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Preliminary group classification of the general quasi-linear wave equation

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Розпочато групову класифікацію квазілінійного хвильового рівняння другого порядку найбільш загальної форми. Знайдено канонічні форми операторів симетрії, які генерують групу інваріантності рівняння, описано перетворення еквівалентності та рівняння, що допускають одно- та двовимірні групи інваріантності.

We begin the group classification of quasi-linear second-order wave equations of the most general form. We find the canonical forms for the symmetry operators which generate the invariance group of the equation, as well as the equivalence group, and we describe those equations which admit one- and two-dimensional invariance groups.

1. Introduction. The group classification of partial differential equations of mathematical physics occupies an important place among the fundamental problems of modern group analysis of differential equations [1–3]. The solution of this problem is of particular interest because one is able to exploit the powerful methods of Lie groups and algebras for the analysis and construction of solutions of equations that model physical processes and which possess non-trivial symmetry properties. The problem derives its importance from the need to choose a differential equation from some general class of differential equations modelling a process, which admit a non-trivial group symmetry. The history of the solution of the problem of group classification of differential equations begins with the work of Sophus Lie. The first article of Lie on this problem was [4]. The modern form of the group classification problem was formulated by Ovsiannikov in his article [5], in which he proposed a procedure for its solution (which we shall call the Lie–Ovsiannikov method) and where he obtained a complete classification of the non-linear heat conductivity equation. After this, the group classification of differential equations became the subject of intensive research. A detailed survey of the work done in this area up to the beginning of the 1990's is given in [6]. In the present article we solve the problem of group classification for non-linear wave equations of the form

$$u_{tt} = F(t, x, u, u_x)u_{xx} + G(t, x, u, u_x), \tag{1}$$

where $F \neq 0$, G are arbitrary smooth functions, $u = u(t, x)$. We note that differential equations of the form (1) are of great importance in mathematical physics and are used in modelling various types of dispersion of waves. They have found applications in differential geometry, hydrodynamics, gas dynamics, as well as in chemical engineering and superconductivity.

Many articles have been written on the subject of group classification of quasi-linear equations of the form (1). A complete solution of the problem of group classification of the general linear equation is given in [4, 7]. The method of Lie–Ovsiannikov has also been applied to give a full solution of the problem for a number of non-linear wave equations:

$$\begin{array}{ll}
u_{tt} = u_{xx} + F(u); & [8\text{–}11] \\
u_{tt} = \left[f(u)u_x\right]_x; & [12, 13] \\
u_{tt} = f(u_x)u_{xx}; & [14] \\
u_{tt} = f(u_x)u_{xx} + G(u_x); & [15] \\
u_{tt} = u_x^m u_{xx} + F(u); & [16] \\
u_{tt} + f(u)u_t = \left[F(u)u_x\right]_x; & [17] \\
u_{tt} + f(u)u_t = \left[F(u)u_x\right]_x + G(u)u_x. & [18]
\end{array}$$

As one can see, the equations listed above involve arbitrary functions of one variable. This is connected with the fact that the standard method of performing a group classification involves solving a defining system of equations for the symmetry operators, and for the equations listed above it is possible to solve these defining equations because the arbitrary elements are functions of just one variable. However, one is confronted with a different situation when these arbitrary functions are functions of two or more variables and the defining equations for the symmetry operators then involve first order partial derivatives of the functions which render difficult or even impossible the complete solution of the defining system of equations. It is this which explains why the classification of the equations

$$\begin{array}{ll} u_{tt} = [f(x,u)u_x]_x\,; & [19] \\ u_{tt} + \lambda u_{xx} = g(u,u_x); & [20,21] \\ u_{tt} = [f(u_x)u_x + g(x,u)]_x\,; & [22–24] \\ u_{tt} = f(x,u_x)u_{xx} + g(x,u_x) & [25] \end{array}$$

is, in the classical sense of Lie, incomplete.

In the differential equations which we study, the arbitrary functions are functions of four variables, and therefore we shall use the method for the solution of the problem of the group classification which was described in [26] and was used in the group classification of the following equations:

$$\begin{array}{ll} u_t = u_{xx} + F(t,x,u,u_x); & [26] \\ u_t = F(t,x,u,u_x)u_{xx} + G(t,x,u,u_x); & [27] \\ u_t = u_{xxx} + F(t,x,u,u_x,u_{xx}); & [28] \\ u_{tt} = u_{xx} + F(t,x,u,u_x). & [29,30] \end{array}$$

A detailed description of the algorithmic method which we use may be found in [26, 27]. Here, we merely note that it differs from the classical method of group classification of differential equations in that we exploit all possible realizations of low-dimensional Lie algebras within the class of vector fields which are infinitesimal symmetries of the equation under study, and this also gives us further specification of the arbitrary functions. We also note that in performing the group classification of equation (1) we exclude all cases which are equivalent, under local changes of coordinates, to a linear equation or to

$$u_{tt} = u_{xx} + F(t,x,u,u_x). \tag{2}$$

2. Preliminary results of group classification of equation (1). Our first task is to determine the form of the vector fields which are symmetry operators for equation (1), and to determine the equivalence group of equation (1) (we define this concept later). We seek these symmetry operators amongst vector fields of the form

$$Q = \tau\partial_t + \xi\partial_x + \eta\partial_u, \tag{3}$$

where $\tau(t,x,u)$, $\xi(t,x,u)$, $\eta(t,x,u)$ are smooth functions from $V=\mathbb{R}^2\times\mathbb{R}$ to $\mathbb{R}$. The variables $(t,x)\in\mathbb{R}^2$ are said to be the independent variables, and $u=u(t,x)\in\mathbb{R}$ is said to be the dependent variable. The condition that such an operator (3) be a symmetry of equation (1) is that

$$\begin{aligned}&\phi^{tt}-\phi^{xx}F-(\tau F_t+\xi F_x+\eta F_u+\phi^x F_{u_x})u_{xx}-\\&\quad-(\tau G_t+\xi G_x+\eta G_u+\phi^x G_{u_x})\big|_{(1)}=0.\end{aligned}\tag{4}$$

By a standard but tedious calculation, equation (4) gives us the following initial information about the coefficients:

$$\begin{aligned}&\tau=a(t,x)u+b(t,x),\quad \xi=\xi(t,x),\\&\eta=a_t(t,x)u^2+c(t,x)u+d(t,x)\end{aligned}$$

where the functions $a(t,x)$, $b(t,x)$, $c(t,x)$, $d(t,x)$, $\xi(t,x)$, F, G satisfy the system of equations

$$\begin{aligned}&\xi_t-(a_xu+b_x+au_x)F=0,\quad 2aF-(a_xu+b_x+au_x)F_{u_x}=0,\\&2\eta_{tu}-\tau_{tt}-3\tau_uG+(\tau_{xx}+2u_x\tau_{xu})F+(\tau_x+u_x\tau_u)G_{u_x}=0,\\&2(\xi_x-\tau_t)F-(\tau F_t+\xi F_x+\eta F_u)-[\eta_x+(\eta_u-\xi_x)u_x]F_{u_x}=0,\\&\eta_{tt}-u_x\xi_{tt}-[\eta_{xx}+(2\eta_{xu}-\xi_{xx})u_x+\eta_{uu}u_x^2)F-(2\tau_t-\eta_u)G-\\&\quad-(\tau G_t+\xi G_x+\eta G_u)-[\eta_x+(\eta_u-\xi_x)u_x]G_{u_x}=0.\end{aligned}\tag{5}$$

From the first two equations of (5) we find that, since $F\neq 0$, then $a=0$. Further, we distinguish three cases: (1) $F_{u_x}\neq 0$, (2) $F_{u_x}=0$, $F_u\neq 0$, (3) $F_u=F_{u_x}=0$.

