Збірник праць Інституту математики НАН України 2019, т. 16, № 1, 181–192

УДК 517.912:512.816

Adjoint solutions and superposition principle for linearizable Krichever–Novikov equation

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Наявність операторної рівності для рівнянь, пов'язаних нелокальними перетвореннями, дозволила запропонувати метод знаходження іншого розв'язку вихідного рівняння, який приєднаний до відомого його розв'язку. Цей підхід застосовано для побудови точних розв'язків лінеаризованого рівняння Кричевера–Новікова та відповідного лінійного рівняння. Виведено формулу нелінійної нелокальної суперпозиції розв'язків, яку використано для розмноження точних розв'язків цього нелінійного рівняння.

Existence of an operator equality for equations connected by nonlocal transformations allowed us to propose a method of finding of a new solution of the initial equation adjoint to its known solution. This approach is used for construction of exact solutions for the linearizable Krichever–Novikov equation and for the corresponding linear equation. The formula of nonlinear nonlocal superposition of solutions for this nonlinear equation is derived and applied to generation of its solutions.

1. Introduction. A wide range of efficient methods for study of nonlinear partial differential equations are being developed at the moment. A considerable part of them are based on a fundamental idea of symmetry and, in particular, on the group-theoretical method suggested by Lie [6, 14, 16]. The most important generalizations of the basic symmetry group approach are realized in the concepts of conditional (nonclassical) symmetries, weak symmetries [5, 9, 15] and nonlocal symmetries of differential equations [1, 2, 3, 4, 7, 10, 12, 13, 17, 18, 21, 23, 24]. Therefore, development of other approaches to seek for new symmetries and methods for investigation of these equations is of importance and stays relevant.

Finite nonlocal transformations are efficiently used to study and solve nonlinear partial differential equations for a long time [8, 10, 21, 23, 24]. In particular, a number of interesting results for nonlinear equations connected among themselves by means of the *nonlocal transformations of variables* were obtained and formulae generating solutions or nonlocal nonlinear superposition were derived [11, 19, 22, 23].

Let us remind here the main concepts and terminology of the nonlocal transformations method. Assume that a given nonlocal transformation

$$\begin{aligned} \mathcal{T}\colon\ & x^i = h^i(y, v_{(k)}), \quad u^K = H^K(y, v_{(k)}), \\ & i = 1, \ldots, n, \qquad K = 1, \ldots, m, \end{aligned} \tag{1}$$

maps an initial (*source*) equation

$$F_0(x, u_{(n)}) = 0 \tag{2}$$

into an equation $\Phi(y, v_{(q)}) = 0$ of order $q = n + k$ that admits factorization to another equation which we call a *target* equation

$$F_1(y, v_{(s)}) = 0, \tag{3}$$

i.e.,

$$\Phi(y, v_{(q)}) = \lambda F_1(y, v_{(s)}). \tag{4}$$

Here λ is a differential operator of order $n + k - s$. This results in algorithms for finding solutions of (2) via known solutions of (3). Existence of factorization equation (4) gives rise to a technique of finding of a special solution to the initial equation (2) from a known solution of the equation $\Phi(y, v_{(q)}) = 0$. The symbol $u_{(r)}$ denotes the tuple of partial derivatives of the function u from order zero up to order r. In the case of two independent variables, we use the special notation of the variables: $x_1 = x$, $x_2 = t$ and thus $u_t = \partial u/\partial t = \partial_t u$, $u_x = \partial u/\partial x = \partial_x u$.

The paper is organized as follows. In the next section we begin with some preliminary remarks on the concept of adjoint solution of the initial equation. Then we apply it to the linearisable Krichever–Novikov equation derived from the linear one via the known nonlocal transformation. In Section 3 this concept is applied to the case of the nonlocal invariance of the linearizable Krichever–Novikov equation. Examples of adjoint solutions are constructed.

2. Adjoint solution of the initial equation. This section is devoted to construction of solutions of the initial equation generated from known solutions of the appropriate inhomogeneous target equation. Existence of a factorization equation (4) gives rise to a technique [20] of construction of the special solution to the initial equation (2). Further we call it an *adjoint* solution.

