Збірник праць Інституту математики НАН України 2015, т. 12, № 5, 267–274

УДК 532.595

The Bateman–Luke variational formalism for sloshing of an ideal incompressible liquid with rotational flows*

A. N. Timokha

Institute of Mathematics of NAS of Ukraine, Kiev, Ukraine;
Centre of Excellence AMOS, Norwegian University of Science and Technology, Trondheim, Norway

The Bateman–Luke variational principle is generalised for sloshing of an ideal incompressible liquid with rotational (non-potential) flows.

Варіаційний принцип Бейтмена–Люка узагальнюється на задачі динаміки ідеальної нестисливої рідини в баках у випадку вихорових (непотенційних) течій.

1. Introduction

Utilising the Bateman–Luke variational principle [1, 6] is a commonly–accepted approach in analytical studies of the nonlinear sloshing [4,7,10]. The studies normally deal with irrotational (potential) flows of an ideal incompressible liquid. When no significant wave breaking occurs, choosing this hydrodynamic model is, generally, supported by experiments for clean (without internal structures) tanks. However, there are exceptions exemplified in experimental observations of Prandtl [8], Hutton [5], and Royon–Lebeaud *et al.* [9] where the resonant steady–state swirl sloshing is accompanied by a circulation (rotational liquid motions like a rigid

* The work was partly supported by the Grant № 0112U001015. The author also acknowledges the financial support of the Centre of Autonomous Marine Operations and Systems (AMOS) whose main sponsor is the Norwegian Research Council (Project number 223254–AMOS).

body). This phenomenon cannot be explained by the angular Stokes drift (mean flow due to the Stokes drift exponentially decays to the tank bottom [4, sect. 9.6.3] that is not detected in the experiments) but rather requires including the rotational (non-potential) flows in the hydrodynamic model.

Using the Clebsch potentials [2, 3] and ideas by Bateman [1, p. 164-166], the present paper generalises the Bateman–Luke variational formalism, which is well known for sloshing of an ideal incompressible liquid with irrotational flows, to the case of *solenoidal (rotational)* liquid motions.

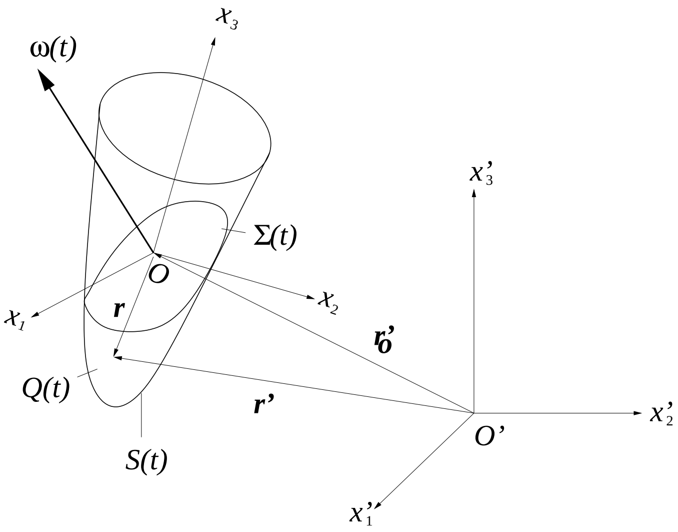

Fig. 1. Sketch of a moving tank. Nomenclature.

2. Notations

A mobile rigid tank is considered partly filled with an inviscid incompressible liquid (the mass density $\rho = \text{const}$). Fig. 1 shows the liquid domain $Q(t)$ bounded by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$, an absolute (inertial) coordinate system $O'x_1'x_2'x_3'$, and a non-inertial (tank-fixed) coordinate system $Ox_1x_2x_3$. The $Ox_1x_2x_3$-system moves (relatively to $O'x_1'x_2'x_3'$) with the absolute translatory velocity $\boldsymbol{v}_O(t)$ and the instant angular velocity $\boldsymbol{\omega}(t)$ so that any fixed point in the $Ox_1x_2x_3$

has the *absolute velocity*

$$\boldsymbol{v}_b = \boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r} \tag{1}$$

where $\boldsymbol{r} = (x_1, x_2, x_3)$ is the tank-fixed radius–vector.