Case 1: $\boldsymbol{F_{u_x}\neq 0}$. If $F_{u_x}\neq 0$ then it follows from the first two equations of (5) that $\xi_t=b_x=0$ and the third equation then becomes $2c_t-b_{tt}=0$, from which we find that $c=\frac{1}{2}b_t+\theta(x)$. Then in (3) we have

$$\tau=b(t),\quad \xi=\xi(x),\quad \eta=\left(\frac{1}{2}b_t+\theta(x)\right)u+d(t,x)\tag{6}$$

and the functions τ, ξ, η, F, G satisfy the last two equations of (5).

Case 2: $\boldsymbol{F_{u_x}=0,\ F_u\neq 0}$. If, in this case, $G_{u_xu_x}\neq 0$ then we obtain the same result as in case 1. So, suppose that $G_{u_xu_x}=0$, which gives us $G=A(t,x,u)u_x+B(t,x,u)$ with A and B being arbitrary smooth functions. If $A=0$ then equation (1) becomes

$$u_{tt}=Fu_{xx}+B,\quad F_u\neq 0\tag{7}$$

and in (3) we find

$$\tau = b(t), \quad \xi = \xi(x),$$
$$\eta = \left[\tfrac{1}{2}(\tau_t + \xi_x) + k\right] u + d(t,x), \quad k \in \mathbb{R},\ k \neq 0.$$

If, on the other hand, $F = \lambda(x)A$, $\lambda(x)A_u \neq 0$, then equation (1) becomes

$$u_{tt} = A[\lambda(x)u_{xx} + u_x] + B \tag{8}$$

and there is then a local change of coordinates $t' = t$, $x' = X(x)$, $u = v(t', x')$, $X_x \neq 0$ with $\lambda X_{xx} + X_x = 0$, which transforms (8) to a wave equation of the form (7). As is well-known (see [1]), from the group-theoretic point of view, such equations are deemed to be equivalent. Finally, if $F \neq \lambda(t,x)A$, $\lambda A \neq 0$ or if $F = \lambda(t,x)A$, $\lambda A \neq 0$, $\lambda_t \neq 0$ then the invariance group of the corresponding wave equation (1) is generated by the operator (3) with τ, ξ, η satisfying (6).

Case 3: $\boldsymbol{F_{u_x} = F_u = 0}$, $\boldsymbol{F \neq 0}$. If, in equation (1) we have $G_{u_x u_x} \neq 0$ or $G = A(t,x,u)u_x + B(t,x,u)$ with $A_u \neq 0$ then the invariance group is generated by operators(3) with coefficients given by (6). This leaves us with the case when equation (1) is of the form

$$u_{tt} = F(t,x)u_{xx} + A(t,x)u_x + B(t,x,u). \tag{9}$$

It is well known from the general theory of partial differential equations that there are invertible changes of coordinates

$$t' = \alpha(t,x), \quad x' = \beta(t,x), \quad \frac{D(t',x')}{D(t,x)} \neq 0$$

which transform the equation (9) either to an equation of hyperbolic type

$$v_{t't'} - v_{x'x'} = \tilde{A}(t',x')v_{x'} + \tilde{C}(t',x')v_{t'} + \tilde{B}(t',x',v) \tag{10}$$

or to an equation of elliptic type

$$v_{t't'} + v_{x'x'} = \tilde{A}(t',x')v_{x'} + \tilde{C}(t',x')v_{t'} + \tilde{B}(t',x',v). \tag{11}$$

However from the point of view of the local analytic theory with analytic coefficients, the elliptic type is equivalent to the hyperbolic type. Therefore there exist corresponding transformations in the complex domain which allow us to obtain equations (10) and (11) from each other. Further, the change of variables $y = t'$, $z = x'$, $\omega(y,z) = \Lambda(t',x')v$, $\Lambda \neq 0$ where

$$\Lambda = \exp\left(-\tfrac{1}{2}\int \tilde{C}(t',x')dt'\right)$$

transforms equation (10) into an equation of the form

$$\omega_{yy} = \omega_{zz} + H(y,z)\omega_z + R(y,z,\omega).$$

This equation belongs to the class of equations (2) and therefore, as we noted earlier, we exclude it from our considerations. From the analysis carried out above, it now follows that, in order to solve our problem of group classification, we need consider only the following cases:

$$u_{tt} = F(t,x,u,u_x)u_{xx} + G(t,x,u,u_x), \quad F_{u_x} \neq 0, \tag{12}$$

$$u_{tt} = F(t,x,u,)u_{xx} + G(t,x,u,u_x), \quad F \neq 0, \quad G_{u_x u_x} \neq 0, \tag{13}$$

$$u_{tt} = F(t,x,u)u_{xx} + G(t,x,u,)u_x + H(t,x,u), \quad F_u \neq 0, \quad F \neq \lambda(t,x)G, \quad \lambda G \neq 0, \tag{14}$$

$$u_{tt} = F(t,x,u)[H(t,x)u_{xx} + u_x] + G(t,x,u), \quad F_u \neq 0, \quad H_t \neq 0, \tag{15}$$

$$u_{tt} = F(t,x)u_{xx} + G(t,x,u)u_x + H(t,x,u), \quad F \neq 0, \quad G_u \neq 0, \tag{16}$$

$$u_{tt} = F(t,x,u)u_{xx} + G(t,x,u), \quad F_u \neq 0. \tag{17}$$

The following result follows from the above considerations.

Proposition 1. *The invariance groups of equations* (12)–(16) *are generated by vector fields of the form*

$$Q = \tau(t)\partial_t + \xi(x)\partial_x + \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right)u + \eta(t,x)\right]\partial_u. \tag{18}$$

Further, for equations (12) *and* (13) *the functions τ, ξ, θ, η, F, G satisfy the system of equations*

$$\begin{aligned}
&2(\xi_x - \tau_t)F - \left(\tau F_t + \xi F_x + \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right)u + \eta(t,x)\right]\right)F_u = \\
&\quad = \left[\theta_x u + \eta_x + u_x\left(\tfrac{1}{2}\tau_t + \theta - \xi_x\right)\right]F_{u_x}, \\
&\tfrac{1}{2}\tau_{ttt}u + \eta_{tt} - [\theta_{xx}u + \eta_{xx} + u_x(2\theta_x - \xi_{xx})]F - \left(\tfrac{3}{2}\tau_t - \theta\right)G - \\
&\quad - \left(\tau G_t + \xi G_x + \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right)u + \eta(t,x)\right]\right)G_u = \\
&\quad = \left[\theta_x u + \eta_x + u_x\left(\tfrac{1}{2}\tau_t + \theta - \xi_x\right)\right]G_{u_x}.
\end{aligned} \tag{19}$$

For equation (14) *the functions* τ, ξ, θ, η, F, G, H *satisfy the system of equations*

$$
\begin{aligned}
&(\xi_x - 2\tau_t)G + (\xi_{xx} - 2\theta_x)F - \big(\tau G_t + \xi G_x + \\
&\quad \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right) u + \eta(t,x)\right] G_u\big) = 0,\\
&2(\xi_x - \tau_t)F - \left(\tau F_t + \xi F_x + \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right) u + \eta(t,x)\right]\right) F_u = 0,\\
&\tfrac{1}{2}\tau_{ttt}u + \eta_{tt} - [\theta_{xx}u + \eta_{xx}]F - (\eta_x + u\theta_x)G - \left(\tfrac{3}{2}\tau_t - \theta\right) H -\\
&\quad - \left(\tau H_t + \xi H_x + \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right) u + \eta\right]\right) H_u = 0.
\end{aligned}
\tag{20}
$$