We assume that a given function $v = f(y)$ *is not a solution* of equation (3), that is, substituting this function into (3), we get another equation with *discrepancy* $w(y)$

$$F_1(y, v_{(s)}) = w(y). \tag{5}$$

Suppose, nevertheless, that equation (4) holds and the equation

$$\lambda(y, v_{(s)}) F_1(y, v_{(s)}) = \lambda(y, v_{(s)}) w(y) = 0 \tag{6}$$

appears to be true. Here $w(y)$ runs through the set of solutions of a *linear* equation $\lambda(y, v_{(s)}) w(y) = 0$ with variable coefficients of spatial form. Solving (6) with respect to the unknown function $w(y, v_{(k)}(y))$, one can find its solution as a function depending on y, $v_{(k)}(y)$

$$w = W(y, v_{(k)}). \tag{7}$$

After substitution of (7) into the equation (5) we obtain an *inhomogeneous* equation for the dependent variable v:

$$F_1(y, v_{(s)}) = W(y, v_{(k)}). \tag{8}$$

Hence the result of transformation (1) takes a form

$$\Phi(y, v_{(q)}) = \lambda F_2(y, v_{(s)}) = \lambda(F_1(y, v_{(s)}) - W(y, v_{(k)}). \tag{9}$$

The function $v(y)$ determined by (9) satisfies the equation $\mathcal{T} F_0(x, u_{(n)}) = \Phi(y, v_{(q)})$. Therefore, substituting $v(y)$ obtained in this way into the formulae of nonlocal transformation $\mathcal{T}$, one can find an appropriate solution of the given equation (2). Moreover, having the information on symmetries of the inhomogeneous equation (8), one can construct a r-parametrical family of solutions for it and, consequently, find the corresponding parametrical sets of solutions to the initial equation (2). If we let in (8) $W(y, v_{(k)}) = 0$ the equality (9) returns us to the connection of solutions of the initial equation and (3). That is why further we call such a special solution an *adjoint*. In what follows, we illustrate this approach by some examples.

3. Adjoint solutions constructed via the linearization. In this section we use the nonlocal transformation that connects the linear equation and the Krichever–Novikov equation of a special form and illustrate the proposed approach by some examples.

It is well-known [8] that the Krichever–Novikov equation

$$u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx} = 0 \tag{10}$$

can be obtained by applying the nonlocal transformation

$$w = \sqrt{u_x} \tag{11}$$

to the homogeneous linear third-order partial differential equation

$$w_t - w_{xxx} = 0. \tag{12}$$

The operator equation (4) connecting these two equations has the form

$$-4u_x^2\partial_x\big(u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx}\big) = 0. \tag{13}$$

Suppose a function $S(x,t)$ (*discrepancy*) is defined such that the inhomogeneous equation

$$u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx} = S(x,t)$$

is satisfied. Then the condition $\partial_x S(x,t) = 0$ follows from (13). In particular, if we let $S(x,t)$ be a linear function of time, i.e., $S(x,t) = ht$, the corresponding equation takes the form

$$u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx} = ht. \tag{14}$$

This equation admits the Lie algebra spanned by the following operators

$$\begin{aligned} &X_1 = \partial_x, \quad X_2 = \partial_t + ht\partial_u, \quad X_3 = \partial_u, \\ &X_4 = (u - ht)\partial_u, \quad X_5 = \tfrac{1}{3}x\partial_x + t\partial_t + ht^2\partial_u. \end{aligned} \tag{15}$$

1) We obtain a simple group-invariant solution of equation (14) solving the characteristic equation generated by the sum of the first two

operators of the algebra (15), $u_x + u_t - ht = 0$. The corresponding Lie ansatz is

$$u(x,t) = -\tfrac{1}{2}hx^2 + hxt + f(t-x).$$

Substituting this expression into (14), we find the reduced ordinary differential equation

$$4f'''(h\omega - f') + 3f''^2 - 6hf'' - 4f'^2 + 8h\omega f' - 4h^2\omega^2 + 3h^2 = 0,$$

where $\omega = t - x$. The general solution of this equation allows us to write down the required solution of (14)

$$\begin{aligned} u(x,t) = {}& \tfrac{1}{2}ht^2 + \tfrac{1}{16}c_2 \sin 2(t - x + c_1) \\ & + \tfrac{1}{2}c_2 \sin(t - x + c_1) + \tfrac{3}{8}c_2(t - x + c_1) + c_3, \end{aligned}$$

where c_1, c_2, c_3 are arbitrary constants.

