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Equivalence groupoid of a class of general Burgers–Korteweg–de Vries equations with space-dependent coefficients

S. Opanasenko

Memorial University of Newfoundland, Canada
E-mail: sopanasenko@mun.ca

Institute of Mathematics of NAS of Ukraine, Kyiv, Ukraine

Описано групоїд еквівалентності класу загальних рівнянь Бюргерса–Кортевега–де Фріза з просторовими коефіцієнтами. Показано, що цей клас зводиться сім'єю перетворень еквівалентності до свого підкласу з чотиривимірною звичайною групою еквівалентності. Прокласифіковано допустимі перетворення цього підкласу та виокремлені підкласи, що допускають максимальні нетривіальні умовні групи еквівалентності. Виявляється, що всі вони мають розмірність більшу за чотири. Зокрема, знайдено декілька нових класів диференціальних рівнянь, нормалізованих в узагальненому сенсі. Жоден з них не допускає єдину ефективну узагальнену групу еквівалентності.

We describe the equivalence groupoid of the class of general Burgers–Korteweg–de Vries equations with space-dependent coefficients. This class is shown to reduce by a family of equivalence transformations to a subclass with a four-dimensional usual equivalence group. Classified are admissible transformations of this subclass and singled out its subclasses admitting maximal nontrivial conditional equivalence groups. All of them turn out to have dimension higher than four. In particular, few new examples of nontrivial cases of normalization in the generalized sense of classes of differential equations appeared this way. Neither of classes discussed possesses a unique effective generalized equivalence group.

1. Introduction. A number of evolution equations that are important in mathematical physics are of the general form

$$u_t + C(t,x)uu_x = \sum_{k=0}^{r} A^k(t,x)u_k + B(t,x). \qquad (1)$$

In particular, this includes Burgers, Korteweg–de Vries (KdV), Kuramoto–Sivashinsky, Kawahara, and generalized Burgers–KdV equations.

Here and in the following the integer parameter r is fixed, and $r \geqslant 2$. We require the condition $CA^r \neq 0$ guaranteeing that equations from the class (1) are nonlinear and of genuine order r. Throughout the paper we use the standard index derivative notation $u_t = \partial u/\partial t$, $u_k = \partial^k u/\partial x^k$.

The class (1) and its various subclasses were subject to studying from the symmetry analysis point of view, see [6] for an extensive list of references. Recently, the class (1) became a source of examples of nontrivial equivalence groups [6]. In fact, the first examples of classes with generalized and extended generalized equivalence groups are of the form (1) (with some additional restrictions). Moreover, detailed studying thereof allowed the authors to introduce the concept of an effective generalized equivalence group of a class of differential equations. Furthermore, the structure of this class is so flexible, that a "reasonable" singled out subclass thereof is likely to possess normalization properties in some sense. Nonetheless, it is not the case for a subclass $\bar{\mathcal{F}}$ of equations with the arbitrary elements being time-independent,

$$u_t + C(x)uu_x = \sum_{k=0}^{r} A^k(x)u_k + B(x), \quad \text{where} \quad A^rC \neq 0. \tag{2}$$

The aim of this paper is to thoroughly study admissible transformations of the class $\bar{\mathcal{F}}$. In a nutshell, the results of this paper comprise the following four facts. Any equation in $\bar{\mathcal{F}}$ is mapped by an equivalence transformation of $\bar{\mathcal{F}}$ to an equation in the subclass $\mathcal{F}$ of reduced general Burgers–Korteweg–de Vries equations with space-dependent coefficients, singled out by conditions $C = 1$ and $A^1 = 0$. The subclass $\mathcal{F}$ is not normalized in any sense, and its usual equivalence group is four-dimensional. Classified are admissible transformations of the class $\mathcal{F}$ and singled out are its subclasses admitting maximal nontrivial conditional equivalence subgroups of the equivalence group of $\mathcal{F}$,

$$\begin{aligned}\hat{\mathcal{F}}_{\mathrm{I},1}\colon \quad u_t + uu_x = {}& \left(\frac{\alpha+2}{a_{01}}b_1 + a_{01}|x+\beta|^\alpha\right)u \\ &+ (x+\beta)\left(b_2|x+\beta|^{2\alpha} + b_1|x+\beta|^\alpha - \frac{b_1^2(\alpha+1)}{a_{01}^2}\right) \\ &+ \sum_{j=2}^{r} a_j(x+\beta)^j|x+\beta|^\alpha u_j \quad \text{with} \quad \alpha a_r a_{01} \neq 0,\end{aligned}$$

$$\hat{\mathcal{F}}_{\mathrm{I},01}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_j(x+\beta)^j|x+\beta|^{\alpha}u_j+a_{00}u$$
$$+(x+\beta)\left(b_2|x+\beta|^{2\alpha}-\frac{\alpha+1}{(\alpha+2)^2}a_{00}^2\right)$$
$$\text{with}\quad (\alpha+2)a_r\neq 0,$$

$$\hat{\mathcal{F}}_{\mathrm{I},00}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_j(x+\beta)^{j-2}u_j+b_0(x+\beta)$$
$$+b_2(x+\beta)^{-5}\quad\text{with}\quad a_r\neq 0,$$

$$\hat{\mathcal{F}}_{\mathrm{II},0}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_j(x+\beta)^j u_j+a_{00}u+b_0\ \text{with}\ \ a_r\neq 0,$$

$$\hat{\mathcal{F}}_{\mathrm{II},1}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_j(x+\beta)^j u_j+(a_{01}\ln|x+\beta|+a_{00})u$$
$$+(x+\beta)\left(-\frac{a_{01}^2}{4}\ln^2|x+\beta|+\left(\frac{a_{01}^2}{4}-\frac{a_{00}a_{01}}{2}\right)\ln|x+\beta|+b_0\right)$$
$$\text{with}\quad a_ra_{01}\neq 0,$$

$$\mathcal{F}_{\mathrm{III}}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_je^{\alpha x}u_j+(a_{01}e^{\alpha x}+a_{00})u+b_2e^{2\alpha x}$$
$$-\frac{a_{00}a_{01}}{\alpha}e^{\alpha x}-\frac{a_{00}^2+a_{00}}{2\alpha}\ \text{with}\ \ \alpha a_r\neq 0,$$

