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On symmetry reduction of the Euler–Lagrange–Born–Infeld equation to linear ODEs

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Вивчається зв'язок між структурними властивостями тривимірних підалгебр алгебри Пуанкаре $\mathfrak{p}(1,4)$ і симетрійною редукцією рівняння Ейлера–Лагранжа–Борна–Інфельда. Основну увагу зосереджено на редукцях за тривимірними підалгебрами, що зводять рівняння Ейлера–Лагранжа—Борна–Інфельда до лінійних диференціальних рівнянь.

Connections between structure properties of three-dimensional subalgebras of the Poincaré algebra $\mathfrak{p}(1,4)$ and Lie reductions of the Euler–Lagrange–Born–Infeld equation are studied. We concentrate our attention on Lie reductions with respect to three-dimensional subalgebras that reduce the Euler–Lagrange–Born–Infeld equation to linear ordinary differential equations.

1. Introduction. Symmetry reduction is the most universal tool for finding exact solutions of partial differential equations (PDEs). We focus our attention on some applications of the classical Lie method to investigation of PDEs with non-trivial symmetry groups. In 1895, Lie [19] considered solutions of PDEs that are invariant with respect to symmetry groups admitted by these PDEs. It turned out that the problem of symmetry reduction and construction of independent invariant solutions for a PDE with a non-trivial symmetry group is reduced to the algebraic problem of classification of inequivalent subalgebras of the Lie invariance algebra of this equation [23, 24].

In 1975, Patera, Winternitz, and Zassenhaus [25] proposed a general method for describing inequivalent subalgebras of Lie algebras with nontrivial ideals. It turned out that reduced equations obtained from inequivalent subalgebras of the same dimension were of different types. Grundland, Harnad and Winternitz [17] were the first who pointed out and studied this phenomenon. Further details can be found in [6, 8, 11, 15, 16, 21, 22]. The results obtained cannot be explained using only the dimension of subalgebras of Lie invariance algebras.

To explain a difference in properties of reduced equations for PDEs with nontrivial symmetry groups, we investigate the relation between structure properties of inequivalent subalgebras of the same dimension of the Lie invariance algebras of those PDEs and properties of the respective reduced equations. By now, we have studied this relation for the case of low-dimensional ($\dim L \leq 3$) inequivalent subalgebras of the same dimension of the algebra $\mathfrak{p}(1,4)$, which is the Lie algebra of the Poincaré group $P(1,4)$, and the eikonal equation [8].

This paper is devoted to the study of the relation between structural properties of low-dimensional ($\dim L \leq 3$) inequivalent subalgebras of the same rank of the algebra $\mathfrak{p}(1,4)$ and properties of reduced equations for the Euler–Lagrange–Born–Infeld (ELBI) equation. By now, this relation has been investigated for three-dimensional subalgebras. We obtained the following types of reduced equations: identities, linear ordinary differential equations, nonlinear ordinary differential equations, partial differential equations. For some subalgebras, it is impossible to construct ansatzes that reduce the ELBI equation.

We focus our attention on reduction of the ELBI equation to linear ODEs. More precisely, we only present the results of symmetry reduction for those types of subalgebras that provide us reductions to linear ODEs.

2. Lie algebra of the Poincaré group $P(1,4)$ and its nonequivalent subalgebras. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. It is the minimal group that contains, as subgroups, the extended Galilei group $\widetilde{G}(1,3)$ [12] and the Poincaré group $P(1,3)$, which are underlying groups of classical and relativistic physics, respectively.

The Lie algebra $\mathfrak{p}(1,4)$ of the group $P(1,4)$ is spanned by 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$, $\mu,\nu = 0,1,2,3,4$, and P_μ, $\mu = 0,1,2,3,4$, which satisfy the commutation relations

$$[P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\sigma] = g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu,$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\sigma}M_{\nu\rho} + g_{\nu\rho}M_{\mu\sigma} - g_{\mu\rho}M_{\nu\sigma} - g_{\nu\sigma}M_{\mu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$.