Applying the nonlocal transformation (11) to the obtained solution, we get the corresponding solution of the linear equation:

$$w(x,t) = \tfrac{1}{4}\sqrt{-2c_2 \cos 2(t - x + c_1) - 8c_2 \cos(t - x + c_1) - 6c_2}.$$

One can compare the above solution with another solution of (12) being obtained in the form $w(x,t) = f(ct - x)$ determining a wave of unchanging profile moving at the constant velocity c:

$$w(x,t) = \bar{c}_1 + \bar{c}_2 \sin(ct - x) + \bar{c}_3 \cos(ct - x).$$

2) Another group-invariant solution of the equation (14) corresponding to the operator X_5 of the Lie algebra (15) has an implicit form

$$\begin{aligned} u(x,t) &= \tfrac{1}{2}ht^2 + \int^{t/x^3} \exp Q(k)\, \mathrm{d}k + c_3, \\ Q(k) &= \tfrac{2}{9}\sqrt{3}c_1 B_1 - \tfrac{16}{3}\sqrt{3}B_2 + \tfrac{4}{3}B_3 + \tfrac{2}{9}\sqrt{3}B_4 \\ &\quad - \tfrac{16}{3}\sqrt{3}c_1 B_5 + \tfrac{4}{3}c_1 B_6 + c_2, \end{aligned} \tag{16}$$

where

$$B_1 = \int^k \frac{Y_{\frac{2}{3}}\big(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\big)}{b^{3/2}\big(c_1 Y_{-\frac{1}{3}}\big(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\big) + J_{-\frac{1}{3}}\big(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\big)\big)}\, \mathrm{d}b,$$

$$B_2 = \int^k \frac{J_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b^{1/2}\left(c_1 Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right) + J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

$$B_3 = \int^k \frac{4\sqrt{3b}J_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right) - J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b\left(c_1 Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right) + J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

$$B_4 = \int^k \frac{J_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b^{3/2}\left(c_1 Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right) + J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

$$B_5 = \int^k \frac{Y_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b^{1/2}\left(c_1 Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right) + J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

$$B_6 = \int^k \frac{4\sqrt{3b}Y_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right) - Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b\left(c_1 Y_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right) + J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,$$

where c_1, c_2, c_3 are arbitrary constants, $J_\alpha(x)$ and $Y_\alpha(x)$ are Bessel functions of the first and the second kinds respectivelly. Applying the formula (11) to this solution, we get such nonstationary solution of the linear equation (12):

$$w(x,t) = \pm\frac{1}{x^2}\sqrt{3t}\mathrm{e}^{\frac{1}{9}\sqrt{3}c_1\bar{B}_1 - \frac{8}{3}\sqrt{3}\bar{B}_2 + \frac{2}{3}\bar{B}_3 + \frac{1}{9}\sqrt{3}\bar{B}_4 - \frac{8}{3}c_1\bar{B}_5 + \frac{2}{3}c_1\bar{B}_6 + \frac{c_2}{2}}.$$

$\bar{B}_i$, $i = 1, \ldots, 6$, are the same as introduced above with $k = \frac{t}{x^3}$.

4. Adjoint solutions found via the nonlocal invariance. Beside a nonlocal linearization, the Krichever–Novikov equation (10) admits the auto-Bäcklund transformation [8]:

$$\begin{aligned} u_x &= v_x^{-1}v_{xx}^2, \\ u_t &= 2v_x^{-1}v_{xx}v_{xxxx} - 2v_x^{-2}v_{xx}^2v_{xxx} + \tfrac{5}{4}v_x^{-3}v_{xx}^4 - v_x^{-1}v_{xxx}^2, \end{aligned} \tag{17}$$

where $v(x,t)$ is another solution of the same equation

$$v_t + \tfrac{3}{4}v_x^{-1}v_{xx}^2 - v_{xxx} = 0.$$

In other words, the equation (10) stays invariant under the nonlocal transformation (17). This connection is realized by means of the operator equality

$$\left(-4v_{xx}v_x^2\partial_x + 8v_x^3\partial_{xx}\right)\cdot\left(v_t + \tfrac{3}{4}v_x^{-1}v_{xx}^2 - v_{xxx}\right) = 0. \tag{18}$$