$$\mathcal{F}_{\mathrm{IV},1}\colon\quad u_t+uu_x=\sum_{j=2}^{r}a_ju_j+a_0u+b_1x+b_0$$
$$\text{with}\quad \alpha a_r\sum_{j=2}^{r-1}|a_j|\neq 0,$$

$$\mathcal{F}_{\mathrm{IV},0}^{r>2}\colon\quad u_t+uu_x=a_ru_r+a_0u+\frac{r-1}{(r-2)^2}a_0^2x+b_0$$
$$\text{with}\quad \alpha a_r\neq 0,\ r>2,$$

$$\mathcal{F}_{\mathrm{IV},0}^{r=2}\colon\quad u_t+uu_x=a_2u_2+b_1x+b_0\quad\text{with}\quad \alpha a_r\neq 0.$$

All these subclasses but $\hat{\mathcal{F}}_{\mathrm{II},0}$ are normalized in the generalized sense. The class $\hat{\mathcal{F}}_{\mathrm{II},0}$ is normalized in the usual sense.

The main result of the paper is described in the following theorem.

Theorem 1. *The usual equivalence group of the class $\mathcal{F}$ of reduced general Burgers–Korteweg–de Vries equations with space-dependent coefficients is four-dimensional. The list of maximal nontrivial conditional equivalence subgroups is exhausted by the generalized equivalence groups of the normalized subclasses $\hat{\mathcal{F}}_{\rm I,1}$, $\hat{\mathcal{F}}_{\rm I,01}$, $\hat{\mathcal{F}}_{\rm I,00}$, $\hat{\mathcal{F}}_{\rm II,1}$, $\mathcal{F}_{\rm III}$, $\mathcal{F}_{\rm IV,1}$, $\mathcal{F}^{r>2}_{\rm IV,0}$, $\mathcal{F}^{r=2}_{\rm IV,0}$ and the usual equivalence group of the normalized subclass $\hat{\mathcal{F}}_{\rm II,0}$. The equivalence groupoid of the class $\mathcal{F}$ is generated by its usual equivalence group and the equivalence groups of the above subclasses.*

For all classes normalized in the generalized sense, we can take their effective generalized equivalence subgroups as maximal conditional equivalence groups. Denote by $\mathcal{F}_0$ the complement to the union of the above subclasses in the class $\mathcal{F}$. It is a normalized class in the usual sense, and its equivalence group coincides with that of $\mathcal{F}$.

Corollary 2. *The class $\mathcal{F}$ is a union of the normalized* (*in either the generalized or the usual sense*) *classes $\hat{\mathcal{F}}_{\rm I,1}$, $\hat{\mathcal{F}}_{\rm I,01}$, $\hat{\mathcal{F}}_{\rm I,00}$, $\hat{\mathcal{F}}_{\rm II,1}$, $\hat{\mathcal{F}}_{\rm II,0}$, $\mathcal{F}_{\rm III}$, $\mathcal{F}_{\rm IV,1}$, $\mathcal{F}^{r>2}_{\rm IV,0}$, $\mathcal{F}^{r=2}_{\rm IV,0}$ and $\mathcal{F}_0$.*

The structure of this paper is as follows. Firstly, we remind in Section 2 theoretical foundations related to equivalence within classes of differential equations. Following [6] in Section 3 we recall the structure of the equivalence groupoids of the superclass of general Burgers–Korteweg–de Vries equations, its subclass of equations with time-independent coefficients and gauging of these classes to the corresponding subclasses of reduced equations. In Section 4 we give the complete classification of admissible transformations of the class $\mathcal{F}$ of reduced general Burgers–KdV equations with space-dependent coefficients. In [6] there were found subclasses of the class $\mathcal{F}$ possessing admissible transformations that are not generated by the equivalence transformations of $\mathcal{F}$. But the question of a structure of equivalence groupoids of these subgroups was not addressed there. Here we fill this gap by comprehensive description of all these subclasses and their equivalence groups (for subclasses normalized in the generalized sense we present either the entire generalized equivalence group, or its effective generalized equivalence group or both of them). By partitioning if necessary these subclasses we achieve a normalization of "subsubclasses" in either usual or generalized sense. Thus we present the superclass $\mathcal{F}$ as a union of normalized classes of differential equations described in Theorem 1. For the two normalized subclasses to be able to have a closed form of group transformations we apply a non-standard approach, the technical crux of which is as follows. First we gauge the class under consideration by a family of equivalence transformations thereof to a nice normalized subclass. Then every equivalence transformation in the class under consideration would be a composition of the gauging mapping, an equivalence transformation within the nice subclass and the inverse of a (not the same as before because we consider not symmetry but equivalence transformations of the superclass) gauging mapping. This procedure may explain an appearance of generalized equivalence groups for most of the considered subclasses. In fact, the determining systems of ODEs are exactly solvable for all but the two equivalence groups and this procedure is only lurking in the background, but we could use it almost everywhere. In this case, even if a nice underlying subclass is normalized in the usual sense, we compose its equivalence transformations with transformations from the families parameterized by arbitrary elements of the superclass, and thus parameterize the equivalence transformations thereof by arbitrary elements, making them generalized.

2. Equivalence of classes of differential equations. We recall the essential notions for the present paper only. See [6, 8, 9] for more details. Let $\mathcal{L}_\theta$ denote a system of differential equations of the form

$$L\big(x, u^{(r)}, \theta\big(x, u^{(r)}\big)\big) = 0,$$

where $x = (x_1, \dots, x_n)$ is the n independent variables, $u = (u^1, \dots, u^m)$ is the m dependent variables, and L is a tuple of differential functions in u. We use the standard short-hand notation $u^{(r)}$ to denote the tuple of derivatives of u with respect to x up to order r, which also includes u as the derivatives of order zero. The system $\mathcal{L}_\theta$ is parameterized by the tuple of functions $\theta = (\theta^1(x, u^{(r)}), \dots, \theta^k(x, u^{(r)}))$, called the arbitrary elements running through the solution set $\mathcal{S}$ of an auxiliary system of differential relations in θ. Thus, the *class of* (*systems of*) *differential equations* $\mathcal{L}|_\mathcal{S}$ is the parameterized family of systems $\mathcal{L}_\theta$, such that θ lies in $\mathcal{S}$.

