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The Narimanov–Moiseev modal equations for sloshing in an annular tank*

A. N. Timokha

Institute of Mathematics of NAS of Ukraine, Kiev, Ukraine;
Centre of Excellence AMOS, Norwegian University of Science and Technology, Trondheim, Norway

A complete weakly-nonlinear modal system of the Narimanov–Moiseev type is derived for sloshing in an upright annular tank by using the derivation scheme proposed by the author for a spherical tank. The modal system couples the two dominant generalised coordinates responsible for the lowest natural sloshing modes and an infinite set of the second- and third-order generalised coordinates.

Виводиться повна слабо-нелінійна модальна система типу Наріманова-Моісєєва, яка описує коливання рідини у вертикальному баці кільцевого перерізу. Використовується схема виводу, яку автор запропонував для сферичного баку. Модальна система пов'язує дві домінантні узагальнені координати, що відповідають за перші власні форми коливання рідини, та нескінченну множину узагальнених координат другого та третього порядків малості.

1. Introduction

The annular upright tank belongs to the historically-first reservoir shapes for which weakly-nonlinear modal systems describing liquid sloshing due to a resonant excitation of the lowest natural sloshing frequencies were derived. Interested readers can find the state-of-the-art of 1990 and 2015 in [5] and [6], respectively. Derivation of these systems adopts,

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normally, the Narimanov–Moiseev asymptotic relationships between the sloshing-related generalised coordinates. For axisymmetric basins, the *complete* weakly-nonlinear modal system of the Narimanov–Moisev type should couple two dominant, first-order generalised coordinates governing the lowest natural sloshing modes amplification and an infinite set of the second- and third-order sloshing-related generalised coordinates [4]. Getting the complete systems is a rather complicated analytical task. As a result, those exist only for the upright circular cylindrical [3] and spherical [2] tanks. The existing Narimanov–Moiseev modal systems for the upright annular tank are almost fully represented by the five-dimensional (three generalised coordinates of the second order) system by Lukovsky [5, 6] and the fourteen-dimensional (six generalised coordinates of the second and third-order, respectively) system by Takahara & Kimura [9]. Using the analytical procedure from [2], the present paper derives the *complete infinite-dimensional* weakly-nonlinear Narimanov–Moiseev modal system for this tank shape. The hydrodynamic coefficients of this system are validated by comparing them with those by Lukovsky [5, 6].

2. Natural sloshing modes and frequencies

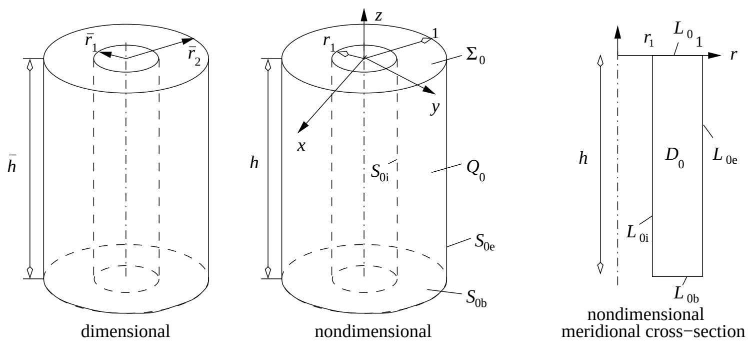

Fig. 1. Dimensional and nondimensional sketches of the mean liquid domain Q_0 in an upright annular tank. The nondimensional mean liquid depth is h, the internal radius equals to r_1 and the external radius is equal to the unit. The mean free surface is Σ_0, the wetted inner and external walls are S_{0i} and S_{0e}, respectively, but S_{0b} is the bottom. The corresponding boundaries in the meridional cross-section are denoted by the L_{0*} symbols.

The nonlinear multimodal method involves the analytically–found natural sloshing modes associated with eigenfunctions of the spectral boundary problem

$$\nabla^2\varphi = 0 \text{ in } Q_0, \quad \frac{\partial\varphi}{\partial n} = 0 \text{ on } S_{0e}, S_{0i}, S_{0b}, \quad \frac{\partial\varphi}{\partial n} = \kappa\varphi \text{ on } \Sigma_0 \qquad (1)$$

in notations of fig. 1. The geometric parameters are normalised by the larger radius $\bar{r}_2$. The $\bar{r}_2$-scaled spectral boundary problem (1) has the analytical solution [1, 6] which is obtained by using separation of spatial variables in the cylindrical coordinate system

$$\varphi_{Mi}(r, z, \theta) = \mathcal{R}_{Mi}(r)\, \mathcal{Z}_{Mi}(z) \begin{smallmatrix}\cos M\theta\\ \sin M\theta\end{smallmatrix}, \quad M = 0, \ldots; \, i = 1, \ldots, \qquad (2)$$

where

$$\mathcal{R}_{Mi}(r) = \alpha_{Mi} \det \begin{vmatrix} J_M(k_{Mi}r) & Y_M(k_{Mi}r) \\ J'_M(k_{Mi}) & Y'_M(k_{Mi}) \end{vmatrix}, \qquad (3a)$$

$$\mathcal{Z}_{Mi}(z) = \frac{\cosh(k_{Mi}(z + h))}{\cosh(k_{Mi}h)}. \qquad (3b)$$

Here, $J_M(\cdot)$ and $Y_M(\cdot)$ are the Bessel functions of the first and second kinds, respectively, the radial wave numbers k_{Mi} are computed from the equations $\mathcal{R}'_{M,i}(r_1) = 0$, and the normalising multipliers α_{Mi} follow from the orthogonality condition [1]

$$\lambda_{(Mi)(Mj)} = \int_{r_1}^{1} r\, \mathcal{R}_{Mi}(r)\, \mathcal{R}_{Mj}(r)\, dr = \delta_{ij}, \quad i, j = 1, \ldots, \qquad (4)$$

where δ_{ij} is the Kronecker delta. The wave numbers and multipliers can be found analytically, by using the Bessel function algebra, or numerically. The spectral parameter κ_{Mi} and the natural sloshing frequencies σ_{Mi} are

$$\kappa_{Mi} = k_{Mi} \tanh(k_{Mi}h), \quad \sigma^2_{Mi} = \kappa_{Mi}\, \bar{g}/\bar{r}_2 = \kappa_{Mi}\, g, \qquad (5)$$

where $\bar{g}$ is the gravity acceleration.

The limit case $r_1 = 0$ (circular cylindrical tank) implies replacing (3a) with $\mathcal{R}_{Mi} = \alpha_{Mi} J_M(k_{Mi}r)$ but other formulas remain the same.

3. The Stokes–Joukowski potentials

The linearised Stokes–Joukowski potentials $\Omega_{0i}(r, z, \theta)$, $i = 1, 2, 3$ are harmonic functions satisfying the Neumann boundary conditions [1, Sect. 5.4.4]:

$$\frac{\partial\Omega_{01}}{\partial n} = -(zn_r - rn_z)\sin\theta, \; \frac{\partial\Omega_{02}}{\partial n} = (zn_r - rn_z)\cos\theta, \; \frac{\partial\Omega_{03}}{\partial n} = 0 \quad (6)$$

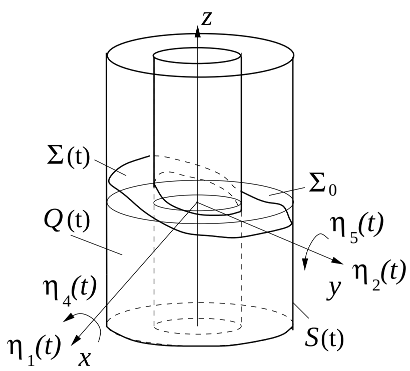

Fig. 2. The liquid volume evolution $Q(t)$ with the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$ are considered in the tank-fixed coordinate system $Oxyz$ whose coordinate plane Oxy coincides with the mean free surface Σ_0 but Oz is the symmetry axis. Small-magnitude tank motions are governed by the generalised coordinates $\eta_1(t)$ (surge), $\eta_4(t)$ (roll), $\eta_2(t)$ (sway), and $\eta_5(t)$ (pitch). The heave and yaw motions are not considered.

on Σ_0, S_{0i}, S_{0e}, and S_{0b} (see, notations in fig. 1). This implies $\Omega_{01} = -F(r,z)\sin\theta$, $\Omega_{02} = F(r,z)\cos\theta$, $\Omega_{03} = 0$, where

$$F(r,z) = rz + \sum_{n=1}^{\infty} c_n \mathcal{R}_{1n}(r) \frac{\sinh(k_{1n}(z+\frac{h}{2}))}{\cosh(k_{1n}\frac{h}{2})}, \tag{7a}$$

$$c_n = -\frac{2}{k_{1n}} P_n, \quad P_n = \int_{r_1}^{1} r^2 \, \mathcal{R}_{1n}(r)\, dr. \tag{7b}$$

Again, c_n can be computed in both analytical and numerical ways.

4. Statement of the problem

We consider sloshing in an upright annular rigid tank performing small-magnitude sway, surge, roll, and pitch motions (heave and yaw are not considered) which are described by the generalised coordinates $\eta_i(t) = O(\epsilon) \ll 1$, $i = 1, 2, 4, 5$ responsible for the tank-fixed coordinate system motions. The geometric notations are given in fig. 2. An inviscid contained liquid with irrotational flows is assumed. The surface patterns $\Sigma(t)$ are governed by $z = \zeta(r, \theta, t)$ and the sloshing flows are defined by velocity potential $\Phi(r,\theta,z,t)$; the both unknowns are given in the tank-fixed, $Oxyz$-equivalent cylindrical coordinate system. The unknowns ζ and Φ should be found from the corresponding free-surface problem or, alternatively, from a variational principle (see, [1, 6] and references therein). Furthermore, the modal solution is used (the natural sloshing modes (2) and their projections on Σ_0 constitute a Fourier basis):

$$\zeta(r,z,\theta) = \sum_{M,i}^{I_a,I_r} \mathcal{R}_{Mi}(r)\cos(M\theta)\,p_{Mi}(t) + \sum_{m,i}^{I_a,I_r} \mathcal{R}_{mi}(r)\cos(m\theta)\,r_{mi}(t), \quad (8a)$$

$$\begin{aligned}\Phi(r,\theta,z,t) = {}& \dot\eta_1(t)\,r\cos\theta + \dot\eta_2(t)\,r\sin\theta + F(r,z)[-\dot\eta_4(t)\sin\theta + \dot\eta_5(t)\cos\theta] \\ &+ \sum_{M,i}^{I_a,I_r} \mathcal{R}_{Mi}(r)\,\mathcal{Z}_{Mi}(z)\,\cos(M\theta)\,P_{Mi}(t) \\ &+ \sum_{m,i}^{I_a,I_r} \mathcal{R}_{mi}(r)\,\mathcal{Z}_{mi}(z)\,\sin(m\theta)\,R_{mi}(t), \quad I_a, I_r \to \infty \quad (8b)\end{aligned}$$

(*henceforth*, the capital summation letter implies the change from zero to I_a but the lower case indices mean the change from one to either I_a or I_r) which introduces the sloshing-related nondimenional generalised coordinates $O(\epsilon) \lesssim p_{Mi}(t), r_{mi}(t)$ and velocities $O(\epsilon) \lesssim P_{mi}(t), R_{mi}(t)$. The $o(\epsilon)$-quantities associated with the nonlinear Stokes–Joukowski potential are omitted so that (8b) linearly depends on $\eta_i(t)$ as in the linear sloshing theories (see, details in [1, Ch. 7 and 5]).

