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Symmetry, equivalence and integrable classes of Abel equations

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Запропоновано підхід для опису інтегровних випадків рівняння Абеля, що базується на підвищенні порядку та використанні перетворень еквівалентності для відповідного звичайного диференціального рівняння другого порядку. Розглянуто проблему лінеаризації розглядуваного класу рівнянь.

We suggest an approach for description of integrable cases of the Abel equations. It is based on increasing of the order of equations up to the second one and using equivalence transformations for the corresponding second-order ordinary differential equations. The problem of linearizability of the equations under consideration is considered.

Introduction. A diversity of methods were developed to date for finding solutions of nonlinear ordinary differential equations (ODE). Everybody who encounters integration of a particular ODE uses, as a rule, the accumulated databases (or reference books) of the classes of ODE and methods for integration of them (e.g. [22, 29]). But if an ODE does not belong to any of the described classes then it does not mean that there is no approaches for finding solutions of this ODE in a closed form.

The symmetry approach is one of the most algorithmic approaches for integration and lowering of the order of ODE that admit a certain nontrivial symmetry (see e.g. books [24, 28] and review papers [19, 35]). In the framework of the symmetry approach (and its modifications) it is possible to obtain many of the known classes of integrable ODE. However, the needs of the applications stimulate new research into development of new methods for construction of ODE solutions in the closed form. The works [2–9, 12–19, 25–27, 29–35] may give an idea of current developments and directions of research in the field of symmetry (algebraic) methods for investigation of ODE.

The problem of finding Lie symmetries for the first-order ODE is equivalent to finding solutions for these equations, and for this reason the direct application of the Lie method is complicated in the general case. On of the well-known approaches in the cases when for a given ODE it is not feasible (or not effective) to apply the Lie method directly, is increasing of the order of the ODE under consideration (in particular, to obtain a second-order ODE related to the respective ODE by a change of variables). For examples of utilisation of such approach we can refer to papers [2–6, 14, 25–27]. In such cases, if the "induced" equation of a higher order admits a non-trivial Lie symmetry (that generated a non-local symmetry for the initial equation), we can speak of so-called hidden symmetries for an initial equation (for more details see [2–4]).

Main results. In this paper we study Abel equations having the form

$$\dot{p}(f_5(y)p + f_0(y)) = p^3 f_4(y) + p^2 f_3(y) + p f_2(y) + f_1(y), \tag{1}$$

where $p = p(y)$, $\dot{p} = \frac{dp}{dy}$, f_i, $i = 0, \dots, 5$, are arbitrary smooth functions (with f_1, f_2, f_3, f_4, f_5 not identically vanishing simultaneously). In view of existence of the gauge transformation of multiplication by an arbitrary function of y, any equation (1) can be reduced to one of the following canonical forms (respectively, Abel equations of the first and the second kind, see e.g. [1, 22, 29]):

$$\dot{p} = p^3 f_4(y) + p^2 f_3(y) + p f_2(y) + f_1(y), \tag{2}$$

$$\dot{p}(p + f_0(y)) = p^3 f_4(y) + p^2 f_3(y) + p f_2(y) + f_1(y). \tag{3}$$

Equations (2), (3) along with the Riccati equation are among the "simplest" nonlinear first-order ODE that have extensive applications. At the same time the problem of description of integrable classes of these equations stays within the focus of current research, and was previously considered in many papers (see e.g. [5, 6, 10–13, 27, 29, 30, 32–34, 36]).

Note that the Abel equations of the first and the second kind (2), (3) are related with each other by a local change of variables (namely, the equation (3) can be reduced to the form (2) by means of the change of variables $p = 1/v(y) - f_0$). Besides, the well-known Riccati equation is a partial case of equation (2).