We consider the canonical realization [13, 14] of $\mathfrak{p}(1,4)$,

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_1 = -\frac{\partial}{\partial x_1}, \quad P_2 = -\frac{\partial}{\partial x_2}, \quad P_3 = -\frac{\partial}{\partial x_3},$$
$$P_4 = -\frac{\partial}{\partial u}, \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad x_4 \equiv u.$$

Hereafter we use the following basis elements

$$G = M_{04}, \quad L_1 = M_{23}, \quad L_2 = -M_{13}, \quad L_3 = M_{12},$$
$$P_a = M_{a4} - M_{0a}, \quad C_a = M_{a4} + M_{0a},$$
$$X_0 = \tfrac{1}{2}(P_0 - P_4), \quad X_k = P_k, \quad X_4 = \tfrac{1}{2}(P_0 + P_4), \quad a, k = 1, 2, 3.$$

Subalgebras of the Lie algebra $\mathfrak{p}(1,4)$ were studied up to $P(1,4)$-conjugation in [4, 5, 10], in particular, the classification of subalgebras of $\mathfrak{p}(1,4)$ of dimensions up to three was given in [7]. Note that the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ is spanned by L_1, L_2, L_3, P_1, P_2, P_3, X_0, X_1, X_2, X_3 and X_4.

3. Classification of symmetry reductions for the Euler–Lagrange–Born–Infeld equation. Born–Infeld-like equations arise in fluid dynamics, theory of continuous medium, general relativity, field theory, theory of minimal surfaces, nonlinear electrodynamics, etc. [1, 2, 3, 18, 26].

We consider the Euler–Lagrange–Born–Infeld (ELBI) equation

$$\Box u\,(1 - u_\nu u^\nu) + u^\mu u^\nu u_{\mu\nu} = 0, \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_\mu \equiv \frac{\partial u}{\partial x_\mu}$, $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}$, $u^\mu = g^{\mu\nu}u_\nu$, $\mu, \nu = 0, 1, 2, 3$, and $\Box$ is the d'Alembert operator.

In 1984, Fushchych and Serov [13] studied symmetry properties of the multidimensional nonlinear Euler–Lagrange equation. These results imply that the Lie invariance algebra of the equation (1) contains, as a subalgebra, the Poincaré algebra $\mathfrak{p}(1,4)$.

We carry out Lie symmetry reductions of the ELBI equation to linear ODEs using subalgebras of $\mathfrak{p}(1,4)$ of the following types: $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,6}$. The notation of three-dimensional algebras is according to Mubarakzyanov's classification of low-dimensional Lie algebras [20].

Among inequivalent subalgebras of the Poincaré algebra $\mathfrak{p}(1,4)$ listed in [7], we select only such subalgebras that do reduce the ELBI equation to linear ODEs with nonlinear solutions since linear solutions are considered to be trivial. For each of the selected subalgebras, we construct an ansatz for u, the corresponding reduced equation, its general solution and the associated family of invariant solutions of the ELBI equation.

Proposition 1. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains* 31 *three-dimensional inequivalent subalgebras of the type* $3A_1$.

1. $\langle P_1\rangle \oplus \langle P_2\rangle \oplus \langle X_3\rangle$:
   the ansatz is $x_0^2 - x_1^2 - x_2^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $\omega^2\varphi'' - 6\omega\varphi' + 6\varphi = 0$;
   the solution of the reduced equation is $\varphi(\omega) = c_1\omega^6 + c_2\omega$;
   the solution of the ELBI equation is
   $$x_0^2 - x_1^2 - x_2^2 - u^2 = c_1(x_0+u)^6 + c_2(x_0+u).$$

2. $\langle P_3\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$:
   the ansatz is $x_0^2 - x_3^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $\omega^2\varphi'' - 4\omega\varphi' + 4\varphi = 0$;
   the solution of the reduced equation is $\varphi(\omega) = c_2\omega^4 + c_1\omega$;
   the solution of the ELBI equation is
   $$x_0^2 - x_3^2 - u^2 = c_2(x_0+u)^4 + c_1(x_0+u).$$