Equivalence of classes of differential equations is based on studying how equations from a given class are mapped to each other. The notion of *admissible transformations*, which constitute the *equivalence groupoid* of the class $\mathcal{L}|_\mathcal{S}$, formalizes this study. An admissible transformation is a triple $(\theta, \tilde\theta, \varphi)$, where $\theta, \tilde\theta \in \mathcal{S}$ are arbitrary-element tuples associated with equations $\mathcal{L}_\theta$ and $\mathcal{L}_{\tilde\theta}$ from the class $\mathcal{L}_\mathcal{S}$ that are similar, and φ is a point transformation in the space of (x,u) that maps $\mathcal{L}_\theta$ to $\mathcal{L}_{\tilde\theta}$.

A related notion of relevance in the group classification of differential equations is that of *equivalence transformations*. Usual equivalence transformations are point transformations in the joint space of independent variables, derivatives of u up to order r and arbitrary elements that are projectable to the space of $(x,u^{(r')})$ for each $r'=0,\dots,r$, with respect the contact structure of the rth order jet space coordinatized by the r-jets $(x,u^{(r)})$ and map every system from the class $\mathcal{L}|_\mathcal{S}$ to a system from the same class. The Lie (pseudo)group constituted by the equivalence transformations of $\mathcal{L}|_\mathcal{S}$ is called the *usual equivalence group* of this class and denoted by $G^\sim$.

Each equivalence transformation $\mathcal{T}\in G^\sim$ generates a family of admissible transformations parameterized by θ,

$$G^\sim\ni\mathcal{T}\to\big\{(\theta,\mathcal{T}\theta,\pi_*\mathcal{T})\,|\,\theta\in\mathcal{S}\big\}\subset\mathcal{G}^\sim,$$

and therefore the usual equivalence group $G^\sim$ gives rise to a subgroupoid of the equivalence groupoid $\mathcal{G}^\sim$. The function π is the projection of the space of $(x,u^{(r)},\theta)$ to the space of equation variables only, $\pi(x,u^{(r)},\theta)=(x,u)$. The pushforward $\pi_*\mathcal{T}$ of $\mathcal{T}$ by π is then just the restriction of $\mathcal{T}$ to the space of (x,u).

The projectability property for equivalence transformations can be neglected. Then these equivalence transformations constitute a Lie pseudogroup $\bar G^\sim$ called the *generalized equivalence group* of the class. See the first discussion of this notion in [3, 4] and the further development in [8, 9]. When the generalized equivalence group coincides with the usual one the situation is considered to be trivial. Similarly to usual equivalence transformations, each element of $\bar G^\sim$ generates a family of admissible transformations parameterized by θ,

$$\bar G^\sim\ni\mathcal{T}\to\big\{(\theta',\mathcal{T}\theta',\pi_*(\mathcal{T}|_{\theta=\theta'(x,u)}))\,|\,\theta'\in\mathcal{S}\big\}\subset\mathcal{G}^\sim,$$

and thus the generalized equivalence group $\bar G^\sim$ also generates a subgroupoid $\bar{\mathcal{H}}$ of the equivalence groupoid $\mathcal{G}^\sim$.

Definition 3. Any minimal subgroup of $\bar G^\sim$ that generates the same subgroupoid of $\mathcal{G}^\sim$ as the entire group $\bar G^\sim$ does is called an *effective generalized equivalence group* of the class $\mathcal{L}|_\mathcal{S}$.

If the entire group $\bar G^\sim$ is effective itself, then its uniqueness is evident. At the same time, there exist classes of differential equations, where effective generalized equivalence groups are proper subgroups of the corresponding generalized equivalence groups that are even not normal. Hence each of these effective generalized equivalence groups is not unique since it differs from some of subgroups non-identically similar to it, and all of these subgroups are also effective generalized equivalence groups of the same class.

The class of differential equations $\mathcal{L}|_{\mathcal{S}}$ is *normalized* in the usual (resp. generalized) sense if the subgroupoid induced by its usual (resp. generalized) equivalence group coincides with the entire equivalence groupoid $\mathcal{G}^{\sim}$ of $\mathcal{L}|_{\mathcal{S}}$. The normalization of $\mathcal{L}|_{\mathcal{S}}$ in the usual sense is equivalent to the following conditions. The transformational part φ of each admissible transformation $(\theta',\theta'',\varphi)\in\mathcal{G}^{\sim}$ does not depend on the fixed initial value θ' of the arbitrary-element tuple θ and, therefore, is appropriate for any initial value of θ.

The normalization properties of the class $\mathcal{L}|_{\mathcal{S}}$ are usually established via computing its equivalence groupoid $\mathcal{G}^{\sim}$, which is realized using the direct method. Here one fixes two arbitrary systems from the class, $\mathcal{L}_\theta\colon L(x,u^{(r)},\theta(x,u^{(r)}))=0$ and $\mathcal{L}_{\tilde\theta}\colon L(\tilde x,\tilde u^{(r)},\tilde\theta(\tilde x,\tilde u^{(r)}))=0$, and aims to find the (nondegenerate) point transformations, φ: $\tilde x_i=X^i(x,u)$, $\tilde u^a=U^a(x,u)$, $i=1,\dots,n$, $a=1,\dots,m$, connecting them. For this, one changes the variables in the system $\mathcal{L}_{\tilde\theta}$ by expressing the derivatives $\tilde u^{(r)}$ in terms of $u^{(r)}$ and derivatives of the functions X^i and U^a as well as by substituting X^i and U^a for $\tilde x_i$ and $\tilde u^a$, respectively. The requirement that the resulting transformed system has to be satisfied identically for solutions of $\mathcal{L}_\theta$ leads to the system of determining equations for the components of the transformation φ.