Further we will consider the following second-order ODE

$$\ddot{y} = \dot{y}^4 f_4(y) + \dot{y}^3 f_3(y) + \dot{y}^2 f_2(y) + \dot{y} f_1(y), \tag{4}$$

$$\ddot{y}(\dot{y} + f_0(y)) = \dot{y}^4 f_4(y) + \dot{y}^3 f_3(y) + \dot{y}^2 f_2(y) + \dot{y} f_1(y), \tag{5}$$

where $y = y(x)$, $\dot{y} = \frac{dy}{dx}$, $\ddot{y} = \frac{d^2y}{dx^2}$, related to the Abel equations (2) and (3).

The substitution $\dot{y} = p(y)$ reduces equations (4) and (5) respectively to the Abel equations (2) and (3) (reduction of the order for equations (4) and (5)). Such reduction is induced by the Lie operator $X_1 = \partial_x$ (that corresponds to invariance of equations (4) and (5) with respect to translations by the variable x). This is exactly the fact that explains why we consider equations (4) and (5).

In the case when (4) or (5) are invariant with respect to another operator (that is when (4) or (5) admit two-dimensional Lie algebras), then equations (4) and (5) are integrable in the framework of the Lie approach. And in this way we can obtain exact solutions of the equations (2) and (3) respectively.

Further we will consider only the equation (5) (since equations (2)–(5) are interconnected – see Remark 3). Let (5) admit a two-dimensional Lie algebra

$$L = \langle X_1, X_2 \rangle, \quad X_1 = \partial_x, \quad X_2 = \xi(x,y)\partial_x + \eta(x,y)\partial_y. \tag{6}$$

We will consider a problem of description of inequivalent equations (5) that are invariant with respect to two-dimensional Lie algebras of the form (6) (non-equivalent realizations of the operator X_2 in the algebra (6) will determine canonical representatives for equation (5)).

It is well-known that any two-dimensional Lie algebra in the general case, by means of choosing the basis operators X_1 and X_2 in an appropriate manner, may be reduced to four nonequivalent cases (see e.g. [19, 24, 28]). In the framework of our problem additional cases arise as we have fixed the form of the operator X_1.

So, it is quite straightforward to show that equation (5) may admit a two-dimensional Lie algebra (6) only of one of the following types:

$$\begin{array}{lll} 1. & [X_1, X_2] = 0, & \operatorname{rank} L = 1; \\ 2. & [X_1, X_2] = 0, & \operatorname{rank} L = 2; \\ 3. & [X_1, X_2] = X_1, & \operatorname{rank} L = 1; \\ 4. & [X_1, X_2] = X_1, & \operatorname{rank} L = 2; \\ 5. & [X_1, X_2] = X_2, & \operatorname{rank} L = 1; \\ 6. & [X_1, X_2] = X_2, & \operatorname{rank} L = 2. \end{array} \tag{7}$$

Further, utilising classification of two-dimensional algebras (7), we obtain that equation (5) may admit only the following realizations of two-dimensional Lie algebras (6):

$$
\begin{aligned}
&1.\ \ X_1=\partial_x, \quad X_2=\xi(y)\partial_x, \quad \xi(y)\not\equiv \mathrm{const};\\
&2.\ \ X_1=\partial_x, \quad X_2=\xi(y)\partial_x+\eta(y)\partial_y,\\
&\qquad\qquad\qquad \xi(y)\not\equiv \mathrm{const}\ \ \text{or}\quad \xi(y)\equiv 0,\quad \eta(y)\neq 0;\\
&3.\ \ X_1=\partial_x, \quad X_2=(x+\xi(y))\partial_x, \quad \xi(y)\not\equiv \mathrm{const}\ \ \text{or}\ \ \xi(y)\equiv 0;\\
&4.\ \ X_1=\partial_x, \quad X_2=(x+\xi(y))\partial_x+\eta(y)\partial_y,\\
&\qquad\qquad\qquad \xi(y)\not\equiv \mathrm{const}\ \ \text{or}\quad \xi(y)\equiv 0,\quad \eta(y)\neq 0;\\
&5.\ \ X_1=\partial_x, \quad X_2=e^x\xi(y)\partial_x, \quad \xi(y)\not\equiv 0;\\
&6.\ \ X_1=\partial_x, \quad X_2=e^x(\xi(y)\partial_x+\eta(y)\partial_y), \quad \eta(y)\neq 0.
\end{aligned}
\tag{8}
$$

