL,}\bar{\lambda}_{(Ab)(nj)(Lk)}\Big], \quad (30b)$$

$$O^{2,pr}_{(Lk),(Mi),(nj)} = \frac{2}{\Lambda_{ll}}\sum_a P_a\Big[\Lambda_{1M,ln}\big(k^2_{1a}\lambda_{(Mi)(1a)(nj)(lk)} - \lambda'_{(nj)(1a),(Mi)(lk)} - \lambda'_{(Mi)(1a),(nj)(lk)}\big) + (n\Lambda_{Mn,1l} - M\Lambda_{,1Mnl})\bar{\lambda}_{(1a)(Mi)(nj)(lk)}\Big] + \sum_{Ab}\kappa^{-1}_{ab}\Big[O^{2,p}_{(Ab),(Mi)}\big(\Lambda_{A,nl}(\lambda'_{(Ab)(nj),(lk)} - k^2_{Ab}\lambda_{(Ab)(nj)(lk)}) - An\Lambda_{n,Al}\bar{\lambda}_{(Ab)(nj)(lk)}\big) + O^{2,r}_{(Ab),(nj)}\big(\Lambda_{M,lA}(\lambda'_{(Ab)(Mi),(lk)} - k^2_{Ab}\lambda_{(Ab)(Mi)(lk)}) - AM\Lambda_{A,Ml}\bar{\lambda}_{(Ab)(Mi)(lk)}\Big]. \quad (30c)$$

3.4 First- and second-order approximate Ω_3

The scalar Stokes-Joukowski potential Ω_3 is also a harmonic function satisfying the zero-Neumann conditio at $z = -h$, $r = 1$ and the non-zero Neumann condition (13c) on the free surface. Using the Taylor expansion by ζ, the latter condition takes in the first (linear) approximation the following form

$$\partial_z\Omega_{13} = -\partial_\theta\zeta = \sum_{mi} m\,\mathcal{R}_{mi}\big[\sin(m\theta)p_{Mi} - \cos(m\theta)r_{mi}\big] \text{ at } z = 0.$$

One can then get the following harmonic

$$\Omega_{13} = \sum_{mi} m\,\mathcal{R}_{mi}(r)\mathcal{Z}_{mi}(z)\big[\sin(m\theta)\,p_{mi} - \cos(m\theta)\,r_{mi}\big]. \quad (31)$$

Further, the second-order harmonic approximation of the Neumann boundary condition (13c) takes the form

$$\partial_z\Omega_{23} = -\partial^2_z\Omega_{13}\,\zeta + \partial_r\Omega_{13}\,\partial_r\zeta + r^{-2}\partial_\theta\Omega_{13}\,\partial_\theta\zeta = \sum_{MiNj} p_{Mi}p_{Nj}M\kappa^{-1}_{Mi}\big[\sin(M\theta)\cos(N\theta)\big(\mathcal{R}'_{Mi}\mathcal{R}'_{Nj} - k^2_{Mi}\mathcal{R}_{Mi}\mathcal{R}_{Nj}\big) - MN\cos(M\theta)\sin(N\theta)r^{-2}\mathcal{R}_{Mi}\mathcal{R}_{Nj}\big]$$

$$+\sum_{mnij} r_{mi}r_{nj}m\,\kappa_{mi}^{-1}\big[\cos(n\theta)\sin(n\theta)\big(k_{mi}^2\mathcal{R}_{mi}\mathcal{R}_{nj}-\mathcal{R}'_{mi}\mathcal{R}'_{nj}\big)$$
$$+mn\sin(m\theta)\cos(n\theta)r^{-2}\mathcal{R}_{mi}\mathcal{R}_{nj}\big]+\sum_{Minj}\Big[\mathcal{R}_{Mi}\mathcal{R}_{nj}$$
$$\times\Big(-\frac{Mk_{Mi}^2}{\kappa_{Mi}}\sin(M\theta)\sin(n\theta)+\frac{nk_{nj}^2}{\kappa_{nj}}\cos(M\theta)\cos(n\theta)\Big)$$
$$+\mathcal{R}'_{Mi}\mathcal{R}'_{nj}\Big(\frac{M}{\kappa_{Mi}}\sin(M\theta)\sin(n\theta)-\frac{n}{\kappa_{nj}}\cos(M\theta)\cos(n\theta)\Big)$$
$$+r^{-2}\mathcal{R}_{Mi}\mathcal{R}_{nj}Mn\Big(\frac{M}{\kappa_{Mi}}\cos(M\theta)\cos(n\theta)$$
$$-\frac{n}{\kappa_{nj}}\sin(M\theta)\sin(n\theta)\Big)\Big]\quad\text{at } z=0.$$