For equation (15) *the functions* τ, ξ, θ, η, F, G, H *satisfy the system of equations*

$$
\begin{aligned}
&[2(\xi_x - \tau_t)H - \tau H_t - \xi H_x]\, F -\\
&\quad - \left\{\tau F_t + \xi F_x + \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right) u + \eta(t,x)\right] F_u\right\} H = 0,\\
&(\xi_{xx} - 2\theta_x)HF - (2\tau_t - \xi_x)F -\\
&\quad - \left\{\tau F_t + \xi F_x + \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right) u + \eta(t,x)\right] F_u\right\} = 0,\\
&\tfrac{1}{2}\tau_{ttt}u + \eta_{tt} - (\eta_{xx} + u\theta_{xx})HF - (u\theta_x + \eta_x)F - \left(\tfrac{3}{2}\tau_t - \theta\right) G -\\
&\quad - \left\{\tau G_t + \xi G_x + \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right) u + \eta(t,x)\right] G_u\right\} = 0.
\end{aligned}
\tag{21}
$$

For equation (16) *the functions* τ, ξ, θ, η, F, G, H *satisfy the system of equations*

$$
\begin{aligned}
&(\xi_{xx} - 2\theta_x)F - (2\tau_t - \xi_x)G -\\
&\quad - \left\{\tau G_t + \xi G_x + \left[\left(\tfrac{1}{2}\tau_t + \theta(x)\right) u + \eta(t,x)\right] G_u\right\} = 0,\\
&2(\xi_x - \tau_t)F - \tau F_t - \xi F_x = 0,\\
&\tfrac{1}{2}\tau_{ttt}u + \eta_{tt} - (\eta_{xx} + u\theta_{xx})F - (u\theta_x + \eta_x)G - \left(\tfrac{3}{2}\tau_t - \theta\right) H -\\
&\quad - \left\{\tau H_t + \xi H_x + \left[\left(\frac{1}{2}\tau_t + \theta(x)\right) u + \eta(t,x)\right] H_u\right\} = 0.
\end{aligned}
\tag{22}
$$

The general infinitesimal operator Q *of the invariance group of equation* (17) *is given by*

$$
Q = \tau(t)\partial_t + \xi(x)\partial_x + \left[\left(\tfrac{1}{2}(\tau_t + \xi_x) + k\right) u + \eta(t,x)\right] \partial_u, \tag{23}
$$

where the functions τ, ξ, η, F, G *and the constant* k *satisfy the system of equations*

$$
2(\xi_x - \tau_t)F - \left(\tau F_t + \xi F_x + \left[\left(\tfrac{1}{2}(\tau_t + \xi_x) +\right) u + \eta\right] F_u\right) = 0,
$$

$$\eta_{tt} + \tfrac{1}{2}\tau_{ttt}u - \left(\tfrac{1}{2}\xi_{xxx}u + \eta_{xx}\right)F - \left(\tfrac{3}{2}\tau_t - \tfrac{1}{2}\xi_x - k\right)G -$$
$$- \left(\tau G_t + \xi G_x + \left[\left(\tfrac{1}{2}(\tau_t + \xi_x) + k\right)u + \eta(t,x)\right]G_u\right) = 0. \qquad (24)$$

A direct calculation shows that when the equations (12)–(17) contain arbitrary functions then the equations do not possess any symmetries in the classical sense of Lie. In what follows, we shall carry out a group classification up to equivalence under the equivalence group (which we denote by $\mathcal{E}$) of the equation under consideration. The equivalence group $\mathcal{E}$ of a given equation consists of those transformations (of the space $V = \mathbb{R}^2 \times \mathbb{R}$) $t' = T(t,x,u)$, $x' = X(t,x,u)$, $v = U(t,x,u)$ which preserve the form of the differential equation (1), that is, transformations of the above type which transform (1) into an equation

$$v_{t't'} = \tilde{F}(t', x', v, v_{x'})v_{x'x'} + \tilde{G}(t', x', v, v_{x'}).$$

In order to determine the transformations of $\mathcal{E}$ one may use the infinitesimal method [31] or the direct method. These calculations are long but standard, and we give only their result.

Proposition 2. *The equivalence group $\mathcal{E}$ of equations* (12)–(16) *consist of the transformations*

$$t' = T(t), \quad x' = X(x), \quad v = U(x)\sqrt{|T_t|}u + Y(t,x) \qquad (25)$$

with the condition $T_t X_x U \neq 0$ and with Y being an arbitrary function. The transformations of the equivalence group $\mathcal{E}$ of equation (17) *consist of the transformations*

$$t' = T(t), \quad x' = X(x), \quad v = \gamma\sqrt{|T_t|}\sqrt{|X_x|}u + Y(t,x) \qquad (26)$$

with $\gamma T_t X_x \neq 0$, $\gamma \in \mathbb{R}$, and

$$t' = T(x), \quad x' = X(t), \quad v = \gamma\sqrt{|T_t|}\sqrt{|X_x|}u + Y(t,x) \qquad (27)$$

with $\gamma T_t X_x \neq 0$, $\gamma \in \mathbb{R}$, and in both cases Y is an arbitrary function.

We now pass to the description of those nonlinear equations which are invariant under low-dimensional Lie algebras.

3. Invariance of equations under one-dimensional Lie algebras. As we showed above, when the functions F, G, H in equations (12)–(17) are allowed to be completely arbitrary, then the equations do not have any symmetry in the sense of Lie. For this reason we begin our classification of those equations which admit symmetries by looking at those which are invariant under one-parameter groups of local transformations (which is equivalent to being invariant under one-dimensional Lie algebras). In order to do this, we first obtain all inequivalent realizations of one-dimensional Lie algebras.

3.1. Realizations of one-dimensional Lie algebras.

Theorem 1. *There exist transformations of the form* (25) *which transform the operator* (18) *into one of the following forms:*

$$
\begin{aligned}
&Q = \partial_t, \quad Q = \partial_x, \quad Q = \partial_t + \partial_x, \\
&Q = f(x)u\partial_u, \quad Q = g(t,x)\partial_u, \quad Q = \partial_t + f(x)u\partial_u,
\end{aligned}
$$

where $f, g \neq 0$.

Proof. Applying the change of coordinates (25) we transform the operator (18) into one of the form

$$
\begin{aligned}
\tilde{Q} = \tau T_t\partial_{t'} + \xi X_x\partial_{x'} + \Big\{ \Big[&\tfrac{1}{2}\epsilon\tau|T|^{-1/2}T_{tt}U + \xi|T_t|^{1/2}U_x + \\
&+ \left(\tfrac{1}{2}\tau_t + \theta\right)|T_t|^{1/2}U \Big] u + \tau Y_t + \xi Y_x + \eta|T_t|^{1/2}U \Big\}\partial_v,
\end{aligned} \tag{28}
$$

where $\epsilon = 1$ if $T_t > 0$ and $\epsilon = -1$ if $T_t < 0$. We now consider three cases: (1) $\tau \neq 0$, (2) $\tau = 0$, $\xi \neq 0$, (3) $\tau = \xi = 0$.