It is clear that using these realizations we can describe equations of the form (5) that are invariant with respect to two-dimensional Lie algebras (similarly as we have discussed in [32]). However, this way is too cumbersome, and thus obtained types of equations (5) will be quite complicated (functions f_i, $i=0,\dots,4$ in (5) will be expressed through coefficients of the operator X_2 from realizations (8)).

It is straightforward to show that the most general transformations that preserve the form of the operator X_1 we look as follows:

$$
t=x+\omega(y), \quad u=g(y),
\tag{9}
$$

where $\omega(y)$, $g(y)$ are arbitrary smooth functions, $g(y)\not\equiv\mathrm{const}$.

After substitution (9) equation (5) takes the form

$$
\begin{aligned}
&\ddot u((1-\omega' f_0)\dot u+f_0 g')g'^2=\\
&\qquad=\big(f_4-\omega''(1-\omega' f_0)-\omega' f_3+\omega'^2 f_2-\omega'^3 f_1\big)\dot u^4\\
&\qquad+\big(g' f_3-\omega'' g' f_0+g''(1-\omega' f_0)-2\omega' g' f_2+3\omega'^2 g' f_1\big)\dot u^3\\
&\qquad+\big(g'^2 f_2+g'' g'^2 f_0-3\omega' g'^2 f_1\big)\dot u^2+f_1 g'^3\dot u,
\end{aligned}
\tag{10}
$$

where $\omega'=\frac{d\omega}{dy}$, $\omega''=\frac{d^2\omega}{dy^2}$, $g'=\frac{dg}{dy}$, $g''=\frac{d^2 g}{dy^2}$ (in addition in (10) all functions of the variable y should be expressed as functions of the variable u).

With $(1-\omega' f_0)\not\equiv 0$ equation (10) belongs again to the class of equations (5).

Remark 1. With $(1-\omega' f_0) \equiv 0$ after the substitution (9), equation (5) is transformed to the equation (4), that is reduced to the Abel equation of the first kind (2).

Remark 2. It is possible to regard that $(1-\omega' f_0) \not\equiv 0$ for the equation (5) as a result of the substitution (9) (we attain that by combination of transformations (9)).

Thus (9) are equivalence transformations for (5), and, besides, these transformations preserve the form of the operator $X_1 = \partial_x$ in the algebra (6).

Remark 3. So, the transformations (9) are equivalence transformations for the class of equations (4)–(5). Moreover, if we prolongate these transformations for $\dot{u} = p$ then they form an equivalence transformation group for (1) and include as a subgroup in the complete equivalence group of class (1), which are formed by the transformations

$$\tilde{y} = F(y), \quad \tilde{p} = \frac{P_1(t)p + Q_1(t)}{P_2(t)p + Q_2(t)},$$

where F, P_1, P_2, Q_1, Q_2 are arbitrary analytic functions, and $P_1Q_2 - P_2Q_1 \neq 0$.

Thus, by means of transformations (9), realizations (8) of the algebra (6) may be reduced to the simplest canonical form. The transformations (9) in that process will not take us out of the class of equations (5).