Using the Bateman–Luke variational principle, the books [1,5,6] derive the fully-nonlinear modal system with respect to the generalised coordinates and velocities. For axisymmetric tanks, the modal system can be rewritten [2] in the form

$$\begin{aligned}\sum_{M,n}^{I_a,I_r} \frac{\partial A^p_{Ab}}{\partial p_{Mn}}\dot p_{Mn} + \sum_{m,n}^{I_a,I_r} \frac{\partial A^p_{Ab}}{\partial r_{mn}}\dot r_{mn} = {}& \sum_{M,n}^{I_a,I_r} A^{pp}_{(Ab)(Mn)} P_{Mn} \\ &+ \sum_{m,n}^{I_a,I_r} A^{pr}_{(Ab),(Mn)} R_{mn}, \quad (9a)\end{aligned}$$

$$\sum_{M,n}^{I_a,I_r} \frac{\partial A^r_{ab}}{\partial p_{Mn}} \dot{p}_{Mn} + \sum_{m,n}^{I_a,I_r} \frac{\partial A^r_{ab}}{\partial r_{mn}} \dot{r}_{mn} = \sum_{M,n}^{I_a,I_r} A^{pr}_{(Mn),(ab)} P_{Mn}$$

$$+ \sum_{m,n}^{I_a,I_r} A^{rr}_{(ab)(mn)} R_{mn}, \quad A = 0, \ldots; \quad a, b = 1, \ldots; \quad I_a, I_r \to \infty \quad \text{(9b)}$$

(the kinematic subsystem) and

$$\sum_{M,n}^{I_a,I_r} \frac{\partial A^p_{Mn}}{\partial p_{Ab}} \dot{P}_{Mn} + \sum_{m,n}^{I_a,I_r} \frac{\partial A^r_{mn}}{\partial p_{Ab}} \dot{R}_{mn} + \frac{1}{2} \sum_{ML,nk}^{I_a,I_r} \frac{\partial A^{pp}_{(Mn)(Lk)}}{\partial p_{Ab}} P_{Mn} P_{Lk}$$

$$+ \sum_{Ml,nk}^{I_a,I_r} \frac{\partial A^{pr}_{(Mn),(lk)}}{\partial p_{Ab}} P_{Mn} R_{lk} + \frac{1}{2} \sum_{ml,nk}^{I_a,I_r} \frac{\partial A^{rr}_{(mn)(lk)}}{\partial P_{Ab}} R_{mn} R_{lk} + g \Lambda_{AA,} p_{Ab}$$

$$+ (\ddot{\eta}_1 - g\eta_5 - S_b \ddot{\eta}_5) \Lambda_{1A,} P_b = 0, \quad \text{(10a)}$$

$$\sum_{M,n}^{I_a,I_r} \frac{\partial A^p_{Mn}}{\partial r_{ab}} \dot{P}_{Mn} + \sum_{m,n}^{I_a,I_r} \frac{\partial A^r_{mn}}{\partial r_{ab}} + \frac{1}{2} \sum_{ML,nk}^{I_a,I_r} \frac{\partial A^{pp}_{(Mn)(Kl)}}{\partial r_{ab}} P_{Mn} P_{Lk}$$

$$+ \sum_{Nl,nk}^{I_a,I_r} \frac{\partial A^{pr}_{(Mn),(lk)}}{\partial r_{ab}} P_{Mn} R_{lk} + \frac{1}{2} \sum_{ml,nk}^{I_a,I_r} \frac{\partial A^{rr}_{(mn)(lk)}}{\partial r_{ab}} R_{mn} R_{lk} + g \Lambda_{,aa} r_{ab}$$

$$+ (\ddot{\eta}_2 + g\eta_4 + S_b \ddot{\eta}_4) \Lambda_{1a,} P_b = 0, \ A = 0, \ldots; \, a, b = 1, \ldots, \quad \text{(10b)}$$

(the dynamic subsystem), $I_a, I_r \to \infty$, where the comma between the indices pairs, alike $(Ab), (Mn)$, means that the pairs do not commutate; coefficients P_b are defined in (7b),

$$S_b = 2\, k_{1b}^{-1} \tanh(k_{1b} \tfrac{h}{2}), \quad \text{(11)}$$

and the Λ-tensor is introduced in section 10. The modal system (9), (10) contains the following nonlinear functions of the sloshing-related generalised coordinates

$$A^{pp}_{(Ab)(Mn)} = \int_{r_1}^{1} \int_{-\pi}^{\pi} r \Big[\cos A\theta \, \cos M\theta \, \mathcal{G}^{(1)}_{(Ab)(Mn)} + \sin A\theta \, \sin M\theta \, \mathcal{G}^{(2)}_{(Ab)(Mn)} \Big] d\theta dr,$$

$$A^{rr}_{(ab)(mn)} = \int_{r_1}^{1} \int_{-\pi}^{\pi} r \left[\sin a\theta \, \sin m\theta \, \mathcal{G}^{(1)}_{(ab)(mn)} + \cos a\theta \, \cos m\theta \, \mathcal{G}^{(2)}_{(ab)(mn)} \right] d\theta dr,$$

$$A^{pr}_{(Ab),(mn)}=\int_{r_1}^{1}\int_{-\pi}^{\pi} r\left[\cos A\theta\,\sin m\theta\,\mathcal{G}^{(1)}_{(Ab)(mn)}-\sin A\theta\,\cos m\theta\,\mathcal{G}^{(2)}_{(Ab)(mn)}\right]d\theta dr,$$

$$A^{p}_{Ab}=\int_{r_1}^{1}\int_{-\pi}^{\pi} r\cos(A\theta)\,\mathcal{G}^{(0)}_{Ab}\,d\theta dr,\;\; A^{r}_{ab}=\int_{r_1}^{1}\int_{-\pi}^{\pi} r\sin(a\theta)\,\mathcal{G}^{(0)}_{Ab}\,d\theta dr, \quad (12)$$

where

$$\mathcal{G}^{(0)}_{Ab}=\mathcal{R}_{Ab}(r)\int_{-h}^{\zeta}\frac{\cosh(k_{Ab}(z+h))}{\cosh(k_{Ab}h)}dz=\mathcal{R}_{Ab}(r)\,I^{(0)}_{(Ab)},$$

$$\mathcal{G}^{(1)}_{(Ab)(Mn)}=\mathcal{R}'_{Ab}(r)\mathcal{R}'_{Mn}(r)I^{(1)}_{(Ab)(Mn)}+\mathcal{R}_{Ab}(r)\mathcal{R}_{Mn}(r)k_{Ab}k_{Mn}I^{(2)}_{(Ab)(Mn)},$$

$$\mathcal{G}^{(2)}_{(Ab)(Mn)}=AM\,r^{-2}\,\mathcal{R}_{Ab}(r)\mathcal{R}_{Mn}(r)\,I^{(1)}_{(Ab)(Mn)}; \quad (13)$$

$$I^{(1)}_{(Ab)(Mn)}=\int_{-h}^{\zeta}\frac{\cosh(k_{Ab}(z+h))\,\cosh(k_{Mn}(z+h))}{\cosh(k_{Ab}h)\,\cosh(k_{Mnh})}dz,$$
$$I^{(2)}_{(Ab)(Mn)}=\int_{-h}^{\zeta}\frac{\sinh(k_{Ab}(z+h))\,\sinh(k_{Mn}(z+h))}{\cosh(k_{Ab}h)\,\cosh(k_{Mnh})}dz. \quad (14)$$

5. Adaptive third-order modal equations

The general adaptive intermodal ordering suggests that all sloshing-related generalised coordinates and velocities in (8) are of the same order $O(\epsilon^{1/3})$. The aim is to derive a weakly-nonlinear infinite-dimensional modal system coupling the sloshing-related generalised coordinates without the generalised velocities. Derivation consists of the following *five steps* [2].

The **first step** suggests the Taylor expansion by ζ of $I^{(0)}_{(Ab)}, I^{(1)}_{(Ab)(Mn)}$, and $I^{(2)}_{(Ab)(Mn)}$ by (13) and (14). By definition, $\zeta = O(\epsilon^{1/3})$. Analysis shows that $I^{(0)}_{(Ab)}$ should be expanded up to the third order but $I^{(1)}_{(Ab)(Mn)}$ and $I^{(2)}_{(Ab)(Mn)}$ require expansion up to the second order, i.e.

$$I^{(0)}_{(Ab)}=k^{-1}_{Ab}\,\tanh(k_{Ab}h)+\zeta+\tfrac{1}{2}\kappa_{Ab}\zeta^2+\tfrac{1}{6}k^2_{Ab}\zeta^3+\ldots, \quad (15a)$$

$$I^{(1)}_{(Ab)(Mn)}=O(1)+\zeta+\tfrac{1}{2}(\kappa_{Ab}+\kappa_{Mn})\zeta^2+\ldots, \quad (15b)$$

$$I^{(2)}_{(Ab)(Mn)}=O(1)+\kappa_{Ab}\kappa_{Mn}\zeta+\tfrac{1}{2}(k^2_{Ab}\kappa_{Mn}+k^2_{Mn}\kappa_{Ab})\zeta^2+\ldots. \quad (15c)$$

Inserting (15b) and (15c) into (13) gives

$$\mathcal{G}^{(1)}_{(Ab)(Mn)} = O(1) + (\mathcal{R}'_{Ab}\mathcal{R}'_{Mn} + \mathcal{R}_{Ab}\mathcal{R}_{Mn}\kappa_{Ab}\kappa_{Mn})\,\zeta + \tfrac{1}{2}[(\kappa_{Ab}+\kappa_{Mn})\mathcal{R}'_{Ab}\mathcal{R}'_{Mn} + \mathcal{R}_{Ab}\mathcal{R}_{Mn}(k^2_{Ab}\kappa_{Mn} + k^2_{Mn}\kappa_{Ab})]\zeta^2, \quad (16a)$$

$$\mathcal{G}^{(2)}_{(Ab)(Mn)} = O(1) + r^{-2}AM\mathcal{R}_{Ab}\mathcal{R}_{Mn}\,\zeta + \tfrac{1}{2}r^{-2}AM(\kappa_{Ab}+\kappa_{Mn})\mathcal{R}_{Ab}\mathcal{R}_{Mn}\,\zeta^2. \quad (16b)$$