3. $\langle P_1\rangle \oplus \langle P_2\rangle \oplus \langle P_3\rangle$:
   the ansatz is $x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $\omega^2\varphi'' - 8\omega\varphi' + 8\varphi = 0$;
   the solution of the reduced equation is $\varphi(\omega) = c_1\omega^8 + c_2\omega$;
   the solution of the ELBI equation is
   $$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = c_1(x_0+u)^8 + c_2(x_0+u).$$

4. $\langle P_1\rangle \oplus \langle P_2 - X_2\rangle \oplus \langle X_3\rangle$:
   the ansatz is $\frac{x_0^2-x_1^2-u^2}{x_0+u} - \frac{x_2^2}{x_0+u+1} = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $(\omega+1)^5\omega^5\left(\omega(\omega+1)\varphi'' - 2(2\omega+1)\varphi'\right) = 0$;
   the solution of the reduced equation is

$\varphi(\omega) = c_2\omega^3\big(6\omega^2 + 15\omega + 10\big) + c_1$;
the solution of the ELBI equation is

$$\frac{x_0^2 - x_1^2 - u^2}{x_0 + u} - \frac{x_2^2}{x_0 + u + 1} = c_2(x_0 + u)^3\big(6(x_0 + u)^2 + 15(x_0 + u) + 10\big) + c_1.$$

5. $\langle P_1\rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0\rangle \oplus \langle P_3 - \gamma X_3, \gamma \neq 0\rangle$:
the ansatz is $2u + \frac{x_1^2}{x_0+u} + \frac{x_2^2}{x_0+u+\alpha} + \frac{x_3^2}{x_0+u+\gamma} = \varphi(\omega),\ \omega = x_0 + u$;
the reduced equation is
$(\omega+\gamma)^5\omega^5(\omega+\alpha)^5\big[\omega\big(\omega^2+(\alpha+\gamma)\omega+\alpha\gamma\big)\varphi'' - 2(3\omega^2+2(\alpha+\gamma)\omega+\alpha\gamma)(\varphi'-1)\big] = 0$;
the solution of the reduced equation is
$\varphi(\omega) = c_1\big[\frac{1}{7}\omega^4 + \frac{1}{3}(\alpha+\gamma)\omega^3 + \frac{1}{5}(\alpha^2 + 4\alpha\gamma + \gamma^2)\omega^2 + \frac{1}{2}\alpha\gamma(\alpha+\gamma)\omega + \frac{1}{3}\alpha^2\gamma^2\big]\omega^3 + \omega + c_2$;
the solution of the ELBI equation is

$$\begin{aligned} &2u + \frac{x_1^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} + \frac{x_3^2}{x_0 + u + \gamma} \\ &= c_1\left[\tfrac{1}{7}(x_0+u)^4 + \tfrac{1}{3}(\alpha+\gamma)(x_0+u)^3 + \tfrac{1}{5}(\alpha^2+4\alpha\gamma+\gamma^2)\right. \\ &\quad \left.\times (x_0+u)^2 + \tfrac{1}{2}\alpha\gamma(\alpha+\gamma)(x_0+u) + \tfrac{1}{3}\alpha^2\gamma^2\right] \\ &\times (x_0+u)^3 + x_0 + u + c_2. \end{aligned}$$

6. $\langle P_1\rangle \oplus \langle P_2 - \alpha X_2, \alpha > 0\rangle \oplus \langle P_3\rangle$:
the ansatz is $2u + \frac{x_1^2+x_3^2}{x_0+u} + \frac{x_2^2}{x_0+u+\alpha} = \varphi(\omega)$, $\omega = x_0 + u$;
the reduced equation is
$(\omega+\alpha)^5\omega^5(\omega(\omega+\alpha)\varphi'' - 2(3\omega+2\alpha)(\varphi'-1)) = 0$;
the solution of the reduced equation is
$\varphi(\omega) = c_1\big(\frac{1}{7}\omega^2 + \frac{\alpha}{3}\omega + \frac{\alpha^2}{5}\big)\omega^5 + \omega + c_2$;
the solution of the ELBI equation is

$$\begin{aligned} &2u + \frac{x_1^2 + x_3^2}{x_0 + u} + \frac{x_2^2}{x_0 + u + \alpha} \\ &= c_1\left(\tfrac{1}{7}(x_0+u)^2 + \tfrac{\alpha}{3}(x_0+u) + \tfrac{\alpha^2}{5}\right)(x_0+u)^5 \\ &\quad + x_0 + u + c_2. \end{aligned}$$