By means of transformations (9) the realizations (8) of two-dimensional Lie algebras (6) admitted for equation (5) are reduced to the following canonical realizations:

$$\begin{aligned}
&1.\ X_1 = \partial_x, \quad X_2 = y\partial_x;\\
&2.\ X_1 = \partial_x, \quad X_2 = \partial_y;\\
&3.\ X_1 = \partial_x, \quad X_2 = x\partial_x;\\
&4.\ X_1 = \partial_x, \quad X_2 = x\partial_x + y\partial_y;\\
&5.\ X_1 = \partial_x, \quad X_2 = e^x\partial_x;\\
&6.\ X_1 = \partial_x, \quad X_2 = e^x(\partial_x + \partial_y).
\end{aligned} \tag{11}$$

In accordance to (11) we obtain the following integrable cases for equation (5) that are non-equivalent with respect to (9):

1. $\ddot{y} = \alpha(y)\dot{y}^3;$

2. $\ddot{y}(\dot{y}+e) = d\dot{y}^4 + c\dot{y}^3 + b\dot{y}^2 + a\dot{y}$;
3. $\ddot{y} = \alpha(y)\dot{y}^2$;
4. $y\ddot{y}(\dot{y}+e) = d\dot{y}^4 + c\dot{y}^3 + b\dot{y}^2 + a\dot{y}$;
5. $\ddot{y}(\dot{y}+\beta(y)) = \alpha(y)\dot{y}^3 + (1-\alpha(y)\beta(y))\dot{y}^2 - \beta(y)\dot{y}$;
6. $a)$ $f_0 = 0:$

$$\ddot{y} = de^y\dot{y}^3 + (-3de^y + c)\dot{y}^2 + (-de^y - (2c+1) + be^{-y})\dot{y} + (-de^y + (c+1) - be^{-y} + ae^{-2y});$$

$b)$ $f_0 \neq 0:$

$$\ddot{y}(\dot{y}+\alpha(y)) = -\dot{y}^3 + (1-\alpha(y))\dot{y}^2 + \alpha(y)\dot{y}, \tag{12}$$

where $\alpha(y)$, $\beta(y)$ are arbitrary smooth functions, a, b, c, d, e are constants.

The case 6a in (12) may be simplified by means of the substitution $t = x$, $u = e^y$ (see (9) and (10)).

Equations (12) determine non-equivalent cases of the form (5) that admit two-dimensional algebras (11) up to equivalence transformations (9).

Thus, summarising the above, we come to the following scheme for integration of the Abel equation (3):

- we increase the order of equation (3), considering a second-order equation (5);
- if a corresponding equation (5) admits a two-dimensional Lie algebra, then we reduce this algebra to one of the canonical forms (11), and thus the equation is reduced to the respective canonical forms (12);
- we integrate the canonical form (12);
- making reverse changes of variables we obtain the solution of the Abel equation (3).

Case of Lie's linearization test. According to results of S. Lie [23] (see also [19–21]) second-order ODEs

$$\ddot{y} = f(x, y, \dot{y}) \tag{13}$$

can be reduced to the form

$$\ddot{u} = 0, \tag{14}$$

by point change of variables

$$t = \varphi(x, y), \quad u = \psi(x, y), \quad u = u(t) \tag{15}$$

if equations (13) is at most cubic in the first derivative, i.e. only if equations (13) has the form

$$\ddot{y} + F_3(x, y)\dot{y}^3 + F_2(x, y)\dot{y}^2 + F_1(x, y)\dot{y} + F(x, y) = 0, \tag{16}$$

where

$$\begin{aligned}
F_3(x, y) &= \frac{\varphi_y \psi_{yy} - \psi_y \varphi_{yy}}{\varphi_x \psi_y - \varphi_y \psi_x},\\
F_2(x, y) &= \frac{\varphi_x \psi_{yy} - \psi_x \varphi_{yy} + 2(\varphi_y \psi_{xy} - \psi_y \varphi_{xy})}{\varphi_x \psi_y - \varphi_y \psi_x},\\
F_1(x, y) &= \frac{\varphi_y \psi_{xx} - \psi_y \varphi_{xx} + 2(\varphi_x \psi_{xy} - \psi_x \varphi_{xy})}{\varphi_x \psi_y - \varphi_y \psi_x},\\
F(x, y) &= \frac{\varphi_x \psi_{xx} - \psi_x \varphi_{xx}}{\varphi_x \psi_y - \varphi_y \psi_x}.
\end{aligned} \tag{17}$$