By the **second step**, A^p_{Ab} and A^r_{ab} should be expanded up to $O(\epsilon)$ in terms to the sloshing-related generalised coordinates. For this purpose, (8a) is inserted into expressions of (15a) and, thereafter, substituted into the corresponding formulas of (12). This gives

$$A^p_{Ab} = \Lambda_{AA,}p_{Ab} + \tfrac{1}{2}\sum_{MN,ij}^{I_a,I_r}\chi^{pp}_{(Mi)(Nj),(Ab)}p_{Mi}p_{Nj} + \tfrac{1}{2}\sum_{mn,ij}^{I_a,I_r}\chi^{rr}_{(mi)(nj),(Ab)}r_{mi}r_{nj} + \tfrac{1}{3}\sum_{MNK,ijl}^{I_a,I_r}\chi^{ppp}_{(Mi)(Nj)(Kl),(Ab)}p_{Mi}p_{Nj}p_{Kl} + \sum_{Mnk,ijl}^{I_a,I_r}\chi^{prr}_{(Mi),(nj)(kl),(Ab)}p_{Mi}r_{nj}r_{kl}, \quad (17a)$$

$$A^r_{ab} = \Lambda_{,aa}r_{ab} + \sum_{Mn,ij}^{I_a,I_r}\chi^{pr}_{(Mi),(nj),(ab)}p_{Mi}r_{nj} + \tfrac{1}{3}\sum_{mnk,ijl}^{I_a,I_r}\chi^{rrr}_{(mi)(nj)(kl),(ab)}r_{mi}r_{nj}r_{kl} + \sum_{MNk,ijl}^{I_a,I_r}\chi^{ppr}_{(Mi)(Nj),(kl),(ab)}p_{Mi}p_{Nj}r_{kl}, \quad I_a, I_r \to \infty, \quad (17b)$$

where

$$\chi^{pp}_{(Mi)(Nj),(Ab)} = \kappa_{Ab}\Lambda_{AMN,}\lambda_{(Ab)(Mi)(Nj)},$$
$$\chi^{rr}_{(Mi)(nj),(Ab)} = \kappa_{Ab}\Lambda_{A,mn}\lambda_{(Ab)(mi)(nj)},$$
$$\chi^{ppp}_{(Mi)(Nj)(Kl),(Ab)} = \tfrac{1}{2}k^2_{Ab}\Lambda_{AMNK,}\lambda_{(Ab)(Mi)(Nj)(Kl)},$$
$$\chi^{prr}_{(Mi),(nj)(kl),(Ab)} = \tfrac{1}{2}k^2_{Ab}\Lambda_{AM,nk}\lambda_{(Ab)(Mi)(nj)(kl)},$$

$$\chi^{pr}_{(Mi),(nj),(ab)} = \kappa_{ab}\Lambda_{M,an}\lambda_{(Mi)(nj)(ab)},$$
$$\chi^{rrr}_{(mi)(nj)(kl),(ab)} = \tfrac{1}{2}k^2_{ab}\Lambda_{,amnk}\lambda_{(mi)(nj)(kl)(ab)},$$
$$\chi^{ppr}_{(Mi)(Nj),(kl),(ab)} = \tfrac{1}{2}k^2_{ab}\Lambda_{MN,ak}\lambda_{(Mi)(Nj)(kl)(ab)}$$

with notations from section 10. The partial derivatives of (17) by the generalised coordinates take the form

$$\frac{\partial A^p_{Ab}}{\partial p_{Df}} = \Lambda_{AD,}\delta_{bf} + \sum_{M,i}^{I_a,I_r} \chi^{pp}_{(Mi)(Df),(Ab)} p_{Mi} + \sum_{NK,jl}^{I_a,I_r} \chi^{ppp}_{(Df)(Nj)(Kl),(Ab)} p_{Nj} p_{Kl} + \sum_{nk,jl}^{I_a,I_r} \chi^{prr}_{(Df),(nj)(kl),(Ab)} r_{nj} r_{kl}, \quad (18a)$$

$$\frac{\partial A^p_{Ab}}{\partial r_{df}} = \sum_{m,i}^{I_a,I_r} \chi^{rr}_{(mi)(df),(Ab)} r_{mi} + 2 \sum_{Mn,ij}^{I_a,I_r} \chi^{prr}_{(Mi),(nj)(df),(Ab)} p_{Mi} r_{nj}, \quad (18b)$$

$$\frac{\partial A^r_{ab}}{\partial p_{Df}} = \sum_{n,j}^{I_a,I_r} \chi^{pr}_{(Df),(nj),(ab)} r_{nj} + 2 \sum_{Mn,ij}^{I_a,I_r} \chi^{ppr}_{(Mi)(Df),(nj),(ab)} p_{Mi} r_{nj}, \quad (18c)$$

$$\frac{\partial A^r_{ab}}{\partial r_{df}} = \Lambda_{,ad}\delta_{bf} + \sum_{M,i}^{I_a,I_r} \chi^{pr}_{(Mi),(df),(ab)} p_{Mi} + \sum_{mn,ij}^{I_a,I_r} \chi^{rrr}_{(mi)(nj)(df),(ab)} r_{mi} r_{nj} + \sum_{MN,ij}^{I_a,I_r} \chi^{ppr}_{(Mi)(Nj),(df),(ab)} p_{Mi} p_{Nj}, \quad I_a, I_r \to \infty. \quad (18d)$$

The **third step** should lead to analogous expressions for $A^{pp}_{(Ab)(Mn)}$, $A^{rr}_{(ab)(mn)}$ and $A^{pr}_{(Ab),(mn)}$ but up to the second-order terms, $O(\epsilon^{2/3})$. The $O(1)$-order term can be taken from the linear modal theory. The result is

$$A^{pp}_{(Ab)(Mn)} = \Lambda_{AM,}\delta_{bn}\kappa_{Ab} + \sum_{K,l}^{I_a,I_r} \Pi^{p,p}_{(Kl),(Ab)(Mn)} p_{Kl} + \sum_{KC,ld}^{I_a,I_r} \Pi^{p,pp}_{(Kl)(Cd),(Ab)(Mn)} p_{Kl} p_{Cd} + \sum_{kc,ld}^{I_a,I_r} \Pi^{p,rr}_{(kl)(cd),(Ab)(Mn)} r_{kl} r_{cd}, \quad (19a)$$

$$A^{rr}_{(ab)(mn)} = \Lambda_{,am}\delta_{bn}\kappa_{ab} + \sum_{K,l}^{I_a,I_r} \Pi^{r,p}_{(Kl),(ab)(mn)} p_{Kl}$$

$$+ \sum_{KC,ld}^{I_a,I_r} \Pi^{r,pp}_{(Kl)(Cd),(ab)(mn)} p_{Kl} p_{Cd} + \sum_{kc,ld}^{I_a,I_r} \Pi^{r,rr}_{(kl)(cd),(ab)(mn)} r_{kl} r_{cd}, \tag{19b}$$

$$A^{pr}_{(Ab),(mn)} = \sum_{k,l}^{I_a,I_r} \Pi^{r}_{(kl),(Ab),(mn)} r_{kl} + \sum_{Kc,ld}^{I_a,I_r} \Pi^{pr}_{(Kl),(cd),(Ab),(mn)} p_{Kl} r_{cd}, \tag{19c}$$

where

$$\Pi^{p,p}_{(Kl),(Ab)(Mn)} = \Lambda_{AMK,} G^{(11)}_{(Ab)(Mn),(Kl)} + \Lambda_{K,AM} G^{(12)}_{(Ab)(Mn),(Kl)},$$
$$\Pi^{r,p}_{(Kl),(ab)(mn)} = \Lambda_{K,am} G^{(11)}_{(ab)(mn),(Kl)} + \Lambda_{amK,} G^{(12)}_{(ab)(mn),(Kl)},$$
$$\Pi^{r}_{(kl),(Ab),(mn)} = \Lambda_{A,mk} G^{(11)}_{(Ab)(mn),(kl)} - \Lambda_{m,Ak} G^{(12)}_{(Ab)(mn),(kl)},$$
$$\Pi^{p,pp}_{(Kl)(Cd),(Ab)(Mn)} = \Lambda_{AMKC,} G^{(21)}_{(Ab)(Mn),(Kl)(Cd)} + \Lambda_{KC,AM} G^{(22)}_{(Ab)(Mn),(Kl)(Cd)},$$
$$\Pi^{p,rr}_{(kl)(cd),(Ab)(Mn)} = \Lambda_{AM,kc} G^{(21)}_{(Ab)(Mn),(kl)(cd)} + \Lambda_{,AMkc} G^{(22)}_{(Ab)(Mn),(kl)(cd)},$$
$$\Pi^{r,pp}_{(Kl)(Cd),(ab)(mn)} = \Lambda_{KC,am} G^{(21)}_{(ab)(mn),(Kl)(Cd)} + \Lambda_{KCam,} G^{(22)}_{(ab)(mn),(Kl)(Cd)},$$
$$\Pi^{r,rr}_{(kl)(cd),(ab)(mn)} = \Lambda_{,amkc} G^{(21)}_{(ab)(mn),(kl)(cd)} + \Lambda_{am,kc} G^{(22)}_{(ab)(mn),(kl)(cd)},$$
$$\Pi^{pr}_{(Kl),(cd),(Ab),(mn)} = 2[\Lambda_{AK,mc} G^{(21)}_{(Ab)(mn),(Kl)(cd)} - \Lambda_{Km,Ac} G^{(22)}_{(Ab)(mn),(Kl)(cd)}];$$
$$G^{(11)}_{(Ab)(Mn),(Kl)} = \lambda'_{(Ab)(Mn),(Kl)} + \kappa_{Ab}\kappa_{Mn}\lambda_{(Ab)(Mn)(Kl)},$$
$$G^{(12)}_{(Ab)(Mn),(Kl)} = AM\bar{\lambda}_{(Ab)(Mn)(Kl)},$$
$$G^{(21)}_{(Ab)(Mn),(Kl)(Cd)} = \tfrac{1}{2}[(\kappa_{Ab} + \kappa_{Mn})\lambda'_{(Ab)(Mn),(Kl)(Cd)}$$
$$+ (k^2_{Ab}\kappa_{Mn} + k^2_{Mn}\kappa_{Ab})\lambda_{(Ab)(Mn)(Kl)(Cd)}],$$
$$G^{(22)}_{(Ab)(Mn),(Kl)(Cd)} = \tfrac{1}{2} AM(\kappa_{Ab} + \kappa_{Mn})\bar{\lambda}_{(Ab)(Mn)(Kl)(Cd)}.$$

The partial derivatives of (19) by the generalised coordinates are

$$\frac{\partial A^{pp}_{(Ab)(Cd)}}{\partial p_{Ef}} = \Pi^{p,p}_{(Ef),(Ab)(Cd)} + 2\sum_{M,i}^{I_a,I_r} \Pi^{p,pp}_{(Mi)(Ef),(Ab)(Cd)} p_{Mi}, \tag{20a}$$

$$\frac{\partial A^{pp}_{(Ab)(Cd)}}{\partial r_{ef}} = 2\sum_{m,i}^{I_a,I_r} \Pi^{p,rr}_{(mi)(ef),(Ab)(Cd)} r_{mi}, \tag{20b}$$

$$\frac{\partial A^{rr}_{(ab)(cd)}}{\partial p_{Ef}} = \Pi^{r,p}_{(Ef),(ab)(cd)} + 2\sum_{M,i}^{I_a,I_r} \Pi^{r,pp}_{(Mi)(Ef),(ab)(cd)} p_{Mi}, \tag{20c}$$

$$\frac{\partial A^{rr}_{(ab)(cd)}}{\partial r_{ef}} = 2\sum_{m,i}^{I_a,I_r} \Pi^{r,rr}_{(mi)(ef),(ab)(cd)} r_{mi}, \tag{20d}$$

$$\frac{\partial A^{pr}_{(Ab),(cd)}}{\partial p_{Ef}} = \sum_{n,j}^{I_a,I_r} \Pi^{pr}_{(Ef),(nj),(Ab)(cd)} r_{nj}, \tag{20e}$$

$$\frac{\partial A^{pr}_{(Ab),(cd)}}{\partial r_{ef}} = \Pi^{r}_{(ef),(Ab),(cd)} + \sum_{M,i}^{I_a,I_r} \Pi^{pr}_{(Mi),(ef),(Ab)(cd)} p_{Mi}. \tag{20f}$$