Proceeding as in the previous sections derives the Fourier solution

$$\Omega_{23}=\sum_{Lk}\mathcal{R}_{Lk}(r)\mathcal{Z}_{Lk}(z)\cos(L\theta)\Bigg[\sum_{Mnij}O^{3,pr}_{(Lk),(Mi),(nj)}p_{Mi}r_{nj}\Bigg]$$
$$+\sum_{lk}\mathcal{R}_{lk}(r)\mathcal{Z}_{Lk}(z)\sin(l\theta)\Bigg[\sum_{MNij}O^{3,pp}_{(lk),(Mi),(Nj)}p_{Mi}p_{Nj}$$
$$+\sum_{mnij}O^{3,rr}_{(lk),(mi),(nj)}r_{mi}r_{nj}\Bigg], \tag{32}$$

where

$$O^{3,pr}_{(Lk),(Mi),(nj)}=\frac{1}{\Lambda_{LL}}\Big[\lambda_{(Mi)(nj)(Lk)}\Big(-\Lambda_{L,Mn}\frac{Mk_{Mi}^2}{\kappa_{Mi}}+\Lambda_{MLn,}\frac{nk_{nj}^2}{\kappa_{nj}}\Big)$$
$$+\lambda'_{(Mi)(nj),(Lk)}\left(\Lambda_{L,Mn}M\kappa_{Mi}^{-1}-\Lambda_{LMn,}n\kappa_{nj}^{-1}\right)$$
$$+\bar\lambda_{(Mi)(nj)(Lk)}Mn\left(\Lambda_{LMn,}M\kappa_{Mi}^{-1}-\Lambda_{L,Mn}n\kappa_{nj}^{-1}\right)\Big], \tag{33a}$$

$$O^{3,pp}_{(lk),(Mi),(Nj)}=\frac{1}{\Lambda_{ll}}\frac{M}{\kappa_{Mi}}\Big[\Lambda_{N,Ml}\left(-k_{Mi}^2\lambda_{(Mi)(Nj)(lk)}+\lambda'_{(Mi)(Nj),(lk)}\right)$$
$$-MN\,\Lambda_{M,Nl}\bar\lambda_{(Mi)(Nj)(lk)}\Big], \tag{33b}$$

$$O^{3,rr}_{(lk),(mi),(nj)} = \frac{1}{\Lambda_{ll}}\frac{m}{\kappa_{mi}}\Big[\Lambda_{m,nl}\left(k^2_{mi}\lambda_{(mi)(nj)(lk)} - \lambda'_{(mi)(nj),(lk)}\right) + mn\,\Lambda_{n,ml}\bar{\lambda}_{(mi)(nj)(lk)}\Big], \quad (33c)$$

in which we adopted notations from Appendix A.

4 Hydrodynamic moment

According to the Lukovsky formula [1, 4], the resulting (dimensional) hydrodynamic moment (relative to the origin O) may be written down ($\boldsymbol{r}_{lC_0} \times \boldsymbol{g}_0 = \boldsymbol{0}$) as

$$\begin{aligned}\boldsymbol{M}_O &= \left[M_l R_0^2\sigma^2\right]\Big(\boldsymbol{r}_{lC} \times (\boldsymbol{g} - \boldsymbol{\omega}\times\boldsymbol{v}_O - \dot{\boldsymbol{v}}_O) - \boldsymbol{J}^1\cdot\dot{\boldsymbol{\omega}} - \dot{\boldsymbol{J}}^1\cdot\boldsymbol{\omega} \\ &\qquad -\boldsymbol{\omega}\times(\boldsymbol{J}^1\cdot\boldsymbol{\omega}) - \ddot{\boldsymbol{l}}_\omega + \dot{\boldsymbol{l}}_{\omega t} - \boldsymbol{\omega}\times\left(\dot{\boldsymbol{l}}_\omega - \boldsymbol{l}_{\omega t}\right)\Big) \\ &= \left[M_l R_0^2\sigma^2\right]\Big(\boxed{\boldsymbol{r}_{lC_0}\times(\boldsymbol{g}-\boldsymbol{g}_0)} - \boldsymbol{r}_{lC_0}\times\dot{\boldsymbol{v}}_O + (\boldsymbol{r}_{lC} - \boldsymbol{r}_{lC_0})\times\boldsymbol{g}_0 \\ &\qquad - \boldsymbol{J}^1_0\cdot\dot{\boldsymbol{\omega}} - \boxed{(\ddot{\boldsymbol{l}}_\omega - \dot{\boldsymbol{l}}_{\omega t})} + o(\epsilon)\Big) \\ &= \left[M_l R_0^2\sigma^2\right]\left(F_4(t)\,\boldsymbol{e}_1(t) + F_5(t)\,\boldsymbol{e}_2(t) + F_6(t)\,\boldsymbol{e}_3(t) + o(\epsilon)\right), \quad (34)\end{aligned}$$