For given function $F_3(x, y)$, $F_3(x, y)$, $F_1(x, y)$ and $F(x, y)$ linearization is possible iff the over-determined system (17) is integrable. S. Lie proved that system (17) is integrable iff the following auxiliary system for w and z

$$\begin{aligned}
\frac{\partial w}{\partial x} &= zw - FF_3 - \frac{1}{3}\frac{\partial F_1}{\partial y} + \frac{2}{3}\frac{\partial F_2}{\partial x},\\
\frac{\partial w}{\partial y} &= -w^2 + F_2 w + F_3 z + \frac{\partial F_3}{\partial x} - F_1 F_3,\\
\frac{\partial z}{\partial x} &= z^2 - Fw - F_1 z + \frac{\partial F}{\partial y} + FF_2,\\
\frac{\partial z}{\partial y} &= -zw + FF_3 - \frac{1}{3}\frac{\partial F_2}{\partial x} + \frac{2}{3}\frac{\partial F_1}{\partial y}
\end{aligned} \tag{18}$$

is compatible. The compatibility conditions for this system have the form

$$\begin{aligned}
&3(F_3)_{xx} - 2(F_2)_{xy} + (F_1)_{yy} =\\
&\qquad = (3F_1F_3 - F_2^2)_x - 3(FF_3)_y - 3F_3F_y + F_2(F_1)_y,\\
&3F_{yy} - 2(F_1)_{xy} + (F_2)_{xx} =
\end{aligned}$$

$$= 3(FF_3)_x - 3(FF_2 - F_1^2)_y + 3F(F_3)_x - F_1(F_2)_x \qquad (19)$$

(subscripts x and y denote differentiations with respect to x and y, respectively).

So, following [20, 21] a necessary and sufficient condition of lineariziation for equations of form (16) is that functions $F_3(x,y)$, $F_2(x,y)$, $F_1(x,y)$ and $F(x,y)$ satisfy the conditions (19).

In case $f_0(y) \equiv 0$ equations (5) is partial case of (16), i.e. have the following form

$$\ddot{y} = \dot{y}^3 f_4(y) + \dot{y}^2 f_3(y) + \dot{y} f_2(y) + f_1(y). \qquad (20)$$

This equations can be linearizable if $f_4(y)$, $f_3(y)$, $f_1(y)$ and $f_1(y)$ satisfy the conditions (following (19))

$$\begin{aligned} &(f_2)_{yy} = 3(f_1 f_4)_y + 3f_4(f_1)_y - f_3(f_2)_y, \\ &3(f_1)_{yy} = 3(f_1 f_3 - f_2^2)_y. \end{aligned} \qquad (21)$$

And point transformations (15) which linearizing (20) can be found from system (17).

Let us note that a second-order ODE is linearizable iff it admits an eight-dimensional Lie algebra. So, any linearizable equation (20) belongs, up to equivalence transformations (9), belong to the set of equations (12).

Conclusion. It is obvious from the above that there is an alternative way for generation of new integrable cases of the Abel equation based on utilisation of the relation between the Abel equations of the first and the second kind, and relation between the equations (4) and (5) by means of the transformations (9). Thus, starting from some integrable Abel equation (that is of such equation for which the solution is known) it is possible to obtain new integrable cases of the Abel equations (solutions of these equations will be related through transformations (9)). It would be possible to use for this purpose even the well-known Riccati equation that is a partial case of the equation (2) (for generation of integrable Riccati equations an approach that is proposed in [30] may be used).

We hope that new results for classification of integrable classes of ODE may be obtained also using our classification of inequivalent realizations of real low-dimensional Lie algebras [31].

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