By the **fourth step**, the kinematic equations (9) should be resolved with respect to the generalised velocities P_{Ab} and R_{ab} by postulating

$$P_{Ab} = \frac{1}{\kappa_{Ab}}\dot{p}_{Ab} + \sum_{MN,ij}^{I_a,I_r} V^{pp}_{(Mi),(Nj),(Ab)} \dot{p}_{Mi} p_{Nj} + \sum_{mn,ij}^{I_a,I_r} V^{rr}_{(mi),(nj),(Ab)} \dot{r}_{mi} r_{nj}$$
$$+ \sum_{MNK,ijl}^{I_a,I_r} V^{ppp}_{(Mi),(Nj),(Kl),(Ab)} \dot{p}_{Mi} p_{Nj} p_{Kl} + \sum_{Mnk,ijl}^{I_a,I_r} V^{prr}_{(Mi),(nj),(kl),(Ab)} \dot{p}_{Mi} r_{nj} r_{kl}$$
$$+ \sum_{Mnk,ijl}^{I_a,I_r} V^{rpr}_{(nj),(Mi),(kl),(Ab)} \dot{r}_{nj} p_{Mi} r_{kl}, \tag{21a}$$

$$R_{ab} = \frac{1}{\kappa_{ab}}\dot{r}_{ab} + \sum_{Mn,ij}^{I_a,I_r} V^{pr}_{(Mi),(nj),(ab)} \dot{p}_{Mi} r_{nj} + \sum_{Mn,ij}^{I_a,I_r} V^{rp}_{(nj),(Mi),(ab)} \dot{r}_{nj} p_{Mi}$$
$$+ \sum_{mnk,ijl}^{I_a,I_r} V^{rrr}_{(mi),(nj),(kl),(ab)} \dot{r}_{mi} r_{nj} r_{kl} + \sum_{MNk,ijl}^{I_a,I_r} V^{rpp}_{(kl),(Mi),(Nj),(ab)} \dot{r}_{kl} p_{Mi} p_{Nj}$$
$$+ \sum_{MNk,ijl}^{I_a,I_r} V^{ppr}_{(Mi),(Nj),(kl),(ab)} \dot{p}_{Mi} p_{Nj} r_{kl}, \tag{21b}$$

substituting (21) into the kinematic subsystem (9), and matching the similar quantities. The procedure derives the V-coefficients as

$$V^{pp}_{(Mi),(Nj),(Ab)} = \frac{1}{\Lambda_{AA}\kappa_{Ab}} \left[\chi^{pp}_{(Nj)(Mi),(Ab)} - \frac{\Pi^{p,p}_{(Nj),(Ab)(Mi)}}{\kappa_{Mi}} \right],$$

$$V^{rr}_{(mi),(nj),(Ab)} = \frac{1}{\Lambda_{AA}\kappa_{Ab}}\left[\chi^{rr}_{(nj)(mi),(Ab)} - \frac{\Pi^{r}_{(nj),(Ab),(mi)}}{\kappa_{mi}}\right],$$

$$V^{rp}_{(nj),(Mi),(ab)} = \frac{1}{\Lambda_{aa}\kappa_{ab}}\left[\chi^{pr}_{(Mi),(nj),(ab)} - \frac{\Pi^{r,p}_{(Mi),(ab)(nj)}}{\kappa_{nj}}\right],$$

$$V^{pr}_{(Mi),(nj),(ab)} = \frac{1}{\Lambda_{aa}\kappa_{ab}}\left[\chi^{pr}_{(Mi),(nj),(ab)} - \frac{\Pi^{r}_{(nj),(Mi),(ab)}}{\kappa_{Mi}}\right],$$

$$V^{ppp}_{(Mi),(Nj),(Kl),(Ab)} = \frac{1}{\Lambda_{AA}\kappa_{Ab}}\left[\chi^{ppp}_{(Mi)(Nj)(Kl),(Ab)} - \frac{\Pi^{p,pp}_{(Nj)(Kl),(Ab)(Mi)}}{\kappa_{Mi}}\right.$$

$$\left. - \sum_{C,d}^{I_a,I_r} V^{pp}_{(Mi),(Nj),(Cd)}\Pi^{p,p}_{(Kl),(Ab)(Cd)}\right]; \quad V^{prr}_{(Mi),(nj),(kl),(Ab)} = \frac{1}{\Lambda_{AA}\kappa_{Ab}}$$

$$\times\left[\chi^{prr}_{(Mi),(nj)(kl),(Ab)} - \frac{\Pi^{p,rr}_{(nj)(kl),(Ab)(Mi)}}{\kappa_{Mi}} - \sum_{c,d}^{I_a,I_r} V^{pr}_{(Mi),(nj),(cd)}\Pi^{r}_{(kl),(Ab),(cd)}\right],$$

$$V^{rrr}_{(mi),(nj),(kl),(ab)} = \frac{1}{\Lambda_{aa}\kappa_{ab}}\left[\chi^{rrr}_{(nj)(kl)(mi),(ab)} - \frac{\Pi^{r,rr}_{(nj)(kl),(ab)(mi)}}{\kappa_{mi}}\right.$$

$$\left. - \sum_{C,d}^{I_a,I_r} V^{rr}_{(mi),(nj),(Cd)}\Pi^{r}_{(kl),(Cd),(ab)}\right]; \quad V^{rpp}_{(kl),(Mi),(Nj),(ab)} = \frac{1}{\Lambda_{aa}\kappa_{ab}}$$

$$\times\left[\chi^{ppr}_{(Mi)(Nj),(kl),(ab)} - \frac{\Pi^{r,pp}_{(Mi)(Nj),(ab)(kl)}}{\kappa_{kl}} - \sum_{c,d}^{I_a,I_r} V^{rp}_{(kl),(Mi),(cd)}\Pi^{r,p}_{(Nj),(ab),(cd)}\right],$$

$$V^{ppr}_{(Mi),(Nj),(kl),(ab)} = \frac{1}{\Lambda_{aa}\kappa_{ab}}\left[2\chi^{ppr}_{(Mi)(Nj),(kl),(ab)} - \frac{\Pi^{pr}_{(Nj),(kl),(Mi)(ab)}}{\kappa_{Mi}}\right.$$

$$\left. - \sum_{C,d}^{I_a,I_r} V^{pp}_{(Mi),(Nj),(Cd)}\Pi^{r}_{(kl),(Cd),(ab)} - \sum_{c,d}^{I_a,I_r} V^{pr}_{(Mi),(kl),(cd)}\Pi^{r,p}_{(Nj),(ab)(cd)}\right],$$

$$V^{rpr}_{(nj),(Mi),(kl),(Ab)} = \frac{1}{\Lambda_{AA}\kappa_{Ab}}\left[2\chi^{prr}_{(Mi),(kl)(nj),(Ab)} - \frac{\Pi^{pr}_{(Mi),(kl),(Ab)(nj)}}{\kappa_{nj}}\right.$$

$$\left. - \sum_{C,d}^{I_a,I_r} V^{rr}_{(nj),(kl),(Cd)}\Pi^{p,p}_{(Mi),(Ab)(Cd)} - \sum_{c,d}^{I_a,I_r} V^{rp}_{(nj),(Mi),(cd)}\Pi^{r}_{(kl),(Ab),(cd)}\right].$$

By the **fifth step**, expressions (18), (20) and (21) are substituted into the dynamic equations (10). Excluding the $o(\epsilon)$-terms gives the *required adaptive weakly-nonlinear modal equations*

$$\sum_{M,i}^{I_a,I_r} \ddot{p}_{Mi} \left[\delta_{ME}\delta_{if} + \sum_{N,j}^{I_a,I_r} d^{pp,(Ef)}_{(Mi),(Nj)} p_{Nj} + \sum_{NK,jl}^{I_a,I_r} d^{ppp,(Ef)}_{(Mi),(Nj),(Kl)} p_{Nj} p_{Kl} \right.$$
$$\left. + \sum_{nk,jl}^{I_a,I_r} d^{prr,(Ef)}_{(Mi),(nj),(kl)} r_{nj} r_{kl} \right] + \sum_{mn,ij}^{I_a,I_r} \ddot{r}_{mi} r_{nj} \left[d^{rr,(Ef)}_{(mi),(nj)} + \sum_{K,l}^{I_a,I_r} d^{rrp,(Ef)}_{(mi),(nj),(Kl)} p_{Kl} \right]$$
$$+ \sum_{MN,ij}^{I_a,I_r} \dot{p}_{Mi} \dot{p}_{Nj} \left[t^{pp,(Ef)}_{(Mi),(Nj)} + \sum_{K,l}^{I_a,I_r} t^{ppp,(Ef)}_{(Mi),(Nj),(Kl)} p_{Kl} \right] + \sigma^2_{Ef} p_{Ef}$$
$$+ \sum_{Mnk,ijl}^{I_a,I_r} t^{prr,(Ef)}_{(Mi),(nj),(kl)} \dot{p}_{Mi} \dot{r}_{nj} r_{kl} + \sum_{mn,ij}^{I_a,I_r} \dot{r}_{mi} \dot{r}_{nj} \left[t^{rr,(Ef)}_{(mi),(nj)} + \sum_{K,l}^{I_a,I_r} t^{rrp,(Ef)}_{(mi),(nj),(Kl)} p_{Kl} \right]$$
$$= -(\ddot{\eta}_1 - g\eta_5 - S_b\ddot{\eta}_5)\delta_{1E}\kappa_{1f} P_f; \quad E = 0, \ldots, I_a; \; f = 1, \ldots, I_r, \qquad (22a)$$

$$\sum_{Mn,ij}^{I_a,I_r} \ddot{p}_{Mi} r_{nj} \left[d^{pr,(ef)}_{(Mi),(nj)} + \sum_{K,l}^{I_a,I_r} d^{prp,(ef)}_{(Mi),(nj),(Kl)} p_{Kl} \right]$$
$$+ \sum_{m,i}^{I_a,I_r} \ddot{r}_{mi} \left[\delta_{me}\delta_{ij} + \sum_{N,j}^{I_a,I_r} d^{rp,(ef)}_{(mi),(Nj)} p_{Nj} + \sum_{NK,jl}^{I_a,I_r} d^{rpp,(ef)}_{(mi),(Nj),(Kl)} p_{Nj} p_{Kl} \right.$$
$$\left. + \sum_{nk,jl}^{I_a,I_r} d^{rrr,(ef)}_{(mi),(nj),(kl)} r_{nj} r_{kl} \right] + \sigma^2_{ef} r_{ef}$$
$$+ \sum_{Mn,ij}^{I_a,I_r} \dot{p}_{Mi} \dot{r}_{nj} \left[t^{pr,(ef)}_{(Mi),(nj)} + \sum_{K,l}^{I_a,I_r} t^{prp,(ef)}_{(Mi),(nj),(Kl)} p_{Kl} \right]$$
$$+ \sum_{MNk,ijl}^{I_a,I_r} t^{ppr,(ef)}_{(Mi),(Nj),(kl)} \dot{p}_{Mi} \dot{p}_{Nj} r_{kl} + \sum_{mnk,ijl}^{I_a,I_r} t^{rrr,(ef)}_{(mi),(nj),(kl)} \dot{r}_{mi} \dot{r}_{nj} r_{kl}$$
$$= -(\ddot{\eta}_2 + g\eta_4 + S_b\ddot{\eta}_4)\delta_{1e}\kappa_{1f} P_f; \quad e = 1, \ldots, I_a; \; f = 1, \ldots, I_r, \qquad (22b)$$

where the natural sloshing frequencies σ_{Ef} are defined by (5), P_f comes from (7b), and

$$d^{pp,(Ef)}_{(Mi),(Nj)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}V^{pp}_{(Mi),(Nj),(Ef)} + \frac{\chi^{pp}_{(Nj)(Ef),(Mi)}}{\kappa_{Mi}}\right],$$

$$d^{ppp,(Ef)}_{(Mi),(Nj),(Kl)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}V^{ppp}_{(Mi),(Nj),(Kl),(Ef)} + \frac{\chi^{ppp}_{(Ef)(Nj)(Kl),(Mi)}}{\kappa_{Mi}} + \sum_{A,b}^{I_a,I_r} V^{pp}_{(Mi),(Nj),(Ab)}\chi^{pp}_{(Kl)(Ef),(Ab)}\right],$$

$$d^{prr,(Ef)}_{(Mi),(nj),(kl)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}V^{prr}_{(Mi),(nj),(kl),(Ef)} + \frac{\chi^{prr}_{(Ef),(nj)(kl),(Mi)}}{\kappa_{Mi}} + \sum_{a,b}^{I_a,I_r} V^{pr}_{(Mi),(nj),(ab)}\chi^{pr}_{(Ef),(kl),(ab)}\right],$$