where $\boldsymbol{J}^1 = J^1_{ij}$ is the non-dimensional liquid inertia tensor ($\boldsymbol{J}^1_0$ is its $O(1)$-order component),

$$J^1_{ij} = \frac{1}{\pi h}\int_{S(t)+\Sigma(t)} \Omega_i\,\partial_n\Omega_j\,dS \quad (35)$$

and

$$\boldsymbol{l}_\omega = \frac{1}{\pi h}\int_{Q(t)} \boldsymbol{\Omega}\,dQ, \quad \boldsymbol{l}_{\omega t} = \frac{1}{\pi h}\int_{Q(t)} \dot{\boldsymbol{\Omega}}\,dQ. \quad (36)$$

The formula (34) contains two framed terms. The first one, $\boldsymbol{r}_{lC_0} \times (\boldsymbol{g}-\boldsymbol{g}_0)$, implies a quasi-static moment relative to O caused by a small instant pivoting of the tank body by η_4 and η_5. The second framed expression, $\ddot{\boldsymbol{l}}_\omega - \dot{\boldsymbol{l}}_{\omega t}$ is a rather complicated function determined by the Stokes-Jukowski potentials. By utilising the Reynolds transport theorem and that the normal relative velocity of the free surface $u_n = \dot{\zeta}/\sqrt{1+(\nabla\zeta)^2}$, the second framed terms re-writes in the form

$$\pi h\,\partial_t\big(\dot{\boldsymbol{l}}_\omega - \boldsymbol{l}_{\omega t}\big) = \partial_t\int_{\Sigma(t)} \boldsymbol{\Omega}\,u_n dS = \partial_t\int_0^1\int_{-\pi}^{\pi} r\left[\boldsymbol{\Omega}|_{z=\zeta}\right]\dot{\zeta}\,d\theta dr. \quad (37)$$

Because $\zeta = O(\epsilon^{1/3})$, getting the $O(\epsilon)$-order component of (37) should adopt the $O(\epsilon^{2/3})$-order solution from the previous section.

The adaptive asymptotic approximation of the hydrodynamic moment also requires the zero-order approximation of the inertia tensor $\boldsymbol{J}_0^1$. Substituting (16) into (35) with $\zeta = 0$ computes $\boldsymbol{J}_0^1 = \{J_{0ij}^1\}$ whose the only non-zero component is

$$J_0 = J_{011}^1 = J_{022}^1 = \frac{h^2}{3} - \frac{3}{4} + \frac{16}{h} \sum_n \frac{\tanh(\frac{1}{2}k_{1n}h)}{k_{1n}^3(k_{1n}^2 - 1)}. \tag{38}$$

4.1 Component $F_4(t)$

Using expressions for the zero- (16), first- (21) and second- (25) order approximations and neglecting the $o(\epsilon)$ terms computes

$$\begin{aligned}
&\int_0^1 \int_{-\pi}^{\pi} r \left(\Omega_{01} + \partial_z \Omega_{01}\zeta + \Omega_{11} + \tfrac{1}{2}\partial_z^2 \Omega_{01}\zeta^2 + \partial_z \Omega_{11}\zeta + \Omega_{21}\right)|_{z=0} \dot{\zeta} d\theta dr \\
&= 2\pi \sum_a \tilde{P}_a \dot{r}_{1a} + \sum_{Njmi} \tilde{O}^{1,pr}_{(Nj),(mi)} \dot{p}_{Nj} r_{mi} + \sum_{Minj} \tilde{O}^{1,rp}_{(nj),(Mi)} \dot{r}_{nj} p_{Mi} \\
&+ \sum_{NjLkmi} \tilde{O}^{1,ppr}_{(Nj),(Lk),(mi)} \dot{p}_{Nj} p_{Lk} r_{mi} + \sum_{njLkMi} \tilde{O}^{1,rpp}_{(nj),(Lk),(Mi)} \dot{r}_{nj} p_{Lk} p_{Mi} \\
&\qquad + \sum_{njlkmi} \tilde{O}^{1,rrr}_{(nj),(lk),(mi)} \dot{r}_{nj} r_{lk} r_{mi},
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
&\tilde{O}^{1,pr}_{(Nj),(mi)} = \Lambda_{N,1m} \sum_a P_a \lambda_{(1a)(mi)(Nj)} + \Lambda_{NN} \kappa_{Nj}^{-1} O^{1,r}_{(Nj),(mi)}, \\
&\tilde{O}^{1,rp}_{(nj),(Mi)} = \Lambda_{M,1n} \sum_a P_a \lambda_{(1a)(Mi)(nj)} + \Lambda_{nn} \kappa_{nj}^{-1} O^{1,p}_{(nj),(Mi)}, \\
&\tilde{O}^{1,ppr}_{(Nj),(Lk),(mi)} = 2\Lambda_{NL,m1} \sum_a \tilde{P}_a \lambda_{(1a)(Nj)(Lk)(mi)} \\
&+ \Lambda_{NN} \kappa_{Nj}^{-1} O^{1,pr}_{(Nj),(Lk),(mi)} + \sum_{Ab} \lambda_{(Ab)(Lk)(Nj)} O^{1,r}_{(Ab),(mi)} \Lambda_{ALN}, \\
&+ \sum_{ab} \lambda_{(ab)(mi)(Nj)} O^{1,p}_{(ab),(Lk)} \Lambda_{N,ma}, \\
&\tilde{O}^{1,rpp}_{(nj),(Lk),(Mi)} = \Lambda_{LM,n1} \sum_a \tilde{P}_a k_{1a}^2 \lambda_{(1a)(nj)(Mi)(Lk)}
\end{aligned}$$