Imposing additional constraints on arbitrary elements of the class, we may single out its subclass whose equivalence group is not contained in the equivalence group of the entire class. Let $\mathcal{L}|_{\mathcal{S}'}$ be the subclass of the class $\mathcal{L}|_{\mathcal{S}}$, which is constrained by the additional system of equations $\mathcal{S}'(x,u^{(r)},\theta^{(q')})=0$ and inequalities $\Sigma'(x,u^{(r)},\theta^{(q')})\neq0$ with respect to the arbitrary elements $\theta=\theta(x,u^{(r)})$. Here $\mathcal{S}'\subset\mathcal{S}$ is the set of solutions of the united system $\mathcal{S}=0$, $\Sigma\neq0$, $\mathcal{S}'=0$, $\Sigma'\neq0$. We assume that the united system is compatible for the subclass $\mathcal{L}|_{\mathcal{S}'}$ to be nonempty.

Definition 4. The equivalence group $G^{\sim}(\mathcal{L}|_{\mathcal{S}'})$ of the subclass $\mathcal{L}|_{\mathcal{S}'}$ is called *a conditional equivalence group* of the entire class $\mathcal{L}|_{\mathcal{S}}$ under the conditions $\mathcal{S}'=0$, $\Sigma'\neq0$. The conditional equivalence group is called *nontrivial* if it is not a subgroup of $G^{\sim}(\mathcal{L}|_{\mathcal{S}})$.

Conditional equivalence groups may be trivial not with respect to the equivalence group of the entire class but with respect to other conditional equivalence groups. Indeed, if $\mathcal{S}' \subset \mathcal{S}''$ and $G^{\sim}(\mathcal{L}|_{\mathcal{S}'}) \subset G^{\sim}(\mathcal{L}|_{\mathcal{S}''})$ then the subclass $\mathcal{L}|_{\mathcal{S}'}$ is not interesting from the conditional symmetry point of view. Therefore, the set of additional conditions on the arbitrary elements can be reduced substantially.

Definition 5. The conditional equivalence group $G^{\sim}_{\mathcal{L}|_{\mathcal{S}'}}$ of the class $\mathcal{L}|_{\mathcal{S}}$ under the additional conditions $\mathcal{S}' = 0$, $\Sigma' \neq 0$ is called maximal if for any subclass $\mathcal{L}|_{\mathcal{S}''}$ of the class $\mathcal{L}|_{\mathcal{S}}$ containing the subclass $\mathcal{L}|_{\mathcal{S}'}$ we have $G^{\sim}_{\mathcal{L}|_{\mathcal{S}'}} \not\subset G^{\sim}_{\mathcal{L}|_{\mathcal{S}''}}$.

3. Preliminary analysis of equivalence groupoid. We start studying admissible transformations of the class $\mathcal{F}$ by presenting the equivalence groupoid of its superclass (1) and then descend therefrom to the class under study.

Proposition 6. *The class* (1) *is normalized in the usual sense. Its usual equivalence group* $G^{\sim}_{(1)}$ *consists of the transformations in the joint space of* (t, x, u, θ) *whose* (t, x, u)*-components are of the form*

$$\tilde{t} = T(t), \quad \tilde{x} = X(t,x), \quad \tilde{u} = U^1(t)u + U^0(t,x),$$

where $T = T(t)$, $X = X(t,x)$, $U^1 = U^1(t)$ *and* $U^0 = U^0(t,x)$ *are arbitrary smooth functions of their arguments such that* $T_t X_x U^1 \neq 0$.

Following [6] we can gauge the arbitrary elements $C = 1$ and $A^1 = 0$ by a family of equivalence transformations of the class (1) and obtain the class of reduced general Burgers–KdV equations

$$u_t + uu_x = \sum_{j=2}^{r} A^j(t,x)u_j + A^0(t,x)u + B(t,x). \tag{3}$$

As before, the arbitrary elements run through the set of smooth functions of (t,x) with $A^r C \neq 0$.

Theorem 7. *The class of reduced* (1+1)*-dimensional general* r*th order Burgers–KdV equations* (3) *is normalized in the usual sense. Its usual equivalence group* $G^{\sim}$ *consists of the transformations of the form*

$$\tilde{t} = T(t), \quad \tilde{x} = X^1(t)x + X^0(t), \quad \tilde{u} = \frac{X^1}{T_t}u + \frac{X^1_t}{T_t}x + \frac{X^0_t}{T_t}, \tag{4}$$

$$\tilde{A}^j = \frac{(X^1)^j}{T_t} A^j, \quad \tilde{A}^0 = \frac{1}{T_t}\left(A^0 + 2\frac{X^1_t}{X^1} - \frac{T_{tt}}{T_t}\right), \tag{5}$$

$$\tilde{B} = \frac{X^1}{(T_t)^2} B + \frac{1}{T_t}\left(\frac{X^1_t}{T_t}\right)_t x + \frac{1}{T_t}\left(\frac{X^0_t}{T_t}\right)_t - \left(\frac{X^1_t}{T_t} x + \frac{X^0_t}{T_t}\right)\tilde{A}^0, \tag{6}$$

where $j = 2, \dots, r$, and $T = T(t)$, $X^1 = X^1(t)$ and $X^0 = X^0(t)$ are arbitrary smooth functions of their arguments with $T_t X^1 \neq 0$.

The subclass $\bar{\mathcal{F}}$ of general Burgers–KdV equations with space-dependent coefficients is singled out from the class (1) by the constraints $A^k_t = 0$, $k = 0, \dots, r$, $B_t = 0$ and $C_t = 0$. Therefore, its usual equivalence group $G^\sim_{\bar{\mathcal{F}}}$ is a subgroup of $G^\sim_{(1)}$ that consists of transformations preserving the above constraints.