The *gravity potential* can be written as

$$U(x_1, x_2, x_3, t) = -\boldsymbol{g} \cdot \boldsymbol{r}', \quad \boldsymbol{r}' = \boldsymbol{r}'_O + \boldsymbol{r},$$

where $\boldsymbol{r}'$ is the radius–vector of a point of the body–liquid system with respect to O', $\boldsymbol{r}'_O$ is the radius–vector of O with respect to O', and $\boldsymbol{g}$ is the gravity acceleration vector.

The free surface $\Sigma(t)$ is implicitly defined in the tank-fixed coordinate system by the equation $Z(x_1, x_2, x_3, t) = 0$ so that the outer normal $\boldsymbol{n}$ to $\Sigma(t)$ is $-\nabla Z/|\nabla Z|$. The function Z is the unknown and satisfies the volume (mass) conservation condition

$$\int_{Q(t)} dQ = V_l = \text{const} \tag{2}$$

treated as a *geometric constraint.*

The liquid motions are described by the three Clebsch potentials $\varphi(x_1, x_2, x_3, t)$, $m(x_1, x_2, x_3, t)$, and $\phi(x_1, x_2, x_3, t)$ so that the *absolute velocity field* $\boldsymbol{v} = (v_1, v_2, v_3, t)$ reads as

$$\boldsymbol{v} = \nabla\varphi + m\nabla\phi. \tag{3}$$

Even though (3) does not give a unique representation of the velocity field (substitution $m := C\,m$, $\phi := \phi/C$, where C is a non-zero constant, confirms that), the Clebsch potentials are henceforth assumed being three independent functions. The case of irrotational flows implies either $m = 0$ or $\phi = \text{const}$.

Remark 2.3. As remarked in [4, p. 47], the spatial derivatives in the introduced inertial (∂'_i) and non-inertial (∂_i) coordinate systems remain the same, but the time-derivatives (∂'_t and ∂_t, respectively) change, i.e.

$$\partial'_i = \partial_i; \;\; \partial'_t = \partial_t - \boldsymbol{v}_b \cdot \nabla; \;\; d'_t = \partial'_t + \boldsymbol{v} \cdot \nabla = \partial_t + (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \nabla. \tag{4}$$

3. The Bateman–Luke variational formulation

Based on relations (4) and [1, p. 164], the following Lagrangian

$$L(\varphi, m, \phi, Z) = \int_{Q(t)} P\, dQ = -\rho \int_{Q(t)} \left[\partial_t' \varphi + m\, \partial_t' \phi + \tfrac{1}{2}|\boldsymbol{v}|^2 + U\right] dQ$$
$$= -\rho \int_{Q(t)} \left[\partial_t \varphi + m\, \partial_t \phi - \boldsymbol{v}_b \cdot \boldsymbol{v} + \tfrac{1}{2}|\boldsymbol{v}|^2 + U\right] dQ \quad (5)$$

and the action

$$W(\varphi, m, \phi, Z) = \int_{t_1}^{t_2} \left[L - p_0 \int_{Q(t)} dQ\right] dt = \int_{t_1}^{t_2} \int_{Q(t)} (P - p_0)\, dQ\, dt \quad (6)$$

are introduced for any fixed instant times $t_1 < t_2$. The action functional (6) acts on the independent Clebsch potentials and Z. The Lagrange multiplier p_0 is a consequence of the volume conservation constraint (2).

Henceforth, the **assumption** is that the Clebsch potentials are smooth functions in $Q(t)$ which admit, for any instant time t, an analytical continuation through the smooth (provided by the admissible Z) free surface $\Sigma(t)$.