$$+\Lambda_{nn}\kappa_{nj}^{-1}O^{1,pp}_{(nj),(Mi),(Lk)}+\sum_{ab}\lambda_{(ab)(Lk)(nj)}O^{1,p}_{(ab),(Mi)}\Lambda_{L,an},$$

$$\tilde{O}^{1,rrr}_{(nj),(lk),(mi)}=\Lambda_{,1nlm}\sum_{a}\tilde{P}_a k_{1a}^2\lambda_{(1a)(nj)(mi)(lk)}$$

$$+\Lambda_{nn}\kappa_{nj}^{-1}O^{1,rr}_{(nj),(mi),(lk)}+\sum_{Ab}\lambda_{(Ab)(lk)(nj)}O^{1,r}_{(Ab),(mi)}\Lambda_{A,ln}.$$

The moment component $F_4(t)$ is then computed as

$$F_4(t)=g\Big[\tfrac{1}{2}h\,\eta_4-h^{-1}\sum_a P_a r_{1a}\Big]-\tfrac{1}{2}h\ddot{\eta}_2-J_0\ddot{\eta}_4-2h^{-1}\sum_a\tilde{P}_a\ddot{r}_{1a}$$
$$+\sum_{Njmi}M^{1,pr}_{(Nj),(mi)}\ddot{p}_{Nj}r_{mi}+\sum_{njMi}M^{1,rp}_{(nj),(Mi)}\ddot{r}_{nj}p_{Mi}+\sum_{Njmi}N^{1,pr}_{(Nj),(mi)}\dot{p}_{Nj}\dot{r}_{mi}$$
$$+\sum_{LkMinj}M^{1,ppr}_{(Lk),(Mi),(lk)}\ddot{p}_{Lk}p_{Mi}r_{nj}+\sum_{lkMiNj}M^{1,rpp}_{(lk),(Mi),(Nj)}\ddot{r}_{lk}p_{Mi}p_{Nj}$$
$$+\sum_{lkminj}M^{1,rrr}_{(lk),(mi),(nj)}\ddot{r}_{lk}r_{mi}r_{nj}+\sum_{LkMinj}N^{1,ppr}_{(Lk),(Mi),(nj)}\dot{p}_{Lk}\dot{p}_{Mi}r_{nj}$$
$$+\sum_{lkminj}N^{1,rrr}_{(lk),(mi),(nj)}\dot{r}_{lk}\dot{r}_{mi}r_{nj}+\sum_{lkMinj}N^{1,rpp}_{(lk),(Mi),(Nj)}\dot{r}_{lk}\dot{p}_{Mi}p_{Nj},\quad(40)$$