Case 1: $\tau \neq 0$**.** Putting $T_t = \tau^{-1}$ in (25) we make the coefficient of $\partial_{t'}$ equal to one. If we also have $\xi \neq 0$ then we may put $X_x = \xi^{-1}$. We also choose U to be a non-trivial solution of $\xi U_x + \theta U = 0$, and Y is taken as a solution of the equation $\tau Y_t + \xi Y_x + \eta|\tau|^{-1/2}U = 0$. In this way we transform the operator given in (28) into the operator

$$\tilde{Q} = \partial_{t'} + \partial_{x'}. \tag{29}$$

If, however, $\xi = 0$, $\theta \neq 0$ then, putting Y equal to a solution of

$$\tau Y_t + \eta|\tau|^{-1/2}U = 0,$$

the transformation (25) takes the operator Q into

$$\tilde{Q} = \partial_{t'} + \theta(x')v\partial_v. \tag{30}$$

If $\xi = \theta = 0$ then, in the same way, we may transform Q into

$$\tilde{Q} = \partial_{t'}. \tag{31}$$

Case 2: $\tau = 0$, $\xi \neq 0$. Put $X_x = \xi^{-1}$ in (25) and choose U to be a non-trivial solution of $\xi U_x + \theta U = 0$ and we choose Y to be a solution of $\xi Y_x + \eta |T_t|^{1/2} U = 0$. This then takes Q into the operator

$$\tilde{Q} = \partial_{x'}. \tag{32}$$

Case 3: $\tau = \xi = 0$. If $\theta \neq 0$ in (18) then we put $T = t$, $Y = \theta^{-1}\eta U$ in (25) and the operator (28) becomes

$$\tilde{Q} = \theta(x')v\partial_v. \tag{33}$$

If we have $\theta = 0$ in (18) then $\eta \neq 0$ and we obtain the operator

$$\tilde{Q} = \tilde{\eta}(t', x')\partial_v. \tag{34}$$

The forms of the operator $\tilde{Q}$ obtained in (29)–(34) is, apart from the notation, that given in the statement of the theorem. All that remains to be done is to verify that these different forms are inequivalent under the action of transformations (25). We show this for the case of the operators $Q_1 = \partial_t$ and $Q_2 = \partial_t + f(x)u\partial_u$, and the other cases are treated in the same way. First, we assume that there is a transformation of the form (25) which takes Q_1 into $\tilde{Q} = \partial_{t'} + \tilde{f}(x')v\partial_v$. Then this implies that we must have $f = 0$ which contradicts the requirement for Q_2. This completes the proof.

Theorem 2. *There exist transformations of the type* (26), (27) *which transform the operator* (23) *into one of the following:*

$$Q = \partial_t + \partial_x, \quad Q = \partial_t, \quad Q = \partial_t + \partial_x + u\partial_u,$$
$$Q = \partial_t + u\partial_u, \quad Q = u\partial_u, \quad Q = g(t,x)\partial_u, \quad g \neq 0.$$

The proof is carried out in the same way as that of Theorem 1.

It follows from these two theorems that, in the classes (18) and (23) of operators, there exist six inequivalent (with respect to the equivalence group of our partial differential equation) types of one-dimensional Lie algebras $A_1 = \langle e_1 \rangle$. We list these below, using a notation which we shall use in the rest of this paper.

One-dimensional Lie algebras of operators of type (18).

$$A_1^1 = \langle \partial_t + \partial_x \rangle, \quad A_1^2 = \langle \partial_t \rangle, \quad A_1^3 = \langle \partial_x \rangle, \quad A_1^4 = \langle \partial_t + f(x)u\partial_u \rangle,$$
$$A_1^5 = \langle f(x)u\partial_u \rangle, \quad A_1^6 = \langle g(t,x)\partial_u \rangle, \ \ f, g \neq 0.$$

One-dimensional Lie algebras of operators of type (23).

$$\tilde{A}_1^1 = \langle \partial_t + \partial_x \rangle, \quad \tilde{A}_1^2 = \langle \partial_t \rangle, \quad \tilde{A}_1^3 = \langle \partial_t + \partial_x + u\partial_u \rangle,$$
$$\tilde{A}_1^4 = \langle \partial_t + u\partial_u \rangle, \quad \tilde{A}_1^5 = \langle u\partial_u \rangle, \quad \tilde{A}_1^6 = \langle g(t,x)\partial_u \rangle, \ \ g \neq 0.$$

3.2. Equations invariant under one-dimensional Lie algebras. We now have to determine for each of the above realizations whether the given operator is an admissible symmetry operator (that is, if there is a wave equation which is invariant under a given realization). To do this we use the system of determining equations Proposition 1. Since the procedure of constructing A_1-invariant equations reduces to integrating systems of first-order partial differential equations, we do not give details: we merely make some remarks and then give a list of our results. All the realizations A_1^i $(i = 1, \ldots, 6)$ are invariance algebras only for equations of the form (12). For equations of the form (13), substituting the values of the coefficients τ, ξ, θ, η in the realizations A_1^5 and A_1^6 into the system (19) leads to the equality $F_u = 0$, which contradicts the condition placed on the equation. We arrive at the same result when we examine these realizations for the equations (14) and (15). For equation (15) we also find that the condition of its invariance under A_1^2 leads to $H_t = 0$, which contradicts the condition placed on the equation. Finally, for equation (16) the invariance under A_1^6 leads to $G_u = 0$, as follows from the second equation in (22). This contradicts the requirements placed on the equation. The realizations $\tilde{A}_1^5$ and $\tilde{A}_1^6$ cannot be invariance algebras of equation (17) since the first equation of the system (24) gives $F_u = 0$ which is a contradiction. Below we give a list of all A_1-invariant equations, and we give the realization of the algebras A_1 which is an invariance algebra of the corresponding equation, and we give the corresponding forms of the functions F, G, H.

***A_1*-invariant equations of type (12).**

$$A_1^1: \ \ F = \tilde{F}(z, u, u_x), \quad G = \tilde{G}(z, u, u_x), \quad z = t - x, \quad \tilde{F}_{u_x} \neq 0;$$
$$A_1^2: \ \ F = \tilde{F}(x, u, u_x), \quad G = \tilde{G}(x, u, u_x), \quad \tilde{F}_{u_x} \neq 0;$$
$$A_1^3: \ \ F = \tilde{F}(t, u, u_x), \quad G = \tilde{G}(t, x, u_x), \quad \tilde{F}_{u_x} \neq 0;$$
$$A_1^4: \ \ G = u\tilde{G}(x, v, \omega) - f^{-1}[f'' u \ln|u| + 2f' u_x \ln|u| - (f')^2 f^{-1} u \ln^2|u|]\tilde{F}, \quad F = \tilde{F}(x, v, \omega),$$
$$v = u\exp(-tf), \quad \omega = u^{-1}u_x - f^{-1}f'\ln|u|, \quad \tilde{F}_\omega \neq 0;$$

$$
\begin{aligned}
A_1^5:\ & G = u\tilde{G}(t,x,\omega) - f^{-1}[f''u\ln|u| + 2f'u_x\ln|u| - \\
& - (f')^2 f^{-1}u\ln^2|u|]\tilde{F}, \quad F = \tilde{F}(t,x,\omega), \quad \tilde{F}_\omega \neq 0, \\
& \omega = u^{-1}u_x - f^{-1}f'\ln|u|; \\
A_1^6:\ & F = \tilde{F}(t,x,\omega), \quad G = \tilde{G}(t,x,\omega) - g^{-1}g_{xx}u\tilde{F} + g^{-1}g_{tt}u, \\
& \omega = gu_x - g_x u, \quad \tilde{F}_\omega \neq 0.
\end{aligned}
$$

A_1-invariant equations of type (13).

$$
\begin{aligned}
A_1^1:\ & F = \tilde{F}(z,u), \quad G = \tilde{G}(z,u,u_x), \quad z = t - x, \quad \tilde{F}_u \neq 0; \\
A_1^2:\ & F = \tilde{F}(x,u,), \quad G = \tilde{G}(x,u,u_x), \quad \tilde{F}_u \neq 0; \\
A_1^3:\ & F = \tilde{F}(t,u,), \quad G = \tilde{G}(t,u,u_x), \quad \tilde{F}_u \neq 0; \\
A_1^4:\ & G = u\tilde{G}(x,v,\omega) - f^{-1}[f''u\ln|u| + 2f'u_x\ln|u| - \\
& - (f')^2 f^{-1}u\ln^2|u|]\tilde{F}, \quad F = \tilde{F}(x,\omega), \\
& \omega = u\exp(-tf), \quad v = u^{-1}u_x - f^{-1}f'\ln|u|, \quad \tilde{F}_\omega \neq 0.
\end{aligned}
$$

A_1-invariant equations of type (14).