We assume existence of the function $v(x,t)$ that is a solution of the inhomogeneous equation

$$v_t + \tfrac{3}{4} v_x^{-1} v_{xx}^2 - v_{xxx} = W(x,t). \tag{19}$$

Solving the partial differential equation generated by (18)

$$-4 v_{xx} v_x^2 W_x(x,t) + 8 v_x^3 W_{xx}(x,t) = 0$$

with respect to $W(x,t,v,v_x,v_{xx})$, we obtain the general solution

$$W(x,t) = f_1(t) + f_2(t) \int \sqrt{v_x}\, \mathrm{d}x. \tag{20}$$

To exclude an integral term in (20), we differentiate (19) with respect to x and set for simplicity $f_1(t) = 0$, $f_2(t) = K$ in (20), where K is a constant. So, instead of (19) we consider the equation

$$\partial_x \big(v_t + \tfrac{3}{4} v_x^{-1} v_{xx}^2 - v_{xxx} \big) - K \sqrt{v_x} = 0. \tag{21}$$

This inhomogeneous equation admits an infinite-dimensional Lie algebra spanned by the following operators

$$\begin{aligned} &X_1 = \partial_t + F_1(t)\partial_v, \quad X_2 = \partial_x + F_2(t)\partial_v, \\ &X_3 = \tfrac{1}{3}\partial_x + t\partial_t + \big(\tfrac{7}{3} v + F_3(t)\big)\partial_v. \end{aligned} \tag{22}$$

Here $F_i(t)$, $i = 1,2,3$, are arbitrary functions of the time variable. This algebra allows us to get a wide range of group-invariant solutions of the equation (21). We choose $v(x,t)$ in the traveling wave solution form $v(x,t) = G(\omega)$, $\omega = x - ht$, where h is a fixed constant. Substituting this expression into (21), we get the reduced equation

$$\begin{aligned} &4G'(\omega)^2 G''''(\omega) - 6G'(\omega)G''(\omega)G'''(\omega) \\ &\quad + 3G''(\omega)^3 + 4hG'(\omega)^2 G''(\omega) + 4KG'(\omega)^{5/2} = 0, \end{aligned} \tag{23}$$

which admits a solution

$$v(x,t) = G(\omega) = \int Y(\omega)\, \mathrm{d}\omega + c_4,$$

where $Y(\omega) = Z(\omega) + \omega + c_3$ and $Z(\omega)$ is the function determined by the equation

$$\int^{Z(\omega)} H^{-1}\, \mathrm{d}f = 0. \tag{24}$$

$H(Z(\omega), f)$ is an implicit solution of any of two equations

$$\mp\big(2\sqrt{f} - c_2 \mp Ph\big)h^{3/2} + Kh\arctan\left(\frac{Hh + K\sqrt{f}}{\sqrt{hf}P}\right) = 0, \tag{25}$$

$$P = \sqrt{-\frac{hH^2 + 2\sqrt{f}KH - c_1 f}{f}}.$$

To use the formula (17) and to verify a new solution, we need an expression for u_x to be a function of $\omega = x - ht$. First we find a solution of the equation (10) differentiated with respect to x. We set $u(x,t) = L(\omega)$ and substitute that into the equation

$$\partial_x\big(u_t + \tfrac{3}{4}u_x^{-1}u_{xx}^2 - u_{xxx}\big) = 0. \tag{26}$$

Implementation of the reduction procedure leads to the ordinary differential equation

$$\begin{aligned} &4L'(\omega)^2L''''(\omega) - 6L'(\omega)L''(\omega)L'''(\omega)\\ &\quad + 3L''(\omega)^3 + 4hL'(\omega)^2L''(\omega) = 0. \end{aligned} \tag{27}$$

An implicit solution of this equation is determined by the integral equation

$$\int^{L(\omega)} Q(a)^{-1}\,\mathrm{d}a - \omega - \bar{c}_4 = 0,$$

where function $Q(a)$ is defined by the equations

$$\mp\int^{Q(a)} \frac{hf}{\sqrt{hf(\bar{c}_1 - 4fh^2 + 4\sqrt{f}h^2\bar{c}_2 - h^2\bar{c}_2^2)}}\,\mathrm{d}f + a + \bar{c}_3 = 0,$$

where $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$ are arbitrary constants.