$$d^{rr,(Ef)}_{(mi),(nj)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}V^{rr}_{(mi),(nj),(Ef)} + \frac{\chi^{pr}_{(Ef),(nj),(mi)}}{\kappa_{mi}}\right],$$

$$d^{rrp,(Ef)}_{(mi),(nj),(Kl)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}V^{rpr}_{(mi),(Kl),(nj),(Ef)} + \frac{2\chi^{ppr}_{(Kl)(Ef),(nj),(mi)}}{\kappa_{mi}} + \sum_{a,b}^{I_a,I_r} V^{rp}_{(mi),(Kl),(ab)}\chi^{pr}_{(Ef),(nj),(ab)} + \sum_{A,b}^{I_a,I_r} V^{rr}_{(mi),(nj),(Ab)}\chi^{pp}_{(Kl)(Ef),(Ab)}\right],$$

$$t^{pp,(Ef)}_{(Mi),(Nj)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}V^{pp}_{(Mi),(Nj),(Ef)} + \frac{\Pi^{p,p}_{(Ef),(Mi)(Nj)}}{2\kappa_{Mi}\kappa_{Nj}}\right],$$

$$t^{ppp,(Ef)}_{(Mi),(Nj),(Kl)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}\bar{V}^{ppp}_{(Mi),(Nj),(Kl),(Ef)} + \frac{\Pi^{p,pp}_{(Kl)(Ef),(Mi)(Nj)}}{\kappa_{Mi}\kappa_{Nj}} + \sum_{A,b}^{I_a,I_r} V^{pp}_{(Mi),(Nj),(Ab)}\chi^{pp}_{(Kl)(Ef),(Ab)} + \sum_{A,b}^{I_a,I_r}\frac{\Pi^{p,p}_{(Ef),(Mi)(Ab)}}{\kappa_{Mi}}V^{pp}_{(Nj),(Kl),(Ab)}\right],$$

$$t^{rr,(Ef)}_{(mi),(nj)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}V^{rr}_{(mi),(nj),(Ef)} + \frac{\Pi^{r,p}_{(Ef),(mi)(nj)}}{2\kappa_{mi}\kappa_{nj}}\right],$$

$$t^{rrp,(Ef)}_{(mi),(nj),(Kl)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}V^{rpr}_{(mi),(Kl),(nj),(Ef)} + \frac{\Pi^{r,pp}_{(Kl)(Ef),(mi)(nj)}}{\kappa_{mi}\kappa_{nj}}\right.$$

$$\left. + \sum_{A,b}^{I_a,I_r} V^{rr}_{(mi),(nj),(Ab)}\chi^{pp}_{(Kl)(Ef),(Ab)} + \sum_{a,b}^{I_a,I_r}\frac{\Pi^{r,p}_{(Ef),(mi)(ab)}}{\kappa_{mi}}V^{rp}_{(nj),(Kl),(ab)}\right],$$

$$t^{prr,(Ef)}_{(Mi),(nj),(kl)} = \frac{\kappa_{Ef}}{\Lambda_{EE}}\left[\Lambda_{EE}\bar{V}^{prr}_{(Mi),(nj),(kl),(Ef)} + \frac{\Pi^{pr}_{(Ef),(kl),(Mi)(nj)}}{\kappa_{Mi}\kappa_{nj}}\right.$$

$$+ \sum_{a,b}^{I_a,I_r}\left(\bar{V}^{pr}_{(Mi),(nj),(ab)}\chi^{pr}_{(Ef),(kl),(ab)} + \frac{1}{\kappa_{nj}}V^{pr}_{(Mi),(kl),(ab)}\Pi^{r,p}_{(Ef),(ab)(nj)}\right)$$

$$\left. + \sum_{A,b}^{I_a,I_r}\frac{\Pi^{p,p}_{(Ef),(Mi)(Ab)}}{\kappa_{Mi}}V^{rr}_{(nj),(kl),(Ab)}\right],$$

$$d^{pr,(ef)}_{(Mi),(nj)} = \frac{\kappa_{ef}}{\Lambda_{ee}}\left[\Lambda_{ee}V^{pr}_{(Mi),(nj),(ef)} + \frac{\chi^{rr}_{(nj),(ef),(Mi)}}{\kappa_{Mi}}\right],$$

$$d^{prp,(ef)}_{(Mi),(nj),(Kl)} = \frac{\kappa_{ef}}{\Lambda_{ee}}\left[\Lambda_{ee}V^{ppr}_{(Mi),(Kl),(nj),(ef)} + \frac{2\chi^{prr}_{(Kl),(nj)(ef),(Mi)}}{\kappa_{Mi}}\right.$$

$$\left. + \sum_{A,b}^{I_a,I_r} V^{pp}_{(Mi),(Kl),(Ab)}\chi^{rr}_{(nj)(ef),(Ab)} + \sum_{a,b}^{I_a,I_r} V^{pr}_{(Mi),(nj),(ab)}\chi^{pr}_{(Kl),(ef),(ab)}\right],$$

$$d^{rp,(ef)}_{(mi),(Nj)} = \frac{\kappa_{ef}}{\Lambda_{ee}}\left[\Lambda_{ee}V^{rp}_{(mi),(Nj),(ef)} + \frac{\chi^{pr}_{(Nj),(ef),(mi)}}{\kappa_{mi}}\right],$$

$$d^{rpp,(ef)}_{(mi),(Nj),(Kl)} = \frac{\kappa_{ef}}{\Lambda_{ee}}\left[\Lambda_{ee}V^{rpp}_{(mi),(Nj),(Kl),(ef)} + \frac{\chi^{ppr}_{(Nj)(Kl),(ef),(mi)}}{\kappa_{mi}}\right.$$

$$\left. + \sum_{a,b}^{I_a,I_r} V^{rp}_{(mi),(Nj),(ab)}\chi^{pr}_{(Kl),(ef),(ab)}\right],$$

$$d^{rrr,(ef)}_{(mi),(nj),(kl)} = \frac{\kappa_{ef}}{\Lambda_{ee}}\left[\Lambda_{ee}V^{rrr}_{(mi),(nj),(kl),(ef)} + \frac{\chi^{rrr}_{(nj)(kl)(ef),(mi)}}{\kappa_{mi}}\right.$$

$$\left. + \sum_{A,b}^{I_a,I_r} V^{rr}_{(mi),(nj),(Ab)}\chi^{rr}_{(kl)(ef),(Ab)}\right],$$

$$t^{pr,(ef)}_{(Mi),(nj)} = \frac{\kappa_{ef}}{\Lambda_{ee}}\left[\Lambda_{ee}\bar{V}^{pr}_{(Mi),(nj),(ef)} + \frac{\Pi^{r}_{(ef),(Mi),(nj)}}{\kappa_{Mi}\kappa_{nj}}\right],$$

$$t^{prp,(ef)}_{(Mi),(nj),(Kl)} = \frac{\kappa_{ef}}{\Lambda_{ee}}\left[\Lambda_{ee}\bar{V}^{rpp}_{(nj),(Mi),(Kl),(ef)} + \frac{\Pi^{pr}_{(Kl),(ef),(Mi)(nj)}}{\kappa_{Mi}\kappa_{nj}}\right.$$

$$+\sum_{a,b}^{I_a,I_r}\bar{V}^{pr}_{(Mi),(nj),(ab)}\chi^{pr}_{(Kl),(ef),(ab)} + \sum_{a,b}^{I_a,I_r}\frac{V^{rp}_{(nj),(Kl),(ab)}}{\kappa_{Mi}}\Pi^{r}_{(ef),(Mi),(ab)}$$

$$\left. + \sum_{A,b}^{I_a,I_r}\frac{V^{pp}_{(Mi),(Kl),(Ab)}}{\kappa_{nj}}\Pi^{r}_{(ef),(Ab),(nj)}\right],$$

$$t^{ppr,(ef)}_{(Mi),(Nj),(kl)} = \frac{\kappa_{ef}}{\Lambda_{ee}}\left[\Lambda_{ee}V^{ppr}_{(Mi),(Nj),(kl),(ef)} + \frac{\Pi^{p,rr}_{(kl)(ef),(Mi)(Nj)}}{\kappa_{Mi}\kappa_{Nj}}\right.$$

$$\left. +\sum_{A,b}^{I_a,I_r}V^{pp}_{(Mi),(Nj),(Ab)}\chi^{rr}_{(kl)(ef),(Ab)} + \sum_{a,b}^{I_a,I_r}\frac{V^{pr}_{(Nj),(kl),(ab)}}{\kappa_{Mi}}\Pi^{r}_{(ef),(Mi),(ab)}\right],$$

$$t^{rrr,(ef)}_{(mi),(nj),(kl)} = \frac{\kappa_{ef}}{\Lambda_{ee}}\left[\Lambda_{ee}\bar{V}^{rrr}_{(mi),(nj),(kl),(ef)} + \frac{\Pi^{r,rr}_{(kl)(ef),(mi)(nj)}}{\kappa_{mi}\kappa_{nj}}\right.$$

$$\left. +\sum_{A,b}^{I_a,I_r}V^{rr}_{(mi),(nj),(Ab)}\chi^{rr}_{(kl)(ef),(Ab)} + \sum_{A,b}^{I_a,I_r}\frac{V^{rr}_{(mi),(kl),(Ab)}}{\kappa_{nj}}\Pi^{r}_{(ef),(Ab),(nj)}\right];$$

$$\bar{V}^{ppp}_{(Mi),(Nj),(Kl),(Ab)} = V^{ppp}_{(Mi),(Nj),(Kl),(Ab)} + V^{ppp}_{(Mi),(Kl),(Nj),(Ab)},$$

$$\bar{V}^{prr}_{(Mi),(nj),(kl),(Ab)} = V^{prr}_{(Mi),(nj),(kl),(Ab)} + V^{prr}_{(Mi),(kl),(nj),(Ab)} + V^{rpr}_{(nj),(Mi),(kl),(Ab)},$$

$$\bar{V}^{pr}_{(Mi),(nj),(ab)} = V^{pr}_{(Mi),(nj),(ab)} + V^{rp}_{(nj),(Mi),(ab)},$$

$$\bar{V}^{rrr}_{(mi),(nj),(kl),(ab)} = V^{rrr}_{(mi),(nj),(kl),(ab)} + V^{rrr}_{(mi),(kl),(nj),(ab)},$$

$$\bar{V}^{rpp}_{(kl),(Mi),(Nj),(ab)} = V^{rpp}_{(kl),(Mi),(Nj),(ab)} + V^{rpp}_{(kl),(Nj),(Mi),(ab)} + V^{ppr}_{(Mi),(Nj),(kl),(ab)}.$$

The formulas suggest finite I_a and I_r but adopting the limit $I_a, I_r \to \infty$ gives the inifinite-dimensional system (22) where computing the hydrodynamic coefficients implies an infinite inner summation.

6. The Narimanov–Moiseev modal equations

Assuming (a) the $O(\epsilon)$-order small-amplitude harmonic excitations with the forcing frequency σ close to the lowest natural sloshing frequency (here, $\sqrt{g\kappa_{11}}$), (b) there are no secondary resonances and other small nondimensional parameters, e.g., shallow liquid depth, Moiseev [7] (i) showed that the dominant sloshing response is then of the order $O(\epsilon^{1/3})$ contributed, exclusively, by the primary excited lowest modes (here, the two generalised coordinates p_{11} and r_{11}) and (ii) derived a necessary (secular) condition of the steady-state (time-periodic) solution to exist. Similar intermodal relations were postulated by Narimanov [8]. For the axisymmetric tanks, due to the trigonometric algebra with respect to the angular coordinate, the Narimanov–Moiseev intermodal relations deduce the following ordering for the generalised coordinates

$$\begin{aligned} &p_{11} \sim r_{11} = O(\epsilon^{1/3}), \quad p_{0j} \sim p_{2j} \sim r_{2j} = O(\epsilon^{2/3}), \\ &\qquad r_{1(j+1)} \sim p_{1(j+1)} \sim p_{3j} \sim r_{3j} = O(\epsilon), \quad j = 1, 2, \dots, \end{aligned} \tag{23}$$

but the other generalised coordinates $r_{kl} \sim p_{kl} = o(\epsilon)$, $k \geq 4$ and, therefore, these can be neglected in the Narimanov–Moiseev asymptotic scheme. The latter means that $I_a = 3$ but I_r may vary from 1 to infinity.