$$
\begin{aligned}
A_1^1:\ & F = \tilde{F}(z,u), \quad G = \tilde{G}(z,u), \quad H = \tilde{H}(z,u), \\
& z = t - x, \quad \tilde{F}_u \neq 0, \quad \tilde{F} \neq \lambda(z)\tilde{G}, \quad \lambda(z)\tilde{G} \neq 0; \\
A_1^2:\ & F = \tilde{F}(x,u), \quad G = \tilde{G}(x,u), \quad H = \tilde{H}(x,u), \\
& \tilde{F}_u \neq 0, \quad \tilde{F} \neq \lambda(x)\tilde{G}, \quad \lambda(x)\tilde{G} \neq 0; \\
A_1^3:\ & F = \tilde{F}(t,u), \quad G = \tilde{G}(t,u), \quad H = \tilde{H}(t,u), \\
& \tilde{F}_u \neq 0, \quad \tilde{F} \neq \lambda(t)\tilde{G}, \quad \lambda(t)\tilde{G} \neq 0; \\
A_1^4:\ & F = \tilde{F}(x,\omega), \quad G = \tilde{G}(x,\omega) - 2f'f^{-1}\ln|u|\tilde{F}, \\
& H = u\tilde{H}(x,\omega) + (f')^2 f^{-2}u\ln^2|u|\tilde{F} - f'f^{-1}u\ln|u|\tilde{G} - \\
& - f''f^{-1}u\ln|u|\tilde{F}, \quad \omega = u\exp(-tf), \quad \tilde{F}_\omega \neq 0; \\
& \text{if } \ f' = 0 \ \text{ then } \quad \tilde{F} \neq \lambda(x)\tilde{G}, \quad \lambda(x)\tilde{G} \neq 0.
\end{aligned}
$$

A_1-invariant equations of type (15).

$$
\begin{aligned}
A_1^1:\ & F = \tilde{F}(z,u), \quad G = \tilde{G}(z,u), \quad H = \tilde{H}(z), \\
& z = t - x, \quad \tilde{F}_u \neq 0, \quad \tilde{H}_z \neq 0; \\
A_1^3:\ & F = \tilde{F}(t,u,), \quad G = \tilde{G}(t,u), \quad H = \tilde{H}(t), \quad \tilde{F}_u \neq 0, \quad \tilde{H}_t \neq 0;
\end{aligned}
$$

$$
\begin{aligned}
A_1^4:\ & F=\tilde F(x,\omega)[\tilde H(x)-2tf'],\quad H=[\tilde H(x)-2tf']^{-1},\\
& G=e^{tf}\tilde G(x,\omega)+u\tilde F\left[\tfrac14(H-2tf')^2-tf''\right],\\
& \omega=u\exp(-tf),\quad \tilde F_\omega\neq 0,\quad f'\neq 0.
\end{aligned}
$$

***A_1*-invariant equations of type (16).**

$$
\begin{aligned}
A_1^1:\ & F=\tilde F(z),\quad G=\tilde G(z,u),\quad H=\tilde H(z,u),\\
& z=t-x,\quad \tilde G_u\neq 0;\\
A_1^2:\ & F=\tilde F(x),\quad G=\tilde G(x,u),\quad H=\tilde H(x,u),\quad \tilde G_u\neq 0;\\
A_1^3:\ & F=\tilde F(t),\quad G=\tilde G(t,u),\quad H=\tilde H(t,u),\quad \tilde G_u\neq 0;\\
A_1^4:\ & F=\tilde F(x),\quad G=\tilde G(x,\omega)-2f'f^{-1}\ln|u|\tilde F,\\
& H=u\tilde H(x,\omega)+(f')^2f^{-2}u\ln^2|u|\tilde F-f'f^{-1}u\ln|u|\tilde G-\\
& -f''f^{-1}u\ln|u|\tilde F,\quad \omega=u\exp(-tf);\\
& \tilde G\quad \text{is arbitrary if}\quad f'\neq 0;\quad \tilde G_\omega\neq 0\quad \text{if}\quad f'=0.\\
A_1^5:\ & F=\tilde F(t,x),\quad G=\tilde G(t,x)-2f'f^{-1}\ln|u|\tilde F,\\
& H=u\tilde H(t,x)-f''f^{-1}u\ln|u|\tilde F+(f')^2f^{-2}u\ln^2|u|\tilde F-\\
& -f'f^{-1}u\ln|u|\tilde G,\quad f'\neq 0.
\end{aligned}
$$

***A_1*-invariant equations of type (17).**

$$
\begin{aligned}
\tilde A_1^1:\ & F=\tilde F(z,u),\quad G=\tilde G(z,u),\quad z=t-x,\quad \tilde F_u\neq 0;\\
\tilde A_1^2:\ & F=\tilde F(x,u),\quad G=\tilde G(x,u),\quad \tilde F_u\neq 0;\\
\tilde A_1^3:\ & F=\tilde F(z,\omega),\quad G=e^t\tilde G(z,\omega),\\
& z=t-x,\quad \omega=u\exp(-t),\quad \tilde F_\omega\neq 0;\\
\tilde A_1^4:\ & F=\tilde F(x,\omega),\quad G=e^t\tilde G(x,\omega),\quad \omega=u\exp(-t),\quad \tilde F_\omega\neq 0.
\end{aligned}
$$

We note that in the above lists, $f'=\frac{df}{dx}$, $f''=\frac{d^2f}{dx^2}$. Also, we note that the Lie algebras given are the maximal invariance algebras of the corresponding equations when the functions $\tilde F$, $\tilde G$, $\tilde H$ are arbitrary.

4. Invariance of equations under two-dimensional Lie algebras. It is well-known (see [32]) that there are, up to isomorphism, only two Lie algebras $A_2=\langle e_1,e_2\rangle$ of dimension two:

$$
A_{2.1}:\quad [e_1,e_2]=0;\quad A_{2.2}:\quad [e_1,e_2]=e_2.
$$

In order to obtain a full list of non-linear equations (1) which are invariant under two-dimensional Lie algebras, we must first construct all possible inequivalent realizations of the Lie algebras $A_{2.1}$ and $A_{2.2}$ in the class of operators (18) and (23). Then we must use the defining system of equations (19)–(22) and (22) in order to pick out those realizations which are invariance algebras for equations of the given type (1). To carry out this construction, we are able to exploit the results of Theorems 1 and 2, which allow us to choose one of the operators of the two-dimensional Lie algebras in one of the canonical forms given in these results.

4.1. $A_{2.1}$-invariant equations. The Lie algebra $A_{2.1}$ is Abelian, so we just add one more operator of the form (18) or (23) which commutes with the first operator chosen from amongst the canonical forms. We do this for the canonical form A_1^1. In this case we put $e_1 = \partial_t + \partial_x$, and we let the operator e_2 have the form (18). Then the commutation relation $[e_1, e_2] = 0$ gives us

$$e_2 = c_1\partial_t + c_2\partial_x + (c_3u + \eta(z))\partial_u, \tag{35}$$

where $c_1, c_2, c_3 \in \mathbb{R}$, $z = t-x$. To give the operator (35) a canonical form we use the subgroup Φ of the equivalence group $\mathcal{E}$ (given by (25)) which maps e_1 to $\lambda e_1'$ with $e_1' = \partial_{t'} + \partial_{x'}$ for some arbitrary choice of constant $\lambda \neq 0$. This is allowed because the commutation relation $[e_1, e_2] = 0$ is preserved. A straightforward calculation gives us the following form of the allowed transformation:

$$t' = \lambda t + \lambda_1, \quad x' = \lambda x + \lambda_2, \quad v = \lambda_3\sqrt{|\lambda|}u + Y(z), \tag{36}$$

where $\lambda, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, $z = t - x$, $\lambda, \lambda_3 \neq 0$. Applying this transformation, we obtain the following possible forms for e_2:

$$\partial_t, \quad \partial_x, \quad \partial_t + u\partial_u, \quad \partial_x + u\partial_u, \quad u\partial_u, \quad g(z)\partial_u,$$

with $g \neq 0$. The extra factor λ gives us flexibility in our calculations so that no other arbitrary constants arise in the canonical forms for e_2. We then have the following forms for the algebra $A_{2.1}$:

$$\langle\partial_t, \partial_x\rangle, \quad \langle\partial_t + \partial_x, u\partial_u\rangle, \quad \langle\partial_t + \partial_x, g(z)\partial_u\rangle, \quad \langle\partial_t + \partial_x, \partial_t + u\partial_u\rangle,$$

where $g \neq 0$, $z = t - x$. Note that the two algebras $\langle\partial_t + \partial_x, \partial_t\rangle$ and $\langle\partial_t + \partial_x, \partial_x\rangle$ are both the same as $\langle\partial_t, \partial_x\rangle$. The other cases are treated in the same manner, and we find the following realizations of the algebras $A^i_{2.1}$ in the class of operators (18):

$$
\begin{aligned}
&A^1_{2.1} = \langle \partial_t, \partial_x \rangle; \quad A^2_{2.1} = \langle \partial_t + \partial_x, u\partial_u \rangle;\\
&A^3_{2.1} = \langle \partial_t + \partial_x, \partial_t + u\partial_u \rangle;\\
&A^4_{2.1} = \langle \partial_t + \partial_x, g(z)\partial_u \rangle, \quad z = t - x,\ g \neq 0; \quad A^5_{2.1} = \langle \partial_t, \partial_u \rangle;\\
&A^6_{2.1} = \langle \partial_x, \partial_t + u\partial_u \rangle; \quad A^7_{2.1} = \langle \partial_x, u\partial_u \rangle; \quad A^8_{2.1} = \langle \partial_x, \partial_u \rangle;\\
&A^9_{2.1} = \langle \partial_t, f(x)u\partial_u \rangle, \quad f \neq 0;\\
&A^{10}_{2.1} = \langle \partial_t + f(x)u\partial_u, e^{tf}\partial_u \rangle, \quad f \neq 0;\\
&A^{11}_{2.1} = \langle \partial_t + f(x)u\partial_u, h(x)u\partial_u \rangle, \ \ fh' - f'h \neq 0;\\
&A^{12}_{2.1} = \langle g(t,x)\partial_u, h(t,x)\partial_u \rangle;\\
&A^{13}_{2.1} = \langle f(x)u\partial_u, h(x)u\partial_u \rangle, \ \ fh' - f'h \neq 0.
\end{aligned}
$$

In the realization of $A^{12}_{2.1}$ the functions h, g are linearly independent (with respect to at least one of the arguments).

The next step is to check whether these realizations can be invariance algebras for equations (12)-(16). We find that all the realizations except $A^{13}_{2.1}$ are invariance algebras for equations of the form given in (12).

$A_{2.1}$-invariant equations of the form (12).

$$
\begin{aligned}
&A^1_{2.1}: \ F = \tilde{F}(u, u_x), \quad G = \tilde{G}(u, u_x), \quad \tilde{F}_{u_x} \neq 0;\\
&A^2_{2.1}: \ F = \tilde{F}(z, \omega), \quad G = u\tilde{G}(z, \omega), \quad z = t - x,\\
&\qquad \omega = u^{-1}u_x, \quad \tilde{F}_\omega \neq 0;\\
&A^3_{2.1}: \ F = \tilde{F}(v, \omega), \quad G = e^z\tilde{G}(v, \omega), \quad v = u\exp(-z),\\
&\qquad \omega = u^{-1}u_x, \quad z = t - x, \quad \tilde{F}_\omega \neq 0;\\
&A^4_{2.1}: \ F = \tilde{F}(z, \omega), \quad G = \tilde{G}(z, \omega) - g^{-1}g''u\tilde{F} + ug^{-1}g'',\\
&\qquad g = g(z) \neq 0, \quad z = t - x, \quad \omega = gu_x + g'u, \quad \tilde{F}_\omega \neq 0;\\
&A^5_{2.1}: \ F = \tilde{F}(x, u_x), \quad G = \tilde{G}(x, u_x), \quad \tilde{F}_{u_x} \neq 0;\\
&A^6_{2.1}: \ F = \tilde{F}(v, \omega), \quad G = u\tilde{G}(v, \omega),\\
&\qquad v = u\exp(-t), \quad \omega = u^{-1}u_x, \quad \tilde{F}_\omega \neq 0;\\
&A^7_{2.1}: \ F = \tilde{F}(t, \omega), \quad G = u\tilde{G}(t, \omega), \quad \omega = u^{-1}u_x, \quad \tilde{F}_\omega \neq 0;\\
&A^8_{2.1}: \ F = \tilde{F}(t, u_x), \quad G = \tilde{G}(t, u_x), \quad \tilde{F}_{u_x} \neq 0;\\
&A^9_{2.1}: \ F = \tilde{F}(x, \omega), \quad \omega = u^{-1}u_x - f^{-1}f'\ln|u|,
\end{aligned}
$$

$$\tilde{F}_\omega \neq 0, \quad G = u\tilde{G}(x,\omega) - f^{-1}[f''u\ln|u| + 2f'u_x\ln|u| - - (f')^2 f^{-1}u\ln|u|]\tilde{F};$$

$A^{10}_{2.1}:$ (1) $f = 1, \quad F = \tilde{F}(x,\omega), \quad G = u + e^t\tilde{G}(x,\omega),$
$\omega = e^{-t}u_x, \quad \tilde{F}_\omega \neq 0;$
(2) $f = x, \quad F = \tilde{F}(x,\omega), \quad \tilde{F}_\omega \neq 0, \quad \omega = [u^{-1}u_x - - x^{-1}\ln|u| - t^{-2}x\ln|u| - 2t + x^{-1}]u\exp(-tx),$
$$G = e^{tx}\tilde{G}(x,\omega) + ux^2 + u[-4t^3x\ln|u| - t^4x^2\ln^2|u| - 5t^2 - - 2tu^{-1}u_x\ln|u| + 2tx^{-1}\ln^2|u| + 2t^3x\ln^2|u| + 6t^2\ln|u| - - 2tx^{-1}\ln|u| + 6tx^{-1} - 2x^{-2} - 2x^{-1}u^{-1}u_x\ln|u| + + x^{-2}\ln^2|u|]\tilde{F};$$

$A^{11}_{2.1}:$ (1) $f = 1, \quad h' \neq 0, \quad F = \tilde{F}(x,\omega), \quad \tilde{F}_\omega \neq 0,$
$\omega = u^{-1}u_x + (h^7 - 1)h't\ln|u|,$
$G = u\tilde{G}(x,\omega) + h^{-1}u[th''\ln|u| + 2h^{-1}(h')^2u - 2h'u_x]\tilde{F};$
(2) $f = x, \quad h \neq 0, \quad \lambda x\,(\lambda \neq 0),$
$F = \tilde{F}(x,\omega), \quad \omega = u^{-1}u_x - (x^{-1} - txh'h^{-1} + t)\ln|u|,$
$$G = u\tilde{G}(x,\omega) + h^{-1}u[(2x^{-1}h' - h^{-1}(h')^2 - hx^{-2})\ln|v| - - h'' + 2(x^{-1}h - h')\omega]\ln|v|\tilde{F} + [x^{-2}u\ln^2|u| - - 2x^{-1}u_x\ln|u|]\tilde{F}, \quad v = u\exp(-tx), \quad \tilde{F}_\omega \neq 0;$$

$A^{12}_{2.1}:$ $g = 1, \quad h = t, \quad F = \tilde{F}(t,x,u_x), \quad G = \tilde{G}(t,x,u_x),$
$\tilde{F}_{u_x} \neq 0.$

In constructing those $A_{2.1}$-invariant equations of the form (13) and (14), we have used the fact that there are no realizations of A^5_1 and A^6_1 which can be symmetry algebras. This allows us to shorten the list of realizations of the algebras of type $A_{2.1}$ to just three: $A^1_{2.1}$, $A^3_{2.1}$, $A^6_{2.1}$. All these algebras are in fact symmetry algebras of equations of type (13) and (14).