where

$M^{1,pr}_{(Nj),(mi)}=-(\pi h)^{-1}\tilde{O}^{1,pr}_{(Nj),(mi)}$, $M^{1,rp}_{(nj),(Mi)}=-(\pi h)^{-1}\tilde{O}^{1,rp}_{(nj),(Mi)}$,
$N^{1,pr}_{(Nj),(mi)}=-(\pi h)^{-1}\big(\tilde{O}^{1,pr}_{(Nj),(mi)}+\tilde{O}^{1,rp}_{(mi),(Nj)}\big)$,
$M^{1,ppr}_{(Lk),(Mi),(lk)}=-(\pi h)^{-1}\tilde{O}^{1,ppr}_{(Lk),(Mi),(nj)}$,
$M^{1,rpp}_{(lk),(Mi),(Nj)}=-(\pi h)^{-1}\tilde{O}^{1,rpp}_{(lk),(Mi),(Nj)}$,
$M^{1,rrr}_{(lk),(mi),(nj)}=-(\pi h)^{-1}\tilde{O}^{1,rrr}_{(lk),(mi),(nj)}$,
$N^{1,ppr}_{(Lk),(Mi),(nj)}=-(\pi h)^{-1}\tilde{O}^{1,ppr}_{(Lk),(Mi),(nj)}$,
$N^{1,rrr}_{(lk),(mi),(nj)}=-(\pi h)^{-1}\big(\tilde{O}^{1,rrr}_{(lk),(mi),(nj)}+\tilde{O}^{1,rrr}_{(lk),(nj),(mi)}\big)$,
$N^{1,rpp}_{(lk),(Mi),(Nj)}=-(\pi h)^{-1}\big(\tilde{O}^{1,rpp}_{(lk),(Mi),(Nj)}+\tilde{O}^{1,ppr}_{(Mi),(Nj),(lk)}+\tilde{O}^{1,rpp}_{(lk),(Nj),(Mi)}\big)$.

4.2 Component $F_5(t)$

In similar to (39), the l_2-related quantity takes the form

$$\int_0^1\int_{-\pi}^{\pi} r\left(\Omega_{02}+\partial_z\Omega_{02}\zeta+\Omega_{12}+\tfrac{1}{2}\partial_z^2\Omega_{02}\zeta^2+\partial_z\Omega_{12}\zeta+\Omega_{22}\right)|_{z=0}\dot\zeta d\theta dr$$
$$=-2\pi\sum_a\tilde P_a\dot p_{1a}+\sum_{NjMi}\tilde O^{2,pp}_{(Nj),(Mi)}\dot p_{Nj}p_{mi}+\sum_{minj}\tilde O^{2,rr}_{(nj),(mi)}\dot r_{nj}r_{mi}$$
$$+\sum_{NjLkMi}\tilde O^{2,ppp}_{(Nj),(Lk),(Mi)}\dot p_{Nj}p_{Lk}p_{Mi}+\sum_{njlkmi}\tilde O^{2,prr}_{(Nj),(lk),(mi)}\dot p_{Nj}r_{lk}r_{mi}$$
$$+\sum_{njLkmi}\tilde O^{2,rpr}_{(nj),(Lk),(mi)}\dot r_{nj}p_{Lk}r_{mi}, \quad (41)$$

where

$$\tilde O^{2,pp}_{(Nj),(Mi)}=-\Lambda_{1MN,}\sum_a P_a\lambda_{(1a)(Mi)(Nj)}+\Lambda_{NN}\kappa^{-1}_{Nj}O^{2,p}_{(Nj),(Mi)},$$
$$\tilde O^{2,rr}_{(nj),(mi)}=-\Lambda_{,1mn}\sum_a P_a\lambda_{(1a)(mi)(nj)}+\Lambda_{nn}\kappa^{-1}_{nj}O^{2,r}_{(nj),(mi)},$$
$$\tilde O^{2,ppp}_{(Nj),(Lk),(Mi)}=-\Lambda_{NLM1,}\sum_a\tilde P_a k^2_{1a}\lambda_{(1a)(Nj)(Lk)(Mi)}$$
$$+\Lambda_{NN}\kappa^{-1}_{Nj}O^{2,pp}_{(Nj),(Mi),(Lk)}+\sum_{Ab}\lambda_{(Ab)(Lk)(Nj)}O^{2,p}_{(Ab),(Mi)}\Lambda_{ALN,},$$
$$\tilde O^{2,prr}_{(Nj),(lk),(mi)}=-\Lambda_{N1,lm}\sum_a\tilde P_a k^2_{1a}\lambda_{(1a)(Nj)(mi)(lk)}$$
$$+\Lambda_{NN}\kappa^{-1}_{Nj}O^{2,rr}_{(Nj),(mi),(lk)}+\sum_{ab}\lambda_{(ab)(lk)(Nj)}O^{2,r}_{(ab),(mi)}\Lambda_{N,al},$$
$$\tilde O^{2,rpr}_{(nj),(Lk),(mi)}=-2\Lambda_{L1,nm}\sum_a\tilde P_a k^2_{1a}\lambda_{(1a)(Lk)(mi)(nj)}$$
$$+\Lambda_{nn}\kappa^{-1}_{nj}O^{2,pr}_{(nj),(Lk),(mi)}+\sum_{Ab}\Lambda_{A,mn}O^{2,p}_{(Ab),(Lk)}\lambda_{(Ab)(mi)(nj)}$$
$$+\sum_{ab}\Lambda_{L,an}O^{2,r}_{(ab),(mi)}\lambda_{(ab)(Lk)(nj)}.$$