7. $\langle P_3 - 2X_0\rangle \oplus \langle X_1\rangle \oplus \langle X_2\rangle$:
the ansatz is

$\frac{1}{6}(x_0+u)^3+x_3(x_0+u)+x_0-u=\varphi(\omega)$, $\omega=(x_0+u)^2+4x_3$;
the reduced equation is $2\omega\varphi''-\varphi'=0$;
the solution of the reduced equation is
$\varphi(\omega)=c_2\omega^{3/2}+c_1$;
the solution of the ELBI equation is

$$\frac{1}{6}(x_0+u)^3+x_3(x_0+u)+x_0-u = c_2\big((x_0+u)^2+4x_3\big)^{3/2}+c_1.$$

8. $\langle P_3-2X_0\rangle\oplus\langle X_1\rangle\oplus\langle X_4\rangle$:
the ansatz is $(x_0+u)^2+4x_3=\varphi(\omega)$, $\omega=x_2$;
the reduced equation is $\varphi''=0$;
the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
the solution of the ELBI equation is

$$u=\varepsilon(c_1x_2-4x_3+c_2)^{1/2}-x_0,\quad \varepsilon=\pm1.$$

Proposition 2. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains* 10 *three-dimensional inequivalent subalgebras of the type* $A_2\oplus A_1$.

1. $\langle -(G+\alpha X_2), X_4, \alpha>0\rangle\oplus\langle X_1\rangle$:
the ansatz is $x_2-\alpha\ln(x_0+u)=\varphi(\omega)$, $\omega=x_3$;
the reduced equation is $\varphi''=0$;
the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
the solution of the ELBI equation is

$$x_2-\alpha\ln(x_0+u)=c_1x_3+c_2.$$

Proposition 3. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains* 17 *three-dimensional inequivalent subalgebras of the type* $A_{3,1}$.

1. $\langle 2\mu X_4, P_3-2X_0, X_1+\mu X_3, \mu>0\rangle$:
the ansatz is $(x_0+u)^2+4x_3-4\mu x_1=\varphi(\omega)$, $\omega=x_2$;
the reduced equation is $\varphi''=0$;
the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
the solution of the ELBI equation is

$$u=\varepsilon\left(4\mu x_1+c_1x_2-4x_3+c_2\right)^{1/2}-x_0,\quad \varepsilon=\pm1.$$

Proposition 4. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains* 3 *three-dimensional nonconjugate subalgebras of the type* $A_{3,2}$.

1. $\langle 2\beta X_4, P_3, G+\alpha X_1+\beta X_3, \alpha>0, \beta>0\rangle$:
   the ansatz is $x_1-\alpha\ln(x_0+u)=\varphi(\omega)$, $\omega=x_2$;
   the reduced equation is $\varphi''=0$;
   the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
   the solution of the ELBI equation is

$$x_1-\alpha\ln(x_0+u)=c_1x_2+c_2.$$

Proposition 5. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains five three-dimensional inequivalent subalgebras of the type* $A_{3,3}$.

1. $\langle P_3, X_4, G+\alpha X_1, \alpha>0\rangle$:
   the ansatz is $x_1-\alpha\ln(x_0+u)=\varphi(\omega)$, $\omega=x_2$;
   the reduced equation is $\varphi''=0$;
   the solution of the reduced equation is $\varphi(\omega)=c_1\omega+c_2$;
   the solution of the ELBI equation is

$$u=\exp\left(\frac{x_1-c_1x_2-c_2}{\alpha}\right)-x_0.$$

Proposition 6. *The Lie algebra* $\mathfrak{p}(1,4)$ *contains* 18 *three-dimensional inequivalent subalgebras of the type* $A_{3,6}$.