Lemma 3.1. *Under the assumption on the smoothness of the Clebsch potentials and the free surface $\Sigma(t)$, the zero first variation*

$$\delta_\varphi W = 0 \quad \textit{subject to} \quad \delta\varphi|_{t=t_1,t_2} = 0 \quad (7)$$

is equivalent to the kinematic relations of the sloshing problem consisting of the continuity equation

$$\boxed{\nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_b) \equiv \nabla \cdot \boldsymbol{v} = 0 \quad in \quad Q(t)} \quad (8)$$

as well as the kinematic boundary conditions

$$\boxed{(\boldsymbol{v} - \boldsymbol{v}_b) \cdot \boldsymbol{n} = 0 \quad on\ S(t)}, \quad \boxed{(\boldsymbol{v} - \boldsymbol{v}_b) \cdot \boldsymbol{n} = -\frac{\partial_t Z}{|\nabla Z|} \quad on \quad \Sigma(t)} \quad (9)$$

expressing that the normal velocity is defined by the rigid wall motions and the fluid particles remain on the free surface $\Sigma(t)$.

Proof. Deriving the first variation by φ is similar (but not the same) to that for the potential flows [4, p. 58-59]. Consequently using the Reynolds transport theorem, the divergence theorems, and the condition $\delta\varphi|_{t=t_1,t_2} = 0$ gives

$$\delta_\varphi W = -\rho \int_{t_1}^{t_2} \int_{Q(t)} (\partial_t(\delta\varphi) + (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \nabla(\delta\varphi))\, dQdt$$
$$= -\rho \int_{t_1}^{t_2} \left(\left[\frac{d}{dt} \int_{Q(t)} \delta\varphi\, dQ + \int_{\Sigma(t)} \frac{\partial_t Z}{|\nabla Z|} \delta\varphi\, dS \right] \right.$$
$$\left. + \left[\int_{S(t)+\Sigma(t)} (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \boldsymbol{n}\, \delta\varphi\, dS - \int_{Q(t)} \nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_b)\, \delta\varphi\, dQ \right] \right) dt$$
$$= -\rho \int_{t_1}^{t_2} \left(\int_{\Sigma(t)} \left[(\boldsymbol{v} - \boldsymbol{v}_b) \cdot \boldsymbol{n} + \frac{\partial_t Z}{|\nabla Z|} \right] \delta\varphi\, dS \right.$$
$$\left. + \int_{S(t)} [(\boldsymbol{v} - \boldsymbol{v}_b) \cdot \boldsymbol{n}]\, \delta\varphi\, dS - \int_{Q(t)} [\nabla \cdot (\boldsymbol{v} - \boldsymbol{v}_b)]\, \delta\varphi\, dQ \right) = 0 \quad (10)$$

which deduces (8) and (9) by using the standard calculus of variables. □

Lemma 3.2. *Under the assumption on the smoothness of the Clebsch potentials and the free surface $\Sigma(t)$, the zero first variation*

$$\delta_m W = 0 \tag{11}$$

is equivalent to the equation

$$\boxed{d'\phi \equiv \partial_t'\phi + \boldsymbol{v} \cdot \nabla\phi \equiv \partial_t\phi + (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \nabla\phi = 0 \quad in \quad Q(t)} \tag{12}$$

which says that the Clebsch potential ϕ remains constant during the motions of a liquid particle (a vortex line moves with the liquid and always contains the same particles).

Proof. The variation by m derives the variational equality

$$\delta_m W = -\rho \int_{t_1}^{t_2} \int_{Q(t)} [\partial_t\phi + (\boldsymbol{v} - \boldsymbol{v}_b) \cdot \nabla\phi]\, \delta m\, dQdt = 0 \tag{13}$$

which proves the lemma. □

Lemma 3.3. *Under the assumption on the smoothness of the Clebsch potentials and the free surface $\Sigma(t)$, the zero first variation*

$$\delta_\phi W = 0 \quad subject\ to \quad \delta\phi|_{t_1,t_2} = 0 \tag{14}$$

and the kinematic problem (8), (9) is equivalent to

$$\boxed{d'm \equiv \partial'_t m + \boldsymbol{v}\cdot\nabla m \equiv \partial_t m + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla m = 0 \quad in \quad Q(t)} \tag{15}$$

which has the same meaing that (12) but for the Clebsch potential m.