$A_{2.1}$-invariant equations of the type (13).

$A^1_{2.1}:$ $F = \tilde{F}(u), \quad G = \tilde{G}(u,u_x), \quad \tilde{F}_u \neq 0;$

$A^3_{2.1}:$ $F = \tilde{F}(v), \quad G = e^z\tilde{G}(v,\omega), \quad \tilde{F}_v \neq 0,$
$v = u\exp(-z), \quad z = t - x, \quad \omega = u^{-1}u_x;$

$$A^6_{2.1}:\ F=\tilde F(\omega),\quad G=u\tilde G(v,\omega),\quad \tilde F_\omega\neq 0,$$
$$\omega=ue^{-t},\quad v=u^{-1}u_x.$$

$A_{2.1}$-invariant equations of the type (14).

$$A^1_{2.1}:\ F=\tilde F(u),\quad G=\tilde G(u),\quad H=\tilde H(u),$$
$$\tilde F_u\neq 0,\quad \tilde F\neq\tilde G,\quad \tilde G\neq 0;$$
$$A^3_{2.1}:\ F=\tilde F(\omega),\quad G=\tilde G(\omega),\quad H=e^z\tilde H(\omega),$$
$$\tilde F_\omega\neq 0,\quad \tilde F\neq\tilde G,\quad \tilde G\neq 0,$$
$$\omega=u\exp(-z),\quad z=t-x,\quad \omega=u^{-1}u_x;$$
$$A^6_{2.1}:\ F=\tilde F(\omega),\quad G=u\tilde G(\omega),\quad H=u\tilde H(\omega),$$
$$\tilde F_\omega\neq 0,\quad \tilde F\neq\tilde G,\quad \tilde G\neq 0,\quad \omega=ue^{-t}.$$

There are no $A_{2.1}$-invariant equations of type (15). First, there are none which are invariant under A^2_1, A^5_1, A^6_1. This then narrows the number of those which are left in the list of $A_{2.1}$ algebras down to just two: $A^3_{2.1}$ and $A^6_{2.1}$. Then we use the system of defining equations (21) and find that requiring $A^3_{2.1}$ invariance leads to $\tilde H_z=0$, which contradicts the conditions of (15); the defining system (21) also leads to $f'=0$ when testing the algebra $A^6_{2.1}$, and this contradicts the condition on f. The same type of procedure is used for examining the symmetry algebras for equation (16): there are just five such equations which are invariant under $A_{2.1}$.

$A_{2.1}$-invariant equations of type (16).

$$A^1_{2.1}:\ F=\lambda,\quad G=\tilde G(u),\quad H=\tilde H(u),\quad \tilde G'\neq 0,\quad \lambda\in\mathbb{R}_*;$$
$$A^3_{2.1}:\ F=\lambda,\quad G=\tilde G(\omega),\quad H=e^z\tilde H(\omega),\quad \tilde G'\neq 0,$$
$$\lambda\in\mathbb{R}_*,\quad z=t-x,\quad \omega=ue^{-z};$$
$$A^6_{2.1}:\ F=\lambda,\quad G=\tilde G(\omega),\quad H=e^t\tilde H(\omega),\quad \tilde G'\neq 0,$$
$$\lambda\in\mathbb{R}_*,\quad \omega=ue^{-t};$$
$$A^9_{2.1}:\ F=\tilde F(x),\quad G=\tilde G(x)-2f'f^{-1}\tilde F\ln|u|,$$
$$H=u\tilde H(x)-f''f^{-1}u\ln|u|\tilde F+(f')^2f^{-2}u\ln^2|u|\tilde F-$$
$$-f'f^{-1}u\ln|u|\tilde G,\quad \tilde F\neq 0,\quad f'\neq 0;$$
$$A^{11}_{2.1}:\ (1)\ f=1,\quad h'\neq 0;\quad F=\tilde F(x)\neq 0,$$

$$
\begin{aligned}
&G = \tilde{G}(x) + 2h'h^{-1}t\ln|u|\tilde{F},\\
&H = u\tilde{H}(x) + t^2h^{-2}u\ln^2|u|\tilde{F} + th''h^{-1}u\ln|u|\tilde{F} +\\
&+ th'h^{-1}u\ln|u|\tilde{G};\\
&(2)\ f = x, \quad h \neq 0, \quad F = \tilde{F}(x) \neq 0,\\
&G = \tilde{G}(x) - 2h^{-1}(x^{-1}h - h')tx\ln|u|\tilde{F} - 2x^{-1}\ln|u|\tilde{F},\\
&H = u\tilde{H}(x) + x^{-2}u\ln^2|u|\tilde{F} - x^{-1}u\ln|u|\tilde{G} +\\
&+ 2th^{-1}(x^{-1}h - h')u\ln^2|u|\tilde{F} + t^2u\ln^2|u|\tilde{F} -\\
&- tu\ln|u|\tilde{G} - x^2t^3h'h^{-1}u\ln^2|u|\tilde{F} - txh^{-1}h''u\ln|u|\tilde{F} -\\
&- t^2x^2(h')^2h^{-2}u\ln^2|u|\tilde{F} + txh^{-1}h'u\ln|u|\tilde{G}.
\end{aligned}
$$

We are now left with the case of $A_{2.1}$-invariant equations of type (17). Having constructed realizations of $A_{2.1}$ in the class of operators (23), and then testing them as symmetry algebras for equations of type (17), we find only three algebras:

$$
\begin{aligned}
&\tilde{A}^1_{2.1} = \langle \partial_t, \partial_x \rangle, \quad \tilde{A}^2_{2.1} = \langle \partial_t, \partial_x + u\partial_u \rangle,\\
&\tilde{A}^3_{2.1} = \langle \partial_t + u\partial_u, \partial_x + u\partial_u \rangle.
\end{aligned}
$$

$\tilde{A}_{2.1}$-invariant equations of type (17).

$$
\begin{aligned}
&\tilde{A}^1_{2.1}: \ F = \tilde{F}(u), \quad G = \tilde{G}(u), \quad \tilde{F}' \neq 0;\\
&\tilde{A}^2_{2.1}: \ F = \tilde{F}(\omega), \quad G = e^x\tilde{G}(\omega), \quad \tilde{F}' \neq 0, \quad \omega = ue^{-x};\\
&\tilde{A}^3_{2.1}: \ F = \tilde{F}(\omega), \quad G = e^{(t+x)}\tilde{G}(\omega), \quad \tilde{F}' \neq 0, \quad \omega = ue^{-(t+x)}.
\end{aligned}
$$

4.2. $A_{2.2}$-invariant equations. As in the case of the $A_{2.1}$-invariant equations, we must first construct all possible inequivalent algebras $A_{2.2}$ in the classes of operators (18) and (23) which do not contain $\tilde{A}^5_1$ or $\tilde{A}^6_1$ as subalgebras (or algebras equivalent to them). In carrying out our construction, we begin with the results of Theorems 1 and 2, according to which we choose one of the basis operators of $A_{2.2}$ (we choose the basis operator e_2 for this) in one of the canonical forms given in these theorems. The calculations are similar to those involved in the construction of the algebras $A_{2.1}$ so we do not dwell on the calculations for this case, and we merely give the list of realizations.