The moment component $F_5(t)$ is then derived as follows

$$F_5(t)=g\left[\tfrac{1}{2}h\eta_5+h^{-1}\sum_a P_a p_{1a}\right]+\tfrac{1}{2}h\ddot\eta_1-J_0\ddot\eta_5-2h^{-1}\sum_a\tilde P_a\ddot p_{1a}$$
$$+\sum_{NjMi}M^{2,pp}_{(Nj),(Mi)}\ddot p_{Nj}p_{mi}+\sum_{njmi}M^{2,rr}_{(nj),(mi)}\ddot r_{nj}p_{mi}$$

$$
\begin{aligned}
&+ \sum_{NjMi} N^{2,pp}_{(Nj),(mi)} \dot{p}_{Nj} \dot{p}_{Mi} + \sum_{njmi} N^{2,rr}_{(nj),(mi)} \dot{r}_{nj} \dot{r}_{mi} \\
&+ \sum_{NjLkNj} M^{2,ppp}_{(Nj),(Lk),(Mi)} \ddot{p}_{Lk} p_{Lk} p_{Mi} + \sum_{Njlkmi} M^{2,prr}_{(Nj),(lk),(mi)} \ddot{p}_{Nj} r_{lki} r_{mi} \\
&+ \sum_{njLkmi} M^{2,rpr}_{(nj),(Lk),(mi)} \ddot{r}_{nj} p_{Lk} r_{mi} + \sum_{NjLkMij} N^{2,ppp}_{(Nj),(Lk),(Mi)} \dot{p}_{Nj} \dot{p}_{Lk} p_{Mi} \\
&+ \sum_{Njlkmi} N^{2,prr}_{(Nj),(lk),(mi)} \dot{p}_{Nj} \dot{r}_{lk} r_{mi} + \sum_{njlkMi} N^{2,rrp}_{(nj),(lk),(Mi)} \dot{r}_{nj} \dot{r}_{lk} p_{Mi},
\end{aligned}
\tag{42}
$$

where

$$
\begin{aligned}
&M^{2,pp}_{(Nj),(Mi)} = N^{2,pp}_{(Nj),(Mi)} = -(\pi h)^{-1} \tilde{O}^{2,pp}_{(Nj),(Mi)}, \\
&M^{2,rr}_{(nj),(mi)} = N^{2,rr}_{(nj),(mi)} = -(\pi h)^{-1} \tilde{O}^{1,rp}_{(nj),(Mi)}, \\
&M^{2,ppp}_{(Nj),(Lk),(Mi)} = -(\pi h)^{-1} \tilde{O}^{2,ppp}_{(Nj),(Lk),(Mi)}, \\
&M^{2,prr}_{(Nj),(lk),(mi)} = -(\pi h)^{-1} \tilde{O}^{2,prr}_{(Nj),(lk),(mi)}, \\
&M^{2,rpr}_{(nj),(Lk),(mi)} = -(\pi h)^{-1} \tilde{O}^{1,rpr}_{(nj),(Lk),(mi)}, \\
&N^{2,ppp}_{(Nj),(Lk),(Mi)} = -(\pi h)^{-1} \Big(\tilde{O}^{2,ppp}_{(Nj),(Lk),(Mi)} + \tilde{O}^{2,ppp}_{(Nj),(Mi),(Lk)} \Big), \\
&N^{2,prr}_{(Nj),(lk),(mi)} = -(\pi h)^{-1} \Big(\tilde{O}^{2,prr}_{(Nj),(lk),(mi)} + \tilde{O}^{2,prr}_{(Nj),(mi),(lk)} \\
&+ \tilde{O}^{2,rpr}_{(lk),(Nj),(mi)} \Big), \quad N^{2,rpr}_{(nj),(lk),(Mi)} = -(\pi h)^{-1} \tilde{O}^{2,rpr}_{(nj),(Mi),(lk)}.
\end{aligned}
$$

4.3 Component $F_6(t)$

The hydrodynamic moment component relative to O is uniquely function of (36), i.e.,

$$
F_6 = -\frac{1}{\pi h} \partial_t \int_0^1 \int_{-\pi}^{\pi} r\, \Omega_3|_{z=\zeta}\, \partial_t \zeta \, d\theta dr. \tag{43}
$$