1. $\langle P_1-X_1, P_2-X_2, -P_3+L_3\rangle$:
   the ansatz is $\frac{x_1^2+x_2^2}{x_0+u+1}+\frac{x_3^2}{x_0+u}+2u=\varphi(\omega)$, $\omega=x_0+u$;
   the reduced equation is
   $\omega^5(\omega+1)^5[\omega(\omega+1)\varphi''-2(3\omega+1)(\varphi'-1)]=0$;
   the solution of the reduced equation is
   $\varphi(\omega)=\frac{c_1}{7}\omega^7+\frac{2}{3}c_1\omega^6+\frac{6}{5}c_1\omega^5+c_1\omega^4+\frac{c_1}{3}\omega^3+\omega+c_2$;
   the solution of the ELBI equation is

$$\frac{x_1^2+x_2^2}{x_0+u+1}+\frac{x_3^2}{x_0+u}+2u=\tfrac{c_1}{7}(x_0+u)^7+\tfrac{2}{3}c_1(x_0+u)^6$$
$$+\tfrac{6}{5}c_1(x_0+u)^5+c_1(x_0+u)^4+\tfrac{c_1}{3}(x_0+u)^3+x_0+u+c_2.$$

2. $\langle P_1, -P_2, -(L_3+\alpha X_3), \alpha>0\rangle$:
   the ansatz is $x_0^2-x_1^2-x_2^2-u^2=\varphi(\omega)$, $\omega=x_0+u$;
   the reduced equation is $\omega^2\varphi''-6\omega\varphi'+6\varphi=0$;
   the solution of the reduced equation is $\varphi(\omega)=c_1\omega^6+c_2\omega$;
   the solution of the ELBI equation is

$$x_0^2-x_1^2-x_2^2-u^2=c_1(x_0+u)^6+c_2(x_0+u).$$

3. $\langle X_1, -X_2, P_3 - L_3 \rangle$:
   the ansatz is $x_0^2 - x_3^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $\omega^2\varphi'' - 4\omega\varphi' + 4\varphi = 0$;
   the solution of the reduced equation is $\varphi(\omega) = c_1\omega^4 + c_2\omega$;
   the solution of the ELBI equation is

$$x_0^2 - x_3^2 - u^2 = c_1(x_0 + u)^4 + c_2(x_0 + u).$$

4. $\langle P_1, P_2, L_3 - P_3 \rangle$:
   the ansatz is $x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = \varphi(\omega)$, $\omega = x_0 + u$;
   the reduced equation is $\omega^2\varphi'' - 8\omega\varphi' + 8\varphi = 0$;
   the solution of the reduced equation is $\varphi(\omega) = c_1\omega^8 + c_2\omega$;
   the solution of the ELBI equation is

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2 = c_1(x_0 + u)^8 + c_2(x_0 + u).$$

5. $\langle X_1, -X_2, P_3 - L_3 - 2\alpha X_0, \alpha > 0 \rangle$:
   the ansatz is $(x_0 + u)^3 + 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) = \varphi(\omega)$, $\omega = (x_0 + u)^2 + 4\alpha x_3$;
   the reduced equation is $2\omega\varphi'' - \varphi' = 0$;
   the solution of the reduced equation is $\varphi(\omega) = c_2\omega^{3/2} + c_1$;
   the solution of the ELBI equation is

$$(x_0 + u)^3 + 6\alpha x_3(x_0 + u) + 6\alpha^2(x_0 - u) = c_2\big((x_0 + u)^2 + 4\alpha x_3\big)^{3/2} + c_1.$$

4. Conclusions. In this paper we focused our attention on Lie reductions of the ELBI equation to linear ODEs. More precisely, we presented results for such three-dimensional subalgebras of $\mathfrak{p}(1,4)$ that give reductions of the ELBI equation to linear ODEs with nonlinear solutions.

It is known [7] that the Lie algebra $\mathfrak{p}(1,4)$ contains three-dimensional inequivalent subalgebras of the following types: $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$, $A_{3,4}$, $A_{3,6}$, $A_{3,7}^a$, $A_{3,8}$, $A_{3,9}$. Results of the paper imply that all the above Lie reductions of the ELBI equation to linear ODEs can be obtained using subalgebras of the types $3A_1$, $A_2 \oplus A_1$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$ and $A_{3,6}$. Moreover, all the subalgebras considered in the paper are also subalgebras of the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$.

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