Realizations of the algebras $A_{2.2}$ in the classes of operators (18).

$$
A^1_{2.2} = \langle -t\partial_t - x\partial_x, \partial_t + \partial_x \rangle;
$$

$$
\begin{aligned}
&A^2_{2.2} = \langle -t\partial_t - x\partial_x + ku\partial_u, \partial_t + \partial_x\rangle;\\
&A^3_{2.2} = \langle -t\partial_t + ku\partial_u, \partial_t\rangle; \quad A^4_{2.2} = \langle -t\partial_t + xu\partial_u, \partial_t\rangle;\\
&A^5_{2.2} = \langle -t\partial_t, \partial_t\rangle; \quad A^6_{2.2} = \langle -t\partial_t + \partial_u, \partial_t\rangle;\\
&A^7_{2.2} = \langle -t\partial_t - x\partial_x, \partial_t\rangle; \quad A^8_{2.2} = \langle -x\partial_x + ku\partial_u, \partial_x\rangle \ (k \neq 0);\\
&A^9_{2.2} = \langle -x\partial_x + \partial_u, \partial_x\rangle; \quad A^{10}_{2.2} = \langle -x\partial_x, \partial_x\rangle;\\
&A^{11}_{2.2} = \langle -t\partial_t - x\partial_x, \partial_x\rangle;\\
&A^{12}_{2.2} = \langle -t\partial_t - x\partial_x + ku\partial_u, \partial_x\rangle \ (k \neq 0);\\
&A^{13}_{2.2} = \langle -t\partial_t + x\partial_x, \partial_t + xu\partial_u\rangle; \quad A^{14}_{2.2} = \langle x\partial_x, xu\partial_u\rangle;\\
&A^{15}_{2.2} = \langle t\partial_t + x\partial_x, xu\partial_u\rangle;\\
&A^{16}_{2.2} = \langle \partial_t + \partial_x, e^t g(z)\partial_u\rangle, \quad z = t - x, \ g \neq 0;\\
&A^{17}_{2.2} = \langle \partial_t, e^t\partial_u\rangle; \quad A^{18}_{2.2} = \langle \partial_x, e^x\partial_u\rangle;\\
&A^{19}_{2.2} = \langle \partial_t + u\partial_u, e^{2t}\partial_u\rangle; \quad A^{20}_{2.2} = \langle \partial_t + xu\partial_u, e^{(1+x)t}\partial_u\rangle;\\
&A^{21}_{2.2} = \langle -u\partial_u, \partial_t, g(t,x)\partial_u\rangle, \ g \neq 0.
\end{aligned}
$$

Realizations of the algebras $A_{2.2}$ in the class of operators (23). In this list we do not include those realizations which contain one-dimensional subalgebras equivalent to $\tilde{A}^5_1$, $\tilde{A}^6_1$. We have:

$$
\begin{aligned}
&\tilde{A}^1_{2.2} = \langle -t\partial_t - x\partial_x + mu\partial_u, \partial_t + \partial_x\rangle, \ m \in \mathbb{R};\\
&\tilde{A}^2_{2.2} = \langle -t\partial_t + \partial_u, \partial_t\rangle; \quad \tilde{A}^3_{2.2} = \langle -t\partial_t + mu\partial_u, \partial_t\rangle;\\
&\tilde{A}^4_{2.2} = \langle -t\partial_t - x\partial_x + mu\partial_u, \partial_t\rangle, \ m \in \mathbb{R}.
\end{aligned}
$$

We remark that when we construct $A_{2.2}$-invariant equations, we put $k = m + 1$ in the system (24) for the realizations of $\tilde{A}^1_{2.2}$, $\tilde{A}^3_{2.2}$, $\tilde{A}^4_{2.2}$. Further, we use only those algebras which do not contain subalgebras equivalent to A^2_1, A^5_1, A^6_1. There are eight such algebras: $A^1_{2.2}$, $A^2_{2.2}$, $A^8_{2.2}$, $A^9_{2.2}$, $A^{10}_{2.2}$, $A^{11}_{2.2}$, $A^{12}_{2.2}$, $A^{13}_{2.2}$. Then, substituting $A^8_{2.2}$, $A^9_{2.2}$, $A^{10}_{2.2}$ into (21), we find that $HF = 0$, which is a contradiction. All the other algebras are invariance algebras of equations of the form (15).

$A_{2.2}$-invariant equations of type (15).

$$
\begin{aligned}
A^1_{2.2}:\ & F = z^{-1}\tilde{F}(\omega), \quad H = \lambda z \quad G = |z|^{-3/2}\tilde{G}(\omega),\\
& \tilde{F}' \neq 0, \quad z = t - x, \quad \omega = |z|^{-1/2}u, \quad \lambda \neq 0,\\
A^2_{2.2}:\ & F = z^{-1}\tilde{F}(\omega), \quad H = \lambda z \quad G = |z|^{k-3/2}\tilde{G}(\omega),
\end{aligned}
$$

$$
\begin{aligned}
& \tilde{F}' \neq 0, \quad z = t - x, \quad \omega = |z|^{k-1/2}u, \quad \lambda k \neq 0, \\
A^{11}_{2.2}: \ & F = t^{-1}\tilde{F}(\omega), \quad H = \lambda t \quad G = |t|^{-3/2}\tilde{G}(\omega), \\
& \tilde{F}' \neq 0, \quad \omega = |t|^{-1/2}u, \quad \lambda \neq 0, \\
A^{12}_{2.2}: \ & F = t^{-1}\tilde{F}(\omega), \quad H = \lambda t \quad G = |t|^{k-3/2}\tilde{G}(\omega), \\
& \tilde{F}' \neq 0, \quad \omega = |t|^{k-1/2}u, \quad \lambda k \neq 0, \\
A^{13}_{2.2}: \ & F = x^3(\lambda - 2tx)\tilde{F}(\omega), \quad H = x8\lambda - 2tx]^{-1}, \\
& G = e^{tx}|x|^{3/2}\tilde{G}(\omega) + \tfrac{1}{4}x^2u(\lambda - 2tx)^2\tilde{F}, \\
& \tilde{F}' \neq 0, \quad \omega = |x|^{1/2}ue^{-tx}, \quad \lambda \in \mathbb{R}.
\end{aligned}
$$

$A_{2.2}$-invariant equations of type (17).

$$
\begin{aligned}
\tilde{A}^{1}_{2.2}: \ & F = \tilde{F}(\omega), \quad G = |z|^{-(m+2)}\tilde{G}(\omega), \\
& \tilde{F}' \neq 0, \quad z = t - x, \quad \omega = |z|^{m}u, \quad m \in \mathbb{R}, \\
\tilde{A}^{2}_{2.2}: \ & F = e^{2u}\tilde{F}(x), \quad G = e^{2u}\tilde{G}(x), \quad \tilde{F} \neq 0, \\
\tilde{A}^{3}_{2.2}: \ & F = |u|^{4/(2m+1)}\tilde{F}(x), \quad G = |u|^{(5+2m)/(2m+1)}\tilde{G}(x), \\
& \tilde{F} \neq 0, \quad m \neq -\tfrac{1}{2}, \\
\tilde{A}^{4}_{2.2}: \ & F = \tilde{F}(\omega), \quad G = |x|^{-(m+2)}\tilde{G}(\omega), \\
& \tilde{F}' \neq 0, \quad \omega = |x|^{m}, \quad m \in \mathbb{R}.
\end{aligned}
$$

Conclusion. It is clear from the above results that the successive increase in dimension of the Lie algebra of invariance of the given equation leads to a corresponding decrease in arbitrariness in the functions entering into the equation. This then allows us, at a certain stage, to use the standard methods to obtain a complete solution of the problem of group classification of equations of type (1). In particular, for equations of the form (15) and (17), the A_2-invariant equations contain arbitrary functions of one variable, which then allows us to use the Lie–Ovsiannikov method in order to obtain a complete list of equations of this type, and whose algebras of invariance are solvable Lie algebras.

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