Proof. The first variation by ϕ reads as

$$\delta_\phi W = -\rho\int_{t_1}^{t_2}\int_{Q(t)} m\left(\partial_t(\delta\phi) + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla(\delta\phi)\right)dQ\,dt$$

$$= -\rho\int_{t_1}^{t_2}\left(\left[\frac{d}{dt}\int_{Q(t)} m\,\delta\phi\,dQ - \int_{Q(t)}\partial_t m\,\delta\phi\,dQ + \int_{\Sigma(t)}\frac{\partial_t Z}{|\nabla Z|}m\,\delta\phi dS\right]\right.$$
$$+\left[\int_{S(t)+\Sigma(t)} m\,(\boldsymbol{v}-\boldsymbol{v}_b)\cdot\boldsymbol{n}\,\delta\phi\,dS\right.$$
$$\left.\left.-\int_{Q(t)}\delta\phi\,\left(m\,\nabla\cdot(\boldsymbol{v}-\boldsymbol{v}_b) + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla m\right)\,dQ\right]\right)dt$$
$$= \rho\int_{t_1}^{t_2}\int_{Q(t)}\delta\phi\,\left[\partial_t m + (\boldsymbol{v}-\boldsymbol{v}_b)\cdot\nabla m\right]dQ\,dt = 0 \tag{16}$$

where the Reynolds transport theorem, the divergence theorems, the zero variation condition (14) at $t = t_1$ and t_2, and the kinematic conditions (8) and (9) were used. The last line of variational equatility (16) proves the lemma. □

Remark 3.4. In contrast to the Bateman–Luke formulation for potential flows, the function P adopted in definition of the Lagrangian (5) is, generally speaking, not the pressure and cannot be treated as the pressure for arbitrary Clebsch potentials. One can show that, the pressure $p = P + f(t)$ ($f(t)$ is an arbitrary function) when (12) and (15) are satisfied. In other words, when assuming (12) and (15), the Euler equation

$$d'\boldsymbol{v} = -\frac{1}{\rho}\left(\nabla P + \nabla U\right) \quad \text{in} \quad Q(t) \tag{17}$$

is formally fulfilled. This fact follows from the expression for the left-hand side of (17)

$$d'(\nabla\varphi + m\nabla\phi) = [\nabla(\partial_t'\varphi) + m\nabla(\partial_t'\phi) + \partial_t' m\nabla\phi] + \underbrace{\boldsymbol{v}\cdot\nabla(\nabla\varphi + m\nabla\phi)}_{\boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla\phi(\nabla m\cdot\boldsymbol{v})}$$

$$= \boxed{\nabla(\partial_t'\varphi) + m\nabla(\partial_t'\phi) + \boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla\phi(\nabla m\cdot\boldsymbol{v})} + \nabla\phi\,[d'm]$$

and the right-hand side (after annihilliating the U-term)

$$\nabla(\partial_t'\varphi + m\partial_t'\phi + \tfrac{1}{2}|\boldsymbol{v}|^2) = [\nabla(\partial_t'\varphi) + m\nabla(\partial_t'\phi) + \partial_t'\phi\nabla m] + \boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla m(\nabla\phi\cdot\boldsymbol{v})$$

$$= \boxed{\nabla(\partial_t'\varphi) + m\nabla(\partial_t'\phi) + \boldsymbol{v}\cdot\nabla\nabla\varphi + m\boldsymbol{v}\cdot\nabla\nabla\phi + \nabla\phi(\nabla m\cdot\boldsymbol{v})} + \nabla m\,[d'\phi],$$

in which the framed terms are identical but the residual terms vanish as (12) and (15) hold true.