Using expressions (31) and (32) and neglecting the $o(\epsilon)$ terms computes

$$
\begin{aligned}
&\int_0^1 \int_{-\pi}^{\pi} r \left(\Omega_{13} \dot{\zeta} + \partial_z \Omega_{13} \zeta \dot{\zeta} + \Omega_{23} \dot{\zeta} \right)|_{z=0} d\theta dr \\
&= \pi \sum_{mi} m \kappa^{-1}_{mi} \big(p_{mi} \dot{r}_{mi} - \dot{p}_{mi} r_{mi} \big) + \sum_{LkMinj} \tilde{O}^{3,ppr}_{(Lk),(Mi),(nj)} \dot{p}_{Lk} p_{Mi} r_{nj}
\end{aligned}
$$

$$+\sum_{lkMiNj}\tilde{O}^{3,rpp}_{(lk),(Mi),(nj)}\dot{r}_{lk}p_{Mi}p_{Nj}+\sum_{lkminj}\tilde{O}^{3,rrr}_{(lk),(mi),(nj)}\dot{r}_{lk}r_{mi}r_{nj}, \quad (44)$$

where

$$\tilde{O}^{3,ppr}_{(Lk),(Mi),(nj)}=\frac{\Lambda_{LL}}{\kappa_{Lk}}O^{3,pr}_{(Lk),(Mi),(nj)}+\lambda_{(Lk)(Mi)(nj)}(M\Lambda_{L,Mn}-n\Lambda_{LMn,}), \quad (45a)$$

$$\tilde{O}^{3,rpp}_{(lk),(Mi),(Nj)}=\Lambda_{ll}\kappa_{lk}^{-1}O^{3,pp}_{(lk),(Mi),(Nj)}+\lambda_{(lk)(Mi)(Nj)}M\Lambda_{N,lM}, \quad (45b)$$

$$\tilde{O}^{3,rrr}_{(lk),(mi),(nj)}=\Lambda_{ll}\kappa_{lk}^{-1}O^{3,rr}_{(lk),(mi),(nj)}-\lambda_{(lk)(Mi)(Nj)}m\Lambda_{m,ln}. \quad (45c)$$

Substituting (44) into (43) derives the final expression for the adaptive Lukovsky moment component

$$F_6(t)=-h^{-1}\sum_{mi}m\kappa_{mi}^{-1}\big(p_{mi}\ddot{r}_{mi}-\ddot{p}_{mi}r_{mi}\big)$$
$$+\sum_{LkMinj}M^{3,ppr}_{(Lk),(Mi),(lk)}\ddot{p}_{Lk}p_{Mi}r_{nj}+\sum_{lkMiNj}M^{3,rpp}_{(lk),(Mi),(Nj)}\ddot{r}_{lk}p_{Mi}p_{Nj}$$
$$+\sum_{lkminj}M^{3,rrr}_{(lk),(mi),(nj)}\ddot{r}_{lk}r_{mi}r_{nj}+\sum_{LkMinj}N^{3,ppr}_{(Lk),(Mi),(nj)}\dot{p}_{Lk}\dot{p}_{Mi}r_{nj}$$
$$+\sum_{lkminj}N^{3,rrr}_{(lk),(mi),(nj)}\dot{r}_{lk}\dot{r}_{mi}r_{nj}+\sum_{lkMinj}N^{3,rpr}_{(lk),(Mi),(nj)}\dot{r}_{lk}\dot{p}_{Mi}r_{nj}, \quad (46)$$

where

$$M^{3,ppr}_{(Lk),(Mi),(lk)}=-(\pi h)^{-1}\tilde{O}^{3,ppr}_{(Lk),(Mi),(nj)},$$
$$M^{3,rpp}_{(lk),(Mi),(Nj)}=-(\pi h)^{-1}\tilde{O}^{3,rpp}_{(lk),(Mi),(Nj)},$$
$$M^{3,rrr}_{(lk),(mi),(nj)}=-(\pi h)^{-1}\tilde{O}^{3,rrr}_{(lk),(mi),(nj)},$$
$$N^{3,ppr}_{(Lk),(Mi),(nj)}=-(\pi h)^{-1}\tilde{O}^{3,ppr}_{(Lk),(Mi),(nj)},$$
$$N^{3,rrr}_{(lk),(mi),(nj)}=-(\pi h)^{-1}\big(\tilde{O}^{3,rrr}_{(lk),(mi),(nj)}+\tilde{O}^{3,rrr}_{(lk),(nj),(mi)}\big),$$
$$N^{3,rpp}_{(lk),(Mi),(Nj)}=-(\pi h)^{-1}\big(\tilde{O}^{3,rpp}_{(lk),(Mi),(Nj)}+\tilde{O}^{3,ppr}_{(Mi),(Nj),(lk)}+\tilde{O}^{3,rpp}_{(lk),(Nj),(Mi)}\big).$$