Using (23) and neglecting the $o(\epsilon)$-terms, tedious but straightforward derivations reduce (22) to the *complete Narimanov–Moseev modal system*

$$\begin{aligned} &\ddot{p}_{11} + \sigma_{11}^2 p_{11} + d_1 p_{11}\left(\ddot{p}_{11}p_{11} + \ddot{r}_{11}r_{11} + \dot{p}_{11}^2 + \dot{r}_{11}^2\right) \\ &\qquad + d_2\left[r_{11}(\ddot{p}_{11}r_{11} - \ddot{r}_{11}p_{11}) + 2\dot{r}_{11}(\dot{p}_{11}r_{11} - \dot{r}_{11}p_{11})\right] \\ &+ \sum_{j=1}^{I_r}\Big[d_3^{(j)}(\ddot{p}_{11}p_{2j} + \ddot{r}_{11}r_{2j} + \dot{p}_{11}\dot{p}_{2j} + \dot{r}_{11}\dot{r}_{2j}) + d_4^{(j)}(\ddot{p}_{2j}p_{11} + \ddot{r}_{2j}r_{11}) \\ &+ d_5^{(j)}(\ddot{p}_{11}p_{0j} + \dot{p}_{11}\dot{p}_{0j}) + d_6^{(j)}\ddot{p}_{0j}p_{11}\Big] = -(\ddot{\eta}_2 + g\eta_4 + S_b\ddot{\eta}_4)\kappa_{11}P_1, \end{aligned} \tag{24a}$$

$$\begin{aligned} &\ddot{r}_{11} + \sigma_{11}^2 r_{11} + d_1 r_{11}\left[\ddot{p}_{11}p_{11} + \ddot{r}_{11}r_{11} + \dot{p}_{11}^2 + \dot{r}_{11}^2\right] \\ &\qquad + d_2\left[p_{11}(\ddot{r}_{11}p_{11} - \ddot{p}_{11}r_{11}) + 2\dot{p}_{11}(\dot{r}_{11}p_{11} - \dot{p}_{11}r_{11})\right] \\ &+ \sum_{j=1}^{I_r}\Big[d_3^{(j)}(\ddot{p}_{11}r_{2j} - \ddot{r}_{11}p_{2j} + \dot{p}_{11}\dot{r}_{2j} - \dot{p}_{2j}\dot{r}_{11}) + d_4^{(j)}(\ddot{r}_{2j}p_{11} - \ddot{p}_{2j}r_{11}) \\ &+ d_5^{(j)}(\ddot{r}_{11}p_{0j} + \dot{r}_{11}\dot{p}_{0j}) + d_6^{(j)}\ddot{p}_{0j}r_{11}\Big] = -(\ddot{\eta}_2 + g\eta_4 + S_b\ddot{\eta}_4)\kappa_{11}P_1; \end{aligned} \tag{24b}$$

$$\ddot{p}_{2k} + \sigma_{2k}^2 p_{2k} + d_{7,k}(\dot{p}_{11}^2 - \dot{r}_{11}^2) + d_{9,k}(\ddot{p}_{11}p_{11} - \ddot{r}_{11}r_{11}) = 0, \tag{25a}$$

$$\ddot{r}_{2k} + \sigma_{2k}^2 r_{2k} + 2d_{7,k}\dot{p}_{11}\dot{r}_{11} + d_{9,k}(\ddot{p}_{11}r_{11} + \ddot{r}_{11}p_{11}) = 0, \tag{25b}$$

$$\ddot{p}_{0k} + \sigma_{0k}^2 p_{0k} + d_{8,k}(\dot{p}_{11}^2 + \dot{r}_{11}^2) + d_{10,k}(\ddot{p}_{11}p_{11} + \ddot{r}_{11}r_{11}) = 0; \tag{25c}$$

$$\ddot{p}_{3k} + \sigma_{3k}^2 p_{3k} + d_{11,k}\left[\ddot{p}_{11}(p_{11}^2 - r_{11}^2) - 2p_{11}r_{11}\ddot{r}_{11}\right] + d_{12,k}\left[p_{11}(\dot{p}_{11}^2 - \dot{r}_{11}^2) - 2r_{11}\dot{p}_{11}\dot{r}_{11}\right] + \sum_{j=1}^{I_r}\Big[d_{13,k}^{(j)}(\ddot{p}_{11}p_{2j} - \ddot{r}_{11}r_{2j}) + d_{14,k}^{(j)}(\ddot{p}_{2j}p_{11} - \ddot{r}_{2j}r_{11}) + d_{15,k}^{(j)}(\dot{p}_{2j}\dot{p}_{11} - \dot{r}_{2j}\dot{r}_{11})\Big] = 0, \tag{26a}$$

$$\ddot{r}_{3k} + \sigma_{3k}^2 r_{3k} + d_{11,k}\left[\ddot{r}_{11}(p_{11}^2 - r_{11}^2) + 2p_{11}r_{11}\ddot{p}_{11}\right] + d_{12,k}\left[r_{11}(\dot{p}_{11}^2 - \dot{r}_{11}^2) + 2p_{11}\dot{p}_{11}\dot{r}_{11}\right] + \sum_{j=1}^{I_r}\Big[d_{13,k}^{(j)}(\ddot{p}_{11}r_{2j} + \ddot{r}_{11}p_{2j}) + d_{14,k}^{(j)}(\ddot{p}_{2j}r_{11} + \ddot{r}_{2j}p_{11}) + d_{15,k}^{(j)}(\dot{p}_{2j}\dot{r}_{11} + \dot{r}_{2j}\dot{p}_{11})\Big] = 0, \quad k = 1, ..., I_r; \tag{26b}$$

$$\begin{aligned}\ddot{p}_{1n} &+ \sigma_{1n}^2 p_{1n} + d_{16,n}(\ddot{p}_{11}p_{11}^2 + r_{11}p_{11}\ddot{r}_{11}) + d_{17,n}(\ddot{p}_{11}r_{11}^2 - r_{11}p_{11}\ddot{r}_{11}) \\ &+ d_{18,n}p_{11}(\dot{p}_{11}^2 + \dot{r}_{11}^2) + d_{19,n}(r_{11}\dot{p}_{11}\dot{r}_{11} - p_{11}\dot{r}_{11}^2) \\ &+ \sum_{j=1}^{I_r}\Big[d_{20,n}^{(j)}(\ddot{p}_{11}p_{2j} + \ddot{r}_{11}r_{2j}) + d_{21,n}^{(j)}(p_{11}\ddot{p}_{2j} + r_{11}\ddot{r}_{2j}) \\ &+ d_{22,n}^{(j)}(\dot{p}_{11}\dot{p}_{2j} + \dot{r}_{11}\dot{r}_{2j}) + d_{23,n}^{(j)}\ddot{p}_{11}p_{0j} + d_{24,n}^{(j)}p_{11}\ddot{p}_{0j} + d_{25,n}^{(j)}\dot{p}_{11}\dot{p}_{0j}\Big] \\ &= -(\ddot{\eta}_1 - g\eta_5 - S_b\ddot{\eta}_5)\kappa_{1n}\,P_n,\end{aligned} \tag{27a}$$

$$\begin{aligned}\ddot{r}_{1n} &+ \sigma_{1n}^2 r_{1n} + d_{16,n}(\ddot{r}_{11}r_{11}^2 + r_{11}p_{11}\ddot{p}_{11}) + d_{17,n}(\ddot{r}_{11}p_{11}^2 - r_{11}p_{11}\ddot{p}_{11}) \\ &+ d_{18,n}r_{11}(\dot{p}_{11}^2 + \dot{r}_{11}^2) + d_{19,n}(p_{11}\dot{p}_{11}\dot{r}_{11} - r_{11}\dot{p}_{11}^2) \\ &+ \sum_{j=1}^{I_r}\Big[d_{20,n}^{(j)}(\ddot{p}_{11}r_{2j} - \ddot{r}_{11}p_{2j}) + d_{21,n}^{(j)}(p_{11}\ddot{r}_{2j} - r_{11}\ddot{p}_{2j}) \\ &+ d_{22,n}^{(j)}(\dot{p}_{11}\dot{r}_{2j} - \dot{r}_{11}\dot{p}_{2j}) + d_{23,n}^{(j)}\ddot{r}_{11}p_{0j} + d_{24,n}^{(j)}r_{11}\ddot{p}_{0j} + d_{25,n}^{(j)}\dot{r}_{11}\dot{p}_{0j}\Big]\end{aligned}$$

$$= -(\ddot{\eta}_2 + g\eta_4 + S_n\ddot{\eta}_4)\kappa_{1n}P_n, \quad n = 2, ..., I_r, \quad (27b)$$

where the hydrodynamic coefficients are computed by the formulas

$$d_1 = d^{ppp,(11)}_{(11),(11),(11)} = d^{rrr,(11)}_{(11),(11),(11)} = t^{ppp,(11)}_{(11),(11),(11)} = t^{rrr,(11)}_{(11),(11),(11)},$$

$$d_2 = d^{prr,(11)}_{(11),(11),(11)} = d^{rpp,(11)}_{(11),(11),(11)} = \tfrac{1}{2}t^{prr,(11)}_{(11),(11),(11)} = \tfrac{1}{2}t^{prp,(11)}_{(11),(11),(11)},$$

$$d_1 - d_2 = d^{rrp,(11)}_{(11),(11),(11)} = d^{prp,(11)}_{(11),(11),(11)},$$

$$d_1 - 2d_2 = t^{rrp,(11)}_{(11),(11),(11)} = t^{ppr,(11)}_{(11),(11),(11)},$$

$$d_3^{(j)} = d^{pp,(11)}_{(11),(2j)} = d^{rr,(11)}_{(11),(2j)} = d^{pr,(11)}_{(11),(2j)} = -d^{rp,(11)}_{(11),(2j)} = t^{pr,(11)}_{(11),(2j)} = -t^{pr,(11)}_{(2j),(11)}$$

$$= t^{pp,(11)}_{(11),(2j)} + t^{pp,(11)}_{(2j),(11)} = t^{rr,(11)}_{(11),(2j)} + t^{rr,(11)}_{(2j),(11)},$$

$$d_4^{(j)} = d^{pp,(11)}_{(2j),(11)} = d^{rr,(11)}_{(2j),(11)} = -d^{pr,(11)}_{(2j),(11)} = d^{rp,(11)}_{(2j),(11)},$$

$$d_5^{(j)} = d^{pp,(11)}_{(11),(0j)} = d^{rp,(11)}_{(11),(0j)} = t^{pr,(11)}_{(0j),(11)} = t^{pp,(11)}_{(0j),(11)} + t^{pp,(11)}_{(11),(0j)},$$

$$d_6^{(j)} = d^{pp,(11)}_{(0j),(11)} = d^{pr,(11)}_{(0j),(11)}, \; d_{7,k} = t^{pp,(2k)}_{(11),(11)} = -t^{rr,(2k)}_{(11),(11)} = \tfrac{1}{2}t^{pr,(2k)}_{(11),(11)},$$