Proposition 8. *The usual equivalence group $G^\sim_{\bar{\mathcal{F}}}$ of the class $\bar{\mathcal{F}}$ of general Burgers–Korteweg–de Vries equations with space-dependent coefficients consists of the transformations in the joint space of (t, x, u, θ) whose (t, x, u)-components are of the form*

$$\tilde{t} = c_1 t + c_2, \quad \tilde{x} = X(x), \quad \tilde{u} = c_3' u + U^0(x),$$

where c_1, c_2 and c_3' are arbitrary constants and $X = X(x)$ and $U^0 = U^0(x)$ are arbitrary smooth functions of x such that $c_1 X_x c_3' \neq 0$.

The existence of classifying conditions [6]

$$\frac{T_t}{(X_x)^r} X_t \tilde{A}^r_{\tilde{x}} + \left(\frac{T_t}{(X_x)^r}\right)_t \tilde{A}^r = 0, \quad \frac{T_t U^1}{X_x} X_t \tilde{C}_{\tilde{x}} + \left(\frac{T_t U^1}{X_x}\right)_t \tilde{C} = 0,$$

for admissible transformations of the class $\bar{\mathcal{F}}$ implies that it is definitely not normalized in any sense. At the same time, we can gauge the arbitrary elements C and A^1 again by means of equivalence transformations of the class $\bar{\mathcal{F}}$ and produce the class $\mathcal{F}$ of reduced general Burgers–KdV equations with space-dependent coefficients,

$$u_t + u u_x = \sum_{j=2}^{r} A^j(x) u_j + A^0(x) u + B(x).$$

Proposition 9. *The usual equivalence group $G^\sim_{\mathcal{F}}$ of the class $\mathcal{F}$ is four-dimensional and consists of transformations of the form*

$$\tilde{t} = c_1 t + c_2, \quad \tilde{x} = c_3 x + c_4, \quad \tilde{u} = \frac{c_3}{c_1} u,$$

$$\tilde{A}^j = \frac{(c_3)^j}{c_1} A^j, \quad \tilde{A}^0 = \frac{1}{c_1} A^0, \quad \tilde{B} = \frac{c_3}{(c_1)^2} B,$$

where $j = 2, \dots, r$, and c's are arbitrary constants with $c_1 c_3 \neq 0$.

Nor the class $\mathcal{F}$ neither its superclass $\bar{\mathcal{F}}$ are normalized in any sense. Thus, the problem of describing the equivalence groupoid $\mathcal{G}^{\sim}_{\mathcal{F}}$ of the class $\mathcal{F}$ should be considered as the classification of admissible transformations up to $G^{\sim}_{\mathcal{F}}$-equivalence, see [9, Sections 2.6 and 3.4]. The class $\mathcal{F}$ is a subclass of the class (3), whence $\mathcal{G}^{\sim}_{\mathcal{F}}$ is a subgroupoid of the equivalence groupoid of the class (3), and the results of Theorem 7 are valid here, although they should be further specified. This is achieved by differentiating the relations (5)–(6), solved with respect to the source arbitrary elements, with respect to t. This gives the classifying conditions for admissible transformations,

$$\big(X^1_t x + X^0_t\big)\tilde{A}^j_{\tilde{x}} + \left(\frac{T_{tt}}{T_t} - j\frac{X^1_t}{X^1}\right)\tilde{A}^j = 0, \tag{7}$$

$$\big(X^1_t x + X^0_t\big)\tilde{A}^0_{\tilde{x}} + \frac{T_{tt}}{T_t}\tilde{A}^0 = \frac{1}{T_t}\left(2\frac{X^1_t}{X^1} - \frac{T_{tt}}{T_t}\right)_t, \tag{8}$$

$$\begin{aligned}\big(X^1_t x + X^0_t\big)\tilde{B}_{\tilde{x}} + \left(2\frac{T_{tt}}{T_t} - \frac{X^1_t}{X^1}\right)\tilde{B} = -\frac{T_t}{X^1}\big(X^1_t x + X^0_t\big)^2\tilde{A}^0_{\tilde{x}} \\ - \frac{X^1}{T_t^2}\left(T_t\frac{X^1_t x + X^0_t}{X^1}\right)_t\tilde{A}^0 + \frac{X^1}{T_t^2}\left(\frac{T_t}{X^1}\left(\frac{X^1_t x + X^0_t}{T_t}\right)_t\right)_t,\end{aligned} \tag{9}$$

where the initial space variable x should be substituted, after expanding all derivatives, by its expression via $\tilde{x}$, $x = (\tilde{x} - X^0)/X^1$. Note that admissible transformations with $T_{tt} = X^0_t = X^1_t = 0$ are generated by the usual equivalence group $G^{\sim}_{\mathcal{F}}$.

4. Nontrivial conditional equivalence subgroups. In [6] with a help of the method of furcate splitting, cf. [5, 7] the classifying conditions (7)–(9) for admissible transformations of the class $\mathcal{F}$ were solved, but the obtained admissible transformations were presented superficially. More precisely, they were parameterized by solutions of some ODEs. Here we study the question in more depth and present explicit forms of group parameters of the nontrivial conditional equivalence groups. Besides, following [6] for simplicity we consider only subclasses of the classes $\mathcal{F}_{\rm I}$ and $\mathcal{F}_{\rm II}$, defined below, admitting proper subgroups of maximal conditional equivalence groups. In fact, these subgroups are the quotients thereof by the space-translations. Note that given in Theorem 1 are the subclasses admitting maximal nontrivial conditional equivalence subgroups.

I. The class $\mathcal{F}_{\rm I}$ of equations

$$u_t+uu_x=\sum_{j=2}^{r}a_jx^j|x|^\alpha u_j+\big(a_{00}+a_{01}|x|^\alpha\big)u \\ +x\big(b_0+b_1|x|^\alpha+b_2|x|^{2\alpha}\big)$$

with $\alpha a_r\neq0$ naturally partitions into two $\mathcal{G}^\sim_{\mathcal{F}_{\rm I}}$-invariant subclasses $\mathcal{F}_{\rm I,0}$ and $\mathcal{F}_{\rm I,1}$ singled out by the conditions $a_{01}=0$ and $a_{01}\neq0$, respectively, since the arbitrary element a_{01} is easily shown to be transformed by the rule $\tilde a_{01}=c_4a_{01}$ under admissible transformations of the class, $c_4\neq0$. The class $\mathcal{F}_{\rm I,1}$ admits additional admissible transformations if and only if $a_{00}=(\alpha+2)b_1/a_{01}$ and $b_0=-b_1^2(1+\alpha)/a_{01}^2$, so we reduce the arbitrary-elements tuple of the class by a_{00} and b_0 and denote the subclass obtained again by $\mathcal{F}_{\rm I,1}$.