Theorem 3.1. *Under the assumption on the smoothness of the Clebsch potentials and the free surface $\Sigma(t)$, the zero first variation of the action (6)*

$$\delta W = \delta_\varphi W + \delta_m W + \delta_\phi W + \delta_Z W = 0 \tag{18}$$

subject to

$$\delta\varphi|_{t_1,t_2} = \delta\phi|_{t_1,t_2} = 0 \tag{19}$$

is equivalent to the sloshing problem which includes the kinematic relations (8) and (9), the two equations (12) and (15) expressing the fact that the Clebsch potentials ϕ and m are constant along the vortex lines as well as the dynamic boundary condition

$$\boxed{p - p_0 = -\rho\left(\partial_t\varphi + m\,\partial_t\phi - \boldsymbol{v}_b\cdot\boldsymbol{v} + \tfrac{1}{2}|\boldsymbol{v}|^2 + U\right) - p_0 = 0 \quad \textit{on } \Sigma(t)} \tag{20}$$

establishing that the pressure equals to the ullage pressure p_0 on the free surface. The volume conservation condition (2) should be added to the sloshing problem.

Proof. The proposition follows from Lemmas 3.1, 3.2 and 3.3, the remark 3.4 establishing that P in the Lagrangian can be treated as the pressure p, and the variational equality

$$\delta_Z W = -\int_{t_1}^{t_2}\int_{\Sigma(t)} (p - p_0)\frac{\delta Z}{|\nabla Z|}\,dQ\,dt = 0 \tag{21}$$

which derives the dynamic boundary condition (20). □

4. Conclusions

Utilising the Clebsch potentials and the Bateman–Luke principle (*the Lagrangian is a "pressure integral"*) makes it possible to derive the full set of governing equations (8), (12), (15) and boundary conditions (9), (20) for sloshing of an ideal incompressible liquid with *rotational flows*. Specifically, the principle (integrand in the Lagrangian (5) is really the pressure) holds true if and only if the vorticity equations (12) and (15) are *a priori* satisfied. The generalised Bateman–Luke formulation can be a background for the nonlinear multimodal method whose second component is the analytically–approximate natural sloshing modes for the rotational sloshing flows.

[1] *Bateman H.* Partial differential equations of mathematical physics.— New York: Dover Publications, 1944.— 420 p.

[2] *Clebsch A.* Über die allgemeine Transformation der hydrodynamischen Gleichungen // J. Reine Angew. Math.— 1857.— **54**.— P. 293–313.

[3] *Clebsch A.* Über die Integration der hydrodynamischen Gleichungen // J. Reine Angew. Math.— 1869.— **56**.— P. 1–10.

[4] *Faltinsen O. M., Timokha A. N.* Sloshing.— Cambridge University Press, 2009.— 686 p.

[5] *Hutton R. E.* Fluid-particle motion during rotary sloshing // Journal of Applied Mechanics, Transactions ASME.— 1964.— **31**, 1.— P. 145–153.

[6] *Luke J. G.* A variational principle for a fluid with a free surface // Journal of Fluid Mechanics.— 1967.— **27**.— P. 395–397.

[7] *Lukovsky I. A.* Nonlinear Dynamics: Mathematical Models for Rigid Bodies with a Liquid.— De Gruyter, 2015.— 400 p.

[8] *Prandtl L.* Erzeugung von Zirkulation beim Schütteln von Gefässen // ZAMM.— 1949.— **29**, 1/2.— P. 8–9.

[9] *Royon-Lebeaud A., Hopfinger E., Cartellier A.* Liquid sloshing and wave breaking in circular and square-base cylindrical containers // Journal of Fluid Mechanics.— 2007.— **577**.— P. 467–494.

[10] *Takahara H., Kimura K.* Frequency response of sloshing in an annular cylindrical tank subjected to pitching excitation // Journal of Sound and Vibration.— 2012.— **331**, 13.— P. 3199–3212.