$$d_{8,k} = t^{pp,(0k)}_{(11),(11)} = t^{rr,(0k)}_{(11),(11)}, \; d_{10,k} = d^{pp,(0k)}_{(11),(11)} = d^{rr,(0k)}_{(11),(11)},$$

$$d_{9,k} = d^{pp,(2k)}_{(11),(11)} = -d^{rr,(2k)}_{(11),(11)} = d^{pr,(2k)}_{(11),(11)} = d^{rp,(2k)}_{(11),(11)},$$

$$d_{11,k} = d^{ppp,(3k)}_{(11),(11),(11)} = -d^{rrr,(3k)}_{(11),(11),(11)} = -\tfrac{1}{2}d^{rrp,(3k)}_{(11),(11),(11)},$$

$$d_{12,k} = t^{ppp,(3k)}_{(11),(11),(11)} = -t^{rrp,(3k)}_{(11),(11),(11)} = -\tfrac{1}{2}t^{prr,(3k)}_{(11),(11),(11)} = t^{ppr,(3k)}_{(11),(11),(11)}$$

$$= -t^{rrr,(3k)}_{(11),(11),(11)} = \tfrac{1}{2}t^{prp,(3k)}_{(11),(11),(11)},$$

$$d_{13,k}^{(j)} = d^{pp,(3k)}_{(11),(2j)} = -d^{rr,(3k)}_{(11),(2j)} = d^{pr,(3k)}_{(11),(2j)} = d^{rp,(3k)}_{(11),(2j)},$$

$$d_{14,k}^{(j)} = d^{pp,(3k)}_{(2j),(11)} = -d^{rr,(3k)}_{(2j),(11)} = d^{pr,(3k)}_{(2j),(11)} = d^{rp,(3k)}_{(2j),(11)},$$

$$d_{15,k}^{(j)} = t^{pp,(3k)}_{(11),(2j)} + t^{pp,(3k)}_{(2j),(11)} = -t^{rr,(3k)}_{(11),(2j)} - t^{rr,(3k)}_{(2j),(11)} = t^{pr,(3k)}_{(11),(2j)} = t^{pr,(3k)}_{(2j),(11)},$$

$$d_{16,n} = d^{ppp,(1n)}_{(11),(11),(11)} = d^{rrr,(1n)}_{(11),(11),(11)}, \quad d_{17,n} = d^{prr,(1n)}_{(11),(11),(11)} = d^{rpp,(1n)}_{(11),(11),(11)},$$

$$d_{18,n} = t^{ppp,(1n)}_{(11),(11),(11)} = t^{rrr,(1n)}_{(11),(11),(11)}, \; d_{19,n} = t^{prr,(1n)}_{(11),(11),(11)} = t^{prp,(1n)}_{(11),(11),(11)},$$

$$d_{16,n} - d_{17,n} = d^{rrp,(1n)}_{(11),(11),(11)} = d^{prp,(1n)}_{(11),(11),(11)},$$

$$d_{18,n} - d_{19,n} = t^{rrp,(1n)}_{(11),(11),(11)} = t^{ppr,(1n)}_{(11),(11),(11)},$$

$$d_{20,n}^{(j)} = d^{pp,(1n)}_{(11),(2j)} = d^{rr,(1n)}_{(11),(2j)} = d^{pr,(1n)}_{(11),(2j)} = -d^{rp,(1n)}_{(11),(2j)},$$

$$d_{21,n}^{(j)} = d_{(2j),(11)}^{pp,(1n)} = d_{(2j),(11)}^{rr,(1n)} = -d_{(2j),(11)}^{pr,(1n)} = d_{(2j),(11)}^{rp,(1n)},$$
$$d_{22,n}^{(j)} = t_{(11),(2j)}^{pp,(1n)} + t_{(2j),(11)}^{pp,(1n)} = t_{(11),(2j)}^{rr,(1n)} + t_{(2j),(11)}^{rr,(1n)} = t_{(11),(2j)}^{pr,(1n)} = -t_{(2j),(11)}^{pr,(1n)},$$
$$d_{23,n}^{(j)} = d_{(11),(0j)}^{pp,(1n)} = d_{(11),(0j)}^{rp,(1n)}; \quad d_{24,n}^{(j)} = d_{(0j),(11)}^{pp,(1n)} = d_{(0j),(11)}^{pr,(1n)},$$
$$d_{25,n}^{(j)} = t_{(11),(0j)}^{pp,(1n)} + t_{(0j),(11)}^{pp,(1n)} = t_{(0j),(11)}^{pr,(1n)}.$$

One should remember that the computational formulas for the hydrodynamic coefficients require, generally speaking, $I_a = 3$ (the special case $I_a = 2$ can be considered as an exception [6] eliminating the differential equations (26) and (27)). Another nonnegative integer I_r determines the number of differential equations in (25)–(27), the summation limit in both the formulas for the hydrodynamic coefficients and the differential equations (24), (26) and (27). The complete Narimanov–Moiseev system implies the limit $I_r \to \infty$.

Specific equalities between the d and t-tensors are due to the trigonometric algebra by the angular coordinate which are associated with the Λ-tensors by (31). The λ-tensors by (32) do not provide any smart properties and, therefore, should be found numerically.

7. The quality control (validation)

The Narimanov–Moiseev modal equations for comparisons can be found in [9] and [5, 6]. Whereas [9] does not present numerical values of the hydrodynamic coefficients, Lukovsky [5,6] computed and tabled them for a broad set of r_1 and h. These coefficients will be used for validation of the derived Narimanov–Moiseev modal equations (24), (25).

Lukovsky [5,6] derived the modal equation, in our notations, for $I_a = 2$, $I_r = 1$ adopting the following approximate truncated modal solution

$$\zeta(r,z,\theta) = \frac{\mathcal{R}_{01}(r)}{\mathcal{R}_0} p_0(t) + \frac{\mathcal{R}_{11}(r)}{\mathcal{R}_1}\left[p_1(t)\cos\theta + r_1(t)\sin\theta\right] + \frac{\mathcal{R}_{21}(r)}{\mathcal{R}_2}\left[p_2(t)\cos 2\theta + r_2(t)\sin 2\theta\right], \quad \mathcal{R}_i = \mathcal{R}_{i1}(1), \; i = 0,1,2 \quad (28)$$

(together with an analogous five-term approximation of the velocity potential) which implies the link between (28) and (8a) expressed by $p_i(t) = \mathcal{R}_i\, p_{i1}(t)$. The five-dimensional modal system by Lukovsky takes the form [6, Eqs. (4.1.15)–(4.1.19)]

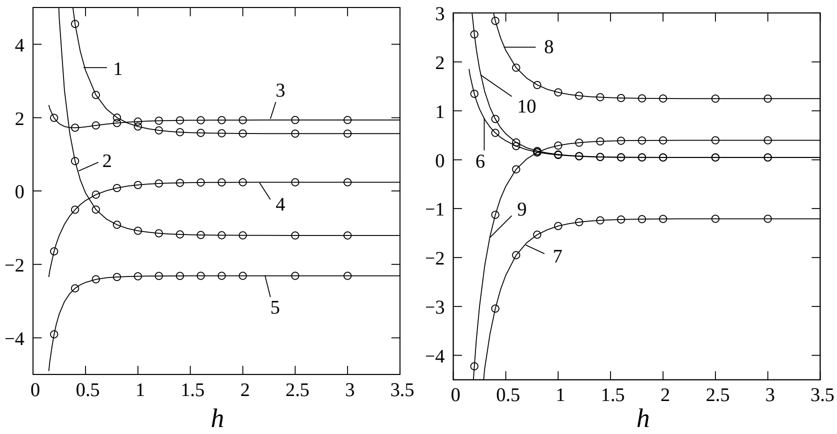

Fig. 3. The hydrodynamic coefficient of the Narimanov–Moiseev modal system within the framework of Lukovsky's [5, 6] five-dimensional approximation ($I_a = 2, I_r = 1$ in our computational formulas) versus the nondimensional liquid depth h. The *circular cross-section*, $r_1 = 0$. The solid lines denote our calculations, but the circles correspond to the tabled values from [5,6] rescaled according to (30). The integers at the graphs denote: 1 – d_1, 2 – d_2, 3 – $d_3^{(1)}$, 4 – $d_4^{(1)}$, 5 – $d_5^{(1)}$, 6 – $d_6^{(1)}$, 7 – $d_{7,1}$, 8 – $d_{8,1}$, 9 – $d_{9,1}$, and 10 – $d_{10,1}$.

$$\mu_1(\ddot{r}_1 + \sigma_1^2 r_1) + \mathrm{d}_1(r_1^2\ddot{r}_1 + r_1\dot{r}_1^2 + r_1 p_1\ddot{p}_1 + r_1\dot{p}_1^2) + \mathrm{d}_2(p_1^2\ddot{r}_1 + 2p_1\dot{r}_1\dot{p}_1 - r_1 p_1\ddot{p}_1 - 2r_1\dot{p}_1^2) - \mathrm{d}_3(r_2\ddot{r}_1 - r_2\ddot{p}_1 + \dot{r}_1\dot{p}_2 - \dot{p}_1\dot{r}_2) + \mathrm{d}_4(r_1\ddot{p}_2 - p_1\ddot{r}_2) + \mathrm{d}_5(p_0\ddot{r}_1 + \dot{r}_1\dot{p}_0) + \mathrm{d}_6 r_1\ddot{p}_0 = -\mathrm{P}\ddot{\eta}_2(t), \quad (29a)$$

$$\mu_1(\ddot{p}_1 + \sigma_1^2 p_1) + \mathrm{d}_1(p_1^2\ddot{p}_1 + r_1 p_1\ddot{r}_1 + p_1\dot{r}_1^2 + p_1\dot{p}_1^2) + \mathrm{d}_2(r_1^2\ddot{p}_1 - r_1 p_1\ddot{r}_1 + 2r_1\dot{r}_1\dot{p}_1 - 2p_1\dot{r}_1^2) + \mathrm{d}_3(p_2\ddot{p}_1 + r_2\ddot{r}_1 + \dot{r}_1\dot{r}_2 + \dot{p}_1\dot{p}_2) - \mathrm{d}_4(p_1\ddot{p}_2 + r_1\ddot{r}_2) + \mathrm{d}_5(p_0\ddot{p}_1 + \dot{p}_1\dot{p}_0) + d_6 p_1\ddot{p}_0 = -\mathrm{P}\ddot{\eta}_1(t), \quad (29b)$$

$$\mu_0(\ddot{p}_0 + \sigma_0^2 p_0) + \mathrm{d}_6(r_1\ddot{r}_1 + p_1\ddot{p}_1) + \mathrm{d}_8(\dot{r}_1^2 + \dot{p}_1^2) = 0, \quad (29c)$$

$$\mu_2(\ddot{r}_2 + \sigma_2^2 r_2) - \mathrm{d}_4(p_1\ddot{r}_1 + r_1\ddot{p}_1) - 2\mathrm{d}_7\dot{r}_1\dot{p}_1 = 0, \quad (29d)$$

$$\mu_2(\ddot{p}_2 + \sigma_2^2 p_2) + \mathrm{d}_4(r_1\ddot{r}_1 - p_1\ddot{p}_1) + \mathrm{d}_7(\dot{r}_1^2 - \dot{p}_1^2) = 0 \quad (29e)$$

that restores our hydrodynamic coefficients (computed with $I_a = 2, I_r = 1$ in all the formulas) as follows

$$d_1 = \frac{\mathrm{d}_1\mathcal{R}_1^2}{\mu_1},\ d_2 = \frac{\mathrm{d}_2\mathcal{R}_1^2}{\mu_1},\ d_3^{(1)} = \frac{\mathrm{d}_3\mathcal{R}_2}{\mu_1},\ d_4^{(1)} = -\frac{\mathrm{d}_4\mathcal{R}_2}{\mu_1},$$