Now we apply the formula (17) to the obtained solution of the equation (23) and find the corresponding expression for $u_x(x,t)$:

$$u_x(x,t) = \hat{L}'(\omega) = G'(\omega)^{-1}G''(\omega)^2.$$

After simplification we get

$$\hat{L}'(\omega) = \frac{B^2}{Y}, \quad Y(\omega) = Z(\omega) + \omega + c_3.$$

Here $B(\omega, Y)$ are the implicit functions determined by the equations

$$\mp\big(2\sqrt{Y} - c_2\big)h^{5/2} + Th^{3/2} + Kh \arctan\left(\frac{Bh + K\sqrt{Y}}{\sqrt{hY}T}\right) = 0,$$

$$T = \sqrt{-\frac{hB^2 + 2\sqrt{Y}KB - c_1 Y}{Y}}.$$

The function $Z(\omega)$ was implicitly determined above by equations (24), (25). Substitution of this solution into the equation (27) takes it to zero.

Knowing the Lie algebra (22) of the inhomogeneous equation (21) we can construct a wide family of group-invariant solutions, and, therefore, obtain various solutions of the equation (10). The new solution of the equation (26) constructed above obviously can be generated via the invariance algebra admitted by this equation or by means of its any other symmetry. The symmetry solutions of the special inhomogeneous target equation allow us to generate different solutions for the initial equation. What type of the symmetry of initial equations do we have in this case? As a target equation is broken by a discrepancy appearance, it seems naturally to call it a forced symmetry.

5. The superposition formula and generation of solutions. We return to the homogeneous linear third-order differential equation (12) and the nonlocal transformation (11), connecting this equation with (10). We choose a linear superposition principle for (12) setting

$$w^{\mathrm{III}}(x,t) \equiv w(x,t) = w^{\mathrm{I}}(x,t) + w^{\mathrm{II}}(x,t).$$

Here $w^{\mathrm{I}}(x,t)$, $w^{\mathrm{II}}(x,t)$ are some known solutions of the linear equation. As equations are connected by the nonlocal transformation (11), the corresponding principle of nonlinear nonlocal superposition of solutions for equation (10) can be constructed.

Theorem 1. *The nonlinear nonlocal superposition formula of solutions for equation* (10) *has the form*

$$u(x,t) = u^{\mathrm{I}}(x,t) + u^{\mathrm{II}}(x,t) + 2\int \sqrt{u^{\mathrm{I}}(x,t)}\sqrt{u^{\mathrm{II}}(x,t)}\,\mathrm{d}x + s(t), \qquad (28)$$

where the arbitrary function $s(t)$ *is defined by the equation*

$$u_t = 2\left(\sqrt{u^{\mathrm{I}}(x,t)} + \sqrt{u^{\mathrm{II}}(x,t)}\right)\partial_x^2\left(\sqrt{u^{\mathrm{I}}(x,t)} + \sqrt{u^{\mathrm{II}}(x,t)}\right)$$

$$-\left(\partial_x\left(\sqrt{u^{\mathrm{I}}(x,t)}+\sqrt{u^{\mathrm{II}}(x,t)}\right)\right)^2. \tag{29}$$

Given solutions u^{I} and u^{II}, the new solution of (10) is found integrating the third term of (28). We get the specialization of the function $s(t)$ substituting the expression (28) into (29) and solving the equations obtained with respect to s.

We illustrate utilization of the proposed superposition formula for generation of solutions of the equation (10).