Proposition 10. *The class $\mathcal{F}_{\rm I,1}$ is normalized in the generalized sense. Its generalized equivalence group consists of the point transformations in the relevant space, which are of the form*

$$\tilde t=\bar T,\quad \tilde x=\bar X^1x,\quad \tilde u=\frac{\bar X^1}{\bar T_t}u-\frac{\bar X^1_t}{\bar T_t}x,$$
$$\tilde\alpha=\alpha,\quad \tilde a_j=\bar c_4a_j,\quad \tilde a_{01}=\bar c_4a_{01},\quad \tilde b_2=\bar c_4^2b_2,\quad \tilde b_1=\bar c_5,$$

where $\bar T$ is a smooth function of t and the arbitrary elements θ,

$$\bar T(t,\theta)=\frac{1}{\bar c_5}\ln\left|\bar c_5\left(c_1\frac{e^{-b_1\alpha t/a_{01}}-1}{-b_1\alpha/a_{01}}+c_2\right)+1\right|,$$
$$\theta=(\alpha,a_j,a_{01},b_2,b_1),$$

taking the form at the singular points

$$\bar T(t,\theta)=\bar c_1\frac{e^{-b_1\alpha t/a_{01}}-1}{-b_1\alpha/a_{01}}+\bar c_2\quad \text{if}\quad \bar c_5=0\ \text{and}\ b_1\neq0,$$
$$\bar T(t,\theta)=\frac{1}{\bar c_5}\ln\big|\bar c_5(\bar c_1t+\bar c_2)\big|\quad \text{if}\quad \bar c_5\neq0\ \text{and}\ b_1=0,$$
$$\bar T(t,\theta)=\bar c_1t+\bar c_2\quad \text{if}\quad (\bar c_5,b_1)=(0,0),$$

$\bar{c}$'s are arbitrary functions of θ with $\bar{c}_1\bar{c}_4\frac{\partial(\tilde{a}_2,\dots,\tilde{a}_r,\tilde{a}_{01},\tilde{b}_1,\tilde{b}_2)}{\partial(a_2,\dots,a_r,a_{01},b_1,b_2)} \neq 0$ as well as $\bar{X}^1(t,\theta) = (\bar{c}_4\bar{T}_t)^{-1/\alpha}$ if α is odd or rational in the reduced form with an odd numerator and $\bar{X}^1(t) = \varepsilon|\bar{c}_4\bar{T}_t|^{-1/\alpha}$ with $\varepsilon = \pm 1$ and $\bar{c}_4\bar{T}_t > 0$ otherwise.

Remark 11. The function T is a solution of an ODE smoothly depending on parameters, so it is a smooth function of these parameters and initial conditions [1, Corollary 6, p. 97] (α, b_1 and a_{01} are the parameters of the equation in this case, c's are the initial conditions). This argumentation is valid for the group parameters in the equivalence groups below, where appropriate, as well. In fact, in these cases it follows from the transformation for $\tilde{A}^0$ (the equation (5)) that the function T satisfies the equation

$$\gamma = \delta\frac{1}{T_t} + \left(\frac{1}{T_t}\right)_t = 0$$

for some constants γ and δ, having the general solution

$$T(t) = \frac{1}{\gamma}\ln\left|\gamma\left(c_1\frac{e^{\delta t}-1}{\delta} + c_2\right) + 1\right|.$$

The continuity of this function is evident and at the singular points the function takes the form

$$\begin{aligned} T(t) &= c_1\frac{e^{\delta t}-1}{\delta} + c_2 && \text{if} \quad \gamma = 0 \text{ and } \delta \neq 0,\\ T(t) &= \frac{1}{\gamma}\ln|\gamma(c_1 t + c_2) + 1| && \text{if} \quad \gamma \neq 0 \text{ and } \delta = 0,\\ T(t) &= c_1 t + c_2 && \text{if} \quad (\gamma,\delta) = (0,0). \end{aligned}$$

The transformations in Proposition 10 indeed form a group, which is straightforward to show. Therefore the equivalence group of the class $\mathcal{F}_{\mathrm{I},1}$ is a local Lie group of transformations (all equivalence group here and below in the paper are finite-dimensional so we do not need to talk about Lie pseudogroups). If the function T is of the form $\frac{1}{\gamma}\ln|\gamma(c_1 t + c_2) + 1|$ and $\gamma c_2 = -1$, then $T(t)$ degenerates into an affine function. To avoid this, in all such situations thereafter we implicitly assume otherwise. The notation $\frac{\partial(\cdot,\dots,\cdot)}{\partial(\cdot,\dots,\cdot)}$ stands for the determinant of the corresponding Jacobian matrix. Thereafter, we will not call attention to these facts.

Since the arbitrary element α is invariant under admissible transformations, it is convenient to consider the two subclasses $\mathcal{F}_{\mathrm{I},00}$ and $\mathcal{F}_{\mathrm{I},01}$ of $\mathcal{F}_{\mathrm{I},0}$ singled out by conditions $\alpha=-2$ and $\alpha\neq-2$, respectively. To achieve an extension of a number of admissible transformations in the latter class we need to consider its subclass (denoted again $\mathcal{F}_{\mathrm{I},01}$) singled out by the conditions $b_1=0$ and $b_0=-(\alpha+1)a_{00}^2/(\alpha+2)^2$.