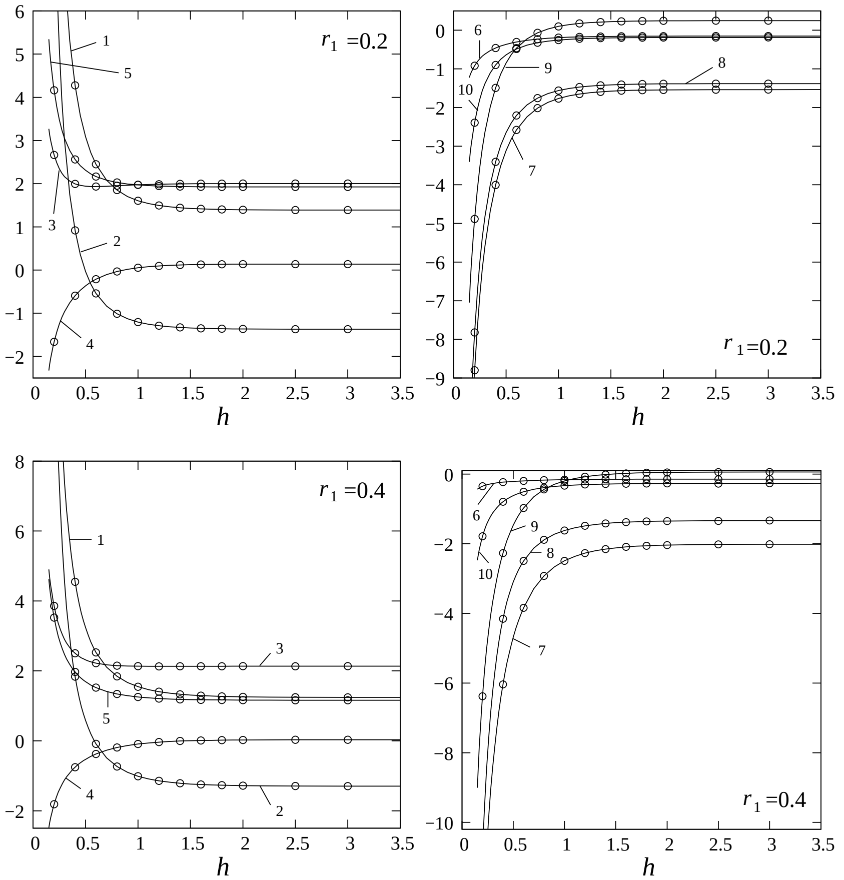

Fig. 4. The same as in fig. 3 but for the annular cross-section with $r_1 = 0.2$ and 0.4.

$$d_5^{(1)} = \frac{\mathrm{d}_5 \mathcal{R}_0}{\mu_1},\ d_6^{(1)} = \frac{\mathrm{d}_6 \mathcal{R}_0}{\mu_1},\ d_{7,1} = -\frac{\mathrm{d}_7 \mathcal{R}_1^2}{\mathcal{R}_2 \mu_2},\ d_{8,1} = \frac{\mathrm{d}_8 \mathcal{R}_1^2}{\mathcal{R}_0 \mu_0},$$
$$d_{9,1} = -\frac{\mathrm{d}_4 \mathcal{R}_1^2}{\mathcal{R}_2 \mu_2},\ d_{10,1} = \frac{\mathrm{d}_6 \mathcal{R}_1^2}{\mathcal{R}_0 \mu_0}. \tag{30}$$

Using our formulas with $I_a = 2, I_r = 1$ and the tabled hydrodynamic coefficients by Lukovsky [5, 6] (rescaled by (30)) gives almost identical results; the difference is always detected being less than 0.1%. This fact

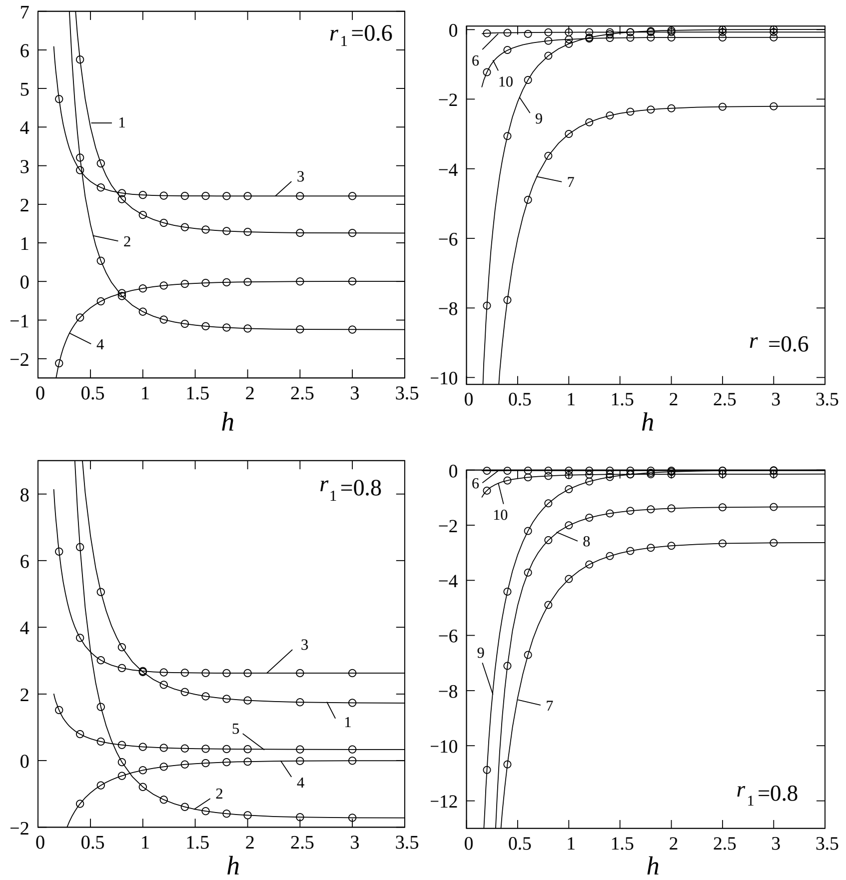

Fig. 5. The same as in fig. 3 but for the annular cross-section with $r_1 = 0.6$ and 0.8. The tabled numbers in [6] for d_5 and d_8 as $r_1 = 0.6$ are clearly wrong demonstrating a discontinuous character on h; these are excluded from our comparisons.

is illustrated in figs. 3–5.

Table 1. Computed hydrodynamic coefficients d_1 and d_2 for $h = 1$ versus I_r.

	$r_1 = 0$		$r_1 = 0.4$		$r_1 = 0.8$	
I_r	d_1	d_2	d_1	d_2	d_1	d_2
1	1.758023	-1.083355	1.549476	-1.011521	2.658499	-0.7886150
2	1.761798	-1.081642	1.670994	-0.894068	2.661058	-0.7860675
3	1.762160	-1.081505	1.677407	-0.889331	2.661086	-0.7860614
4	1.762240	-1.081476	1.678871	-0.888116	2.661098	-0.7860503
5	1.762266	-1.081467	1.679219	-0.887935	2.661100	-0.7860501
	...	...	...	...	...	...
14	1.762287	-1.081460	1.679490	-0.887773	2.661101	-0.7860490
15	1.762287	-1.081460	1.679491	-0.887773	2.661101	-0.7860490
16	1.762288	-1.081460	1.679492	-0.887771	2.661101	-0.7860490
	...	...	...	...	...	...
20	1.762288	-1.081460	1.679493	-0.887771	2.661101	-0.7860490
25	1.762288	-1.081460	1.679493	-0.887771	2.661101	-0.7860490

8. The modal systems of larger dimensions

The comparative study from the previous section validated our derivations of the Narimanov–Moiseev modal system for the very particular, $I_a = 2$ and $I_r = 1$. An extra analysis is needed to ensure us that the derived expressions are valid for a larger system dimension. A difficulty is that increasing $I_p \geq 2$ yields s series of new differential equations and changes the hydrodynamic coefficients at the cubic polynomial terms with respect to the generalised coordinates and their derivatives. These hydrodynamic coefficients, $d_1, d_2, d_{11,k}, d_{12,k}, d_{16,k}$ and $d_{18,k}$, are functions of the upper summation limit I_r. This means, in particular, that d_1 and d_2 in the Lukovsky modal system (29) cannot be used for validation of our Narimanov–Moiseev modal systems with a larger dimension, as $I_r \geq 2$.

The coefficients d_1 and d_2 versus I_r are illustrated in Table 1 for the nondimensional liquid depth $h = 1$. The table confirms that d_1 and d_2 change with I_r, but not dramatically. Lukovsky's computations [6] with $I_r = 1$ can be adopted as a rough approximation and $I_r = 2$ stabilises, at least, two significant figures of these hydrodynamic coefficients.

9. Conclusions

The complete Narimanov-Moiseev weakly-nonlinear modal system is derived for sloshing in an upright annular tank. The derivations are validated by comparison with numerical results on the hydrodynamic coefficients by Lukovsky [6]. Further studies should focus on derivations of the corresponding formulas for the hydrodynamic forces and moments. The general Lukovsky formulas would facilitate that. Another problem is a study of the modal system applicability. This should include an analysis of the secondary resonance occurrence, an estimate of damping due to the flow separation at the central pile as well as establishing a clear strategy on how many higher modes (driven by I_r) are needed to approximate steady-state and transient wave patterns. The latter task implies a quantitative comparison with experiments which can be found, e.g. in [9].

10. Notations

Axisymmetric shape yields two algebras for the natural sloshing modes, in angular and radial directions. The angular components leads to the Λ-tensor whose elements are

$$\Lambda_{M\ldots N,i\ldots j} = \int_{-\pi}^{\pi} \cos(A\theta)\ldots\cos(M\theta) \cdot \sin(i\theta)\ldots\sin(j\theta)\, d\theta. \tag{31}$$

They can be computed by recursive formulas

$$\Lambda_{M,i} = 0, \quad \Lambda_{,ij} = \pi\delta_{ij}, \quad \Lambda_{MN,} = \pi\delta_{MN}, \; M^2 + N^2 \neq 0, \; \Lambda_{00,} = 2\pi,$$
$$\Lambda_{M\ldots NK,i\ldots j} = \tfrac{1}{2}(\Lambda_{M\ldots|N-K|,i\ldots j} + \Lambda_{M\ldots|N+K|,i\ldots j}),$$
$$\Lambda_{M\ldots N,i\ldots ljk} = \tfrac{1}{2}(\Lambda_{M\ldots|j-k|,i\ldots l} - \Lambda_{M\ldots|j+k|,i\ldots l})$$

following from the corresponding trigonometrical relations.

The radial component introduces the λ-tensors defined by the formulas

$$\lambda_{(Ab)\ldots(Mn)} = \int_{r_1}^{1} r\mathcal{R}_{Ab}(r)\ldots\mathcal{R}_{Mn}(r)\, dr,$$
$$\lambda'_{(Ab)(Mn),(Cd)\ldots(Ef)} = \int_{r_1}^{1} r\mathcal{R}'_{Ab}(r)\mathcal{R}'_{Mn}(r) \cdot \mathcal{R}_{Cd}(r)\ldots\mathcal{R}_{Ef}(r)\, dr,$$
$$\bar{\lambda}_{(Ab)\ldots(Mn)} = \int_{r_1}^{1} r^{-1}\, \mathcal{R}_{Ab}(r)\ldots\mathcal{R}_{Mn}(r)\, dr \tag{32}$$

which should, generally speaking, be computed, numerically, except for the case (4).

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