1) It can be easily verified that

$$\begin{aligned}
&u^{\mathrm{I}}=\tfrac{1}{1280c_1}\left(x^5+5c_2x^4+10c_2^2x^3+10c_2^3x^2+5c_2^4x+c_2^5\right)+c_3,\\
&u^{\mathrm{II}}=k_1x^5
\end{aligned}$$

are time-independent solutions of the equation (10). Applying the formula (28) adduced in Theorem 1 we find a time-dependent solution

$$\begin{aligned}
u^{\mathrm{III}}&=\tfrac{1}{3840{c_1}^{3/2}}\big(a_1x^5+a_2x^4+a_3x^3+30\sqrt{c_1}c_2^3x^2\\
&\quad+15\sqrt{c_1}c_2^4x+960c_1c_2^2\sqrt{5k_1}t+\kappa\big),\\
a_1&=3\sqrt{c_1}+96\sqrt{5k_1}c_1+3840k_1{c_1}^{3/2},\\
a_2&=15c_2\sqrt{c_1}+240\sqrt{5k_1}c_1c_2,\\
a_3&=160\sqrt{5k_1}c_1c_2^2+30\sqrt{c_1}c_2^2,\\
\kappa&=3\sqrt{c_1}{c_2}^5+380c_1^{3/2}(c_3+c_4).
\end{aligned}$$

2) Choosing $c_1=0$ in (16), we obtain a simpler solution

$$\begin{aligned}
u^{\mathrm{I}}&=\int^{\frac{y}{x^3}}\exp\left(-\tfrac{16}{3}\sqrt{3}\tilde{B}_1+\tfrac{4}{3}\tilde{B}_2+\tfrac{2}{9}\sqrt{3}\tilde{B}_3+c_2\right)\mathrm{d}k+c_3,\\
\tilde{B}_1&=\int^k J_{\frac{2}{3}}\left(\tfrac{1}{9}\tfrac{\sqrt{3}}{\sqrt{b}}\right)b^{-1/2}J_{-\frac{1}{3}}\left(\tfrac{1}{9}\tfrac{\sqrt{3}}{\sqrt{b}}\right)^{-1}\mathrm{d}b,\\
\tilde{B}_2&=\int^k \frac{4\sqrt{3b}J_{\frac{2}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)-J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)}{b\left(J_{-\frac{1}{3}}\left(\frac{1}{9}\frac{\sqrt{3}}{\sqrt{b}}\right)\right)}\,\mathrm{d}b,\\
\tilde{B}_3&=\int^k J_{\frac{2}{3}}\left(\tfrac{1}{9}\tfrac{\sqrt{3}}{\sqrt{b}}\right)b^{-3/2}J_{-\frac{1}{3}}\left(\tfrac{1}{9}\tfrac{\sqrt{3}}{\sqrt{b}}\right)^{-1}\mathrm{d}b.
\end{aligned} \tag{30}$$

Let the second solution be

$$u^{\mathrm{II}} = k_1 x^5.$$

Then

$$u_x^{\mathrm{III}} = -3tx^{-4} \exp\left(-\tfrac{16}{3}\sqrt{3}\hat{B}_1 + \tfrac{4}{3}\hat{B}_2 + \tfrac{2}{9}\sqrt{3}\hat{B}_3 + c_2\right) + 5k_1x^4$$
$$+ 2\sqrt{5}\sqrt{k_1x^4}\sqrt{-3tx^{-4}\exp\left(-\tfrac{16}{3}\sqrt{3}\hat{B}_1 + \tfrac{4}{3}\hat{B}_2 + \tfrac{2}{9}\sqrt{3}\hat{B}_3 + c_2\right)},$$

where $\hat{B}_i$, $i = 1, 2, 3$, are the same as those introduced in (30) but $k = \frac{t}{x^3}$. One can easily verify that obtained expression satisfies (26). More solutions may be constructed by utilization of the previous theorem and application of the Lie symmetry transformations or any other formula generating solutions.

6. Conclusion. The concept of an adjoint solution of the initial equation was developed in this paper, and used for construction of new solutions of linearizable Krichever–Novikov equation and for the connected linear one. Some of them were obtained in an explicit form, while others have a parametrical representations with functional parameters given in implicit form. The Lie symmetry solutions of the special inhomogeneous target equation allowed us to generate appropriate solutions for the given initial equation. The superposition formula was derived in the present paper and applied for the generation of solutions to the equation (10). All the found solutions can be naturally extended by means of the Lie symmetry transformations or any other formula generating new solutions. The results obtained for the equation (10) can be extended to similar classes of equations.

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