Proposition 12. *The class $\mathcal{F}_{\mathrm{I},01}$ is normalized in the generalized sense. Its generalized equivalence group consists of the point transformations of the form*

$$\tilde t=\bar T(t),\quad \tilde x=\bar X^1(t)x,\quad \tilde u=\frac{\bar X^1}{\bar T_t}u-\frac{\bar X^1_t}{\bar T_t}x,$$
$$\tilde\alpha=\alpha,\quad \tilde a_j=\bar c_4a_j,\quad \tilde a_{00}=\bar c_5,\quad \tilde b_2=\bar c_4^2b_2,$$

where $\bar T$ is a smooth function of t and the arbitrary elements θ,

$$\bar T(t,\theta)=\frac{1}{\bar c_5}\ln\left|\bar c_5\left(\bar c_1\frac{e^{a_{00}\alpha t/(\alpha+2)}-1}{a_{00}\alpha/(\alpha+2)}+\bar c_2\right)+1\right|.$$

The function $\bar T$ takes at the singular points the following forms

$$\bar T(t,\theta)=\bar c_1\frac{e^{a_{00}\alpha t/(\alpha+2)}-1}{a_{00}\alpha/(\alpha+2)}+\bar c_2\quad \text{if}\quad \bar c_5=0 \text{ and } a_{00}\neq0,$$
$$\bar T(t,\theta)=\frac{1}{\bar c_5}\ln|\bar c_5(\bar c_1t+\bar c_2)+1|\quad \text{if}\quad \bar c_5\neq0 \text{ and } a_{00}=0,$$
$$\bar T(t,\theta)=\bar c_1t+\bar c_2\quad \text{if}\quad (\bar c_5,a_{00})=(0,0).$$

Here $\bar c$'s are arbitrary functions of θ with $\bar c_1\bar c_4\frac{\partial(\tilde a_2,\dots,\tilde a_r,\tilde a_{00},\tilde b_2)}{\partial(a_2,\dots a_r,a_{00},b_2)}\neq0$ as well as $\bar X^1(t,\theta)=(\bar c_4\bar T_t)^{-1/\alpha}$ if α is odd or rational in the reduced form with an odd numerator and $\bar X^1(t,\theta)=\varepsilon|\bar c_4\bar T_t|^{-1/\alpha}$ with $\varepsilon=\pm1$ and $\bar c_4\bar T_t>0$ otherwise.

A description of the equivalence group of the class $\mathcal{F}_{\mathrm{I},00}$ is more complicated and we present its equivalence groupoid first. In accordance with our standard approach we consider its subclass singled out by the conditions $a_{00}=b_1=0$.

Proposition 13. *A point transformation connects the two equations in the class $\mathcal{F}_{\mathrm{I},00}$ if and only if its components are of the form*

$$\tilde t=T(t),\quad \tilde x=X^1(t)x,\quad \tilde u=\frac{X^1}{T_t}u-\frac{X^1_t}{T_t}x,$$

where $(X^1(t))^2 = c_4 T_t$ *and the smooth function* T *of* t *satisfies the equation*

$$\left(\frac{T_{tt}}{T_t}\right)_t - \frac{1}{2}\left(\frac{T_{tt}}{T_t}\right)^2 = 2\tilde{b}_0 T_t^2 - 2b_0.$$

Here c_4 *is an arbitrary constant and* $c_4 T_t > 0$*.*

The last equation is an autonomous ordinary differential equation on T which can be integrated in quadratures with standard techniques, but proceeding this way one can write an explicit form of the general solution only for specific values of parameters. On the other hand, for any equation in $\mathcal{F}_{\mathrm{I},00}$ there is an equivalent one to it in the subclass $\mathcal{F}_{\mathrm{I},00}^{b_0=0}$ singled out by the condition $b_0 = 0$. The corresponding point transformation is $\tilde{t} = T(t)$, $\tilde{x} = \sqrt{T_t}x$, $\tilde{u} = u/\sqrt{T_t} - T_{tt}x/(2\sqrt{(T_t)^3})$, where a smooth function T of t is a solution of the equation $(T_{tt}/T_t)_t - \frac{1}{2}\left(T_{tt}/T_t\right)^2 + 2b_0 = 0$, for which the general solution can be found explicitly, although a particular solution will suffice for our purposes. Thus, if $b_0 = b^2 > 0$, then $T(t) = e^{2bt}$ is a particular solution; if $b_0 = -b^2 < 0$, then $T(t) = \tan(bt)$ is a particular solution, $b > 0$ in both cases.

Proposition 14. *The class* $\mathcal{F}_{\mathrm{I},00}^{b_0=0}$ *is normalized in the usual sense. Its usual equivalence group is constituted by the point transformations of the form*

$$\tilde{t} = T(t), \quad \tilde{x} = X^1(t)x, \quad \tilde{u} = \frac{X^1}{T_t}u - \frac{X^1_t}{T_t}x,$$
$$\tilde{a}_2 = c_4 a_2, \quad \tilde{b}_2 = c_4^2 b_2,$$

where $X^1(t) = \varepsilon\sqrt{c_4 T_t}$ *with* $\varepsilon = \pm 1$, $T = (c_1 t + c_2)/(c_3 t + c_0)$ *and* c*'s are arbitrary constants, with* $\delta = c_1 c_0 - c_2 c_3 \neq 0$ *and* c_0, c_1, c_2 *and* c_3 *being defined up to a nonzero constant, and* $c_4\delta > 0$*.*

On the other hand, any admissible transformation of the class $\mathcal{F}_{\mathrm{I},00}$ can be represented as a composition of an admissible transformation with a source equation in $\mathcal{F}_{\mathrm{I},00}$ and a target equation in $\mathcal{F}_{\mathrm{I},00}^{b_0=0}$, an admissible transformation generated by an equivalence transformation in $\mathcal{F}_{\mathrm{I},00}^{b_0=0}$ and an admissible transformation back. In this way we avoid implicit quadrature expressions arising in a previous approach. Note that the parameter-function T is defined as a solution of a third-order ODE parameterized by b_0 and $\tilde{b}_0$ and thus should be parameterized by three constants to agree with the Picard–Lindelöf theorem. This is indeed the case.