5 Conclusions

The derivation scheme from [2] was generalised to derive adaptive weakly-nonlinear expressions for the resulting hydrodynamic force and moments caused by the pressure loads on the wetted surface of an upright circular container partly filled with a liquid. The expressions follow from the so-called Lukovsky formulas, which give the force and moment written down in terms of the hydrodynamic generalised coordinates. They are consistent with asymptotic relations of the adaptive modal equations in [2]. The forthcoming derivations should focus on simplifications of the weakly-nonlinear expressions to handle the case of the Narimanov-Moiseev–type modal theory [2,3].

A Tensors

To derive the adaptive modal equations, [2] introduced a set of tensors, which imply an algebra in the angular and radial coordinates. The first, Λ-type tensor reads as

$$\Lambda_{M...N,i...j} = \int_{-\pi}^{\pi} \cos(M\theta)...\cos(N\theta)\cdot\sin(i\theta)...\sin(j\theta)\,d\theta, \tag{47}$$

whose computations are best done by using the recursive formulas

$$\begin{aligned}
&\Lambda_{M,i} = 0;\ \Lambda_{ij} = \Lambda_{,ij} = \Lambda_{ij,} = \pi\delta_{ij};\\
&\Lambda_{MN,} = \pi\delta_{MN}\,(M+N\neq 0),\ \Lambda_{MN,} = 2\pi\delta_{MN}\,(M=N=0);\\
&\Lambda_{M...NK,i...j} = \tfrac{1}{2}\left(\Lambda_{M...|N-K|,i...j} + \Lambda_{M...|N+K|,i..j}\right);\\
&\Lambda_{M...N,i...jkl} = \tfrac{1}{2}\left(\Lambda_{M...|j-k|,i...l} - \Lambda_{M...|j+k|,i..l}\right).
\end{aligned}$$

The second, the radial components and functions $\mathcal{R}_{Mi}(r)$ yield the λ-tensors:

$$\lambda_{(Mi)...(Nk)} = \int_0^1 r\mathcal{R}_{Mi}(r)...\mathcal{R}_{Nk}(r)\,dr, \tag{48}$$

$$\lambda'_{(Mi)(Nk),(Cd)...(Ef)} = \int_0^1 r\mathcal{R}'_{Mi}(r)\mathcal{R}'_{Nk}(r)\cdot\mathcal{R}_{Cd}(r)...\mathcal{R}_{Ef}(r)\,dr, \tag{49}$$

$$\bar{\lambda}_{(Mi)...(Nk)} = \int_0^1 \frac{1}{r^2}\mathcal{R}_{Mi}(r)...\mathcal{R}_{Nk}(r)\,dr. \tag{50}$$

B Differentiation rules

The analysis suggests the hydrodynamic moments relative to axes of the Cartesian coordinate system $Oxyz$ (with the coordinate units $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$ and

$\hat{\boldsymbol{z}}$) but the natural sloshing modes and associate derivations deal with the cylindrical coordinates $Or\theta z$ (the coordinate units $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$). This implies the gradients

$$\nabla_{xyz} = \hat{\boldsymbol{x}}\,\partial_x + \hat{\boldsymbol{y}}\,\partial_y + \hat{\boldsymbol{z}}\,\partial_z, \quad \nabla_{r\theta z} = \hat{\boldsymbol{r}}\,\partial_r + \hat{\boldsymbol{\theta}}\,r^{-1}\partial_\theta + \hat{\boldsymbol{z}}\,\partial_z, \tag{51}$$

the differentiation rule

$$\partial_x = \cos\theta\,\partial_r - r^{-1}\sin\theta\,\partial_\theta, \quad \partial_y = \sin\theta\,\partial_r + r^{-1}\cos\theta\,\partial_\theta, \tag{52}$$

which, in particular, deduce the expression

$$\nabla_{xyz}[z - \zeta(r,\theta)] = \hat{\boldsymbol{x}}\,[-\cos\theta\,\partial_r\zeta + r^{-1}\sin\theta\,\partial_\theta\zeta] \\ + \hat{\boldsymbol{y}}\,[-\sin\theta\,\partial_r\zeta - r^{-1}\cos\theta\,\partial_\theta\zeta] + \hat{\boldsymbol{z}}\,[1] \tag{53}$$

used to identify the outer normal unit on the free surface $\Sigma(t)$.

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