Proposition 15. *The class $\mathcal{F}_{\mathrm{I},00}$ is normalized in the generalized sense. Its effective generalized equivalence group is constituted by the point transformations of the form*

$$
\begin{aligned}
&\tilde t = P^2(T(P^1(t))), \quad \tilde x = \sqrt{P^2_{\bar t}P^1_t}\,X^1(\hat t)x,\\
&\tilde u = \frac{1}{P^2_{\bar t}}\left(\frac{X^1}{T_{\hat t}P^1_t}u - \left(\frac{X^1P^1_{tt}}{2T_{\hat t}(P^1_t)^{3/2}} + \frac{X^1_{\hat t}\sqrt{P^1_t}}{T_{\hat t}} + \frac{P^2_{\bar t\bar t}X^1\sqrt{P^1_t}}{2P^2_{\bar t}}\right)x\right),\\
&\tilde a_j = c_4a_j, \quad \tilde b_2 = c_4^2b_2,\\
&\tilde b_0 = \frac{1}{(P^2_{\bar t})^2}\left(\frac{1}{(P^1_t)^2}\left(b_0 - \left(\frac{P^1_{tt}}{2P^1_t}\right)^2 + \frac12\left(\frac{P^1_{tt}}{P^1_t}\right)_t\right)\right.\\
&\qquad\left. - \left(\frac{P^2_{\bar t\bar t}}{2P^2_{\bar t}}\right)^2 + \frac12\left(\frac{P^2_{\bar t\bar t}}{P^2_{\bar t}}\right)_{\bar t}\right),
\end{aligned}
$$

where $\hat t = P^1(t)$, $\bar t = T(\hat t)$, $\tilde t = P^2(\bar t)$, $X^1(\hat t) = \varepsilon(c_4T_{\hat t})^{1/2}$, $T = (c_1\hat t + c_2)/(c_3\hat t + c_0)$, with $\delta = c_1c_0 - c_2c_3 \neq 0$, c's are arbitrary constants,

$$
P^1(t) = \begin{cases} t & \text{if} \quad b_0 = 0,\\ \tan(\sqrt{-b_0}\,t) & \text{if} \quad b_0 < 0,\\ e^{2\sqrt{b_0}\,t} & \text{if} \quad b_0 > 0; \end{cases}
$$

$P^2(\bar t)$ runs through the set of smooth functions $\{\bar t, \frac{1}{c_5}\ln|\bar t|, \frac{1}{2c_5}\arctan\bar t\}$, with $c_4\delta > 0$, c_i, $i = 0,1,2,3$, are defined up to a nonzero constant, and $P^2_{\bar t} > 0$ and $\varepsilon = \pm1$.

The arbitrary element $\tilde b_0$ of the target equation takes the value of c_5^2 if $P^2(y) = \frac{1}{c_5}\ln|y|$, of $-c_5^2$ if $P^2(y) = \frac{1}{2c_5}\arctan y$ and of 0 otherwise. The functions $P^2(T(P^1(t)))$ give a three-parameter family of solutions to the nonlinear third-order equation on T above parameterized by b_0 and $\tilde b_0$.

Remark 16. The point transformations in Proposition 15 form a group by construction, and thus constitute an effective generalized equivalence group of the class $\mathcal{F}_{\mathrm{I},00}$. To obtain the entire generalized equivalence group thereof one allows c's to vary through the set of arbitrary smooth functions of the arbitrary elements of the class.

II. A class $\mathcal{F}_{\rm II}$ of differential equations of the form

$$u_t + uu_x = \sum_{j=2}^{r} a_j x^j u_j + (a_{01} \ln|x| + a_{00})u \\ + x\left(-\frac{a_{01}^2}{4}\ln^2|x| + \left(\frac{a_{01}^2}{4} - \frac{a_{00}a_{01}}{2}\right)\ln|x| + b_0\right)$$

is partitioned into two subclasses $\mathcal{F}_{\rm II,0}$ and $\mathcal{F}_{\rm II,1}$ that are singled out by conditions $a_{01} = 0$ and $a_{01} \neq 0$, respectively, and invariant under the admissible transformations of the class $\mathcal{F}_{\rm II}$.

Proposition 17. *The class $\mathcal{F}_{\rm II,0}$ is normalized in the usual sense. Its equivalence group is constituted by the point transformations of the form*

$$\tilde t = c_1 t + c_2, \quad \tilde x = c_4 e^{c_3 t} x, \quad \tilde u = \frac{c_4 e^{c_3 t}}{c_1}(u + c_3 x),$$
$$\tilde a_j = \frac{a_j}{c_1}, \quad \tilde a_{00} = \frac{a_{00} + 2c_3}{c_1}, \quad \tilde b_0 = \frac{b_0 - c_3^2}{(c_1)^2},$$

where c's are arbitrary constants with $c_1c_4 \neq 0$.

The class $\mathcal{F}_{\rm II,0}$ is the only owner of a conditional group normalized in the usual sense.

Proposition 18. *The class $\mathcal{F}_{\rm II,1}$ is normalized in the generalized sense. Its generalized equivalence group $\bar G^{\sim}_{\rm II,1}$ is constituted by the point transformations of the form*

$$\tilde t = \bar c_1 t + \bar c_2, \quad \tilde x = \bar X^1 x, \quad \tilde u = \frac{\bar X^1}{\bar c_1}\left(u + \frac{\bar c_4 a_{01}}{2} e^{a_{01}t/2} x\right),$$
$$\tilde a_j = \frac{a_j}{\bar c_1}, \quad \tilde a_{01} = \frac{a_{01}}{\bar c_1}, \quad \tilde a_{00} = \frac{1}{\bar c_1}(a_{00} - a_{01}\bar c_3),$$
$$\tilde b_0 = \frac{1}{4\bar c_1^2}\left(4b_0 - a_{01}^2(\bar c_3^2 + \bar c_3) + 2a_{00}a_{01}\bar c_3\right),$$

where $\bar X^1 := \exp\left(\bar c_3 + \bar c_4 \exp(\frac{a_{01}t}{2})\right)$, and $\bar c$'s are smooth functions of the arbitrary elements a_{00}, a_{01}, a_j and b_0 with $\bar c_1 \frac{\partial(\tilde a_2,\dots,\tilde a_r,\tilde a_{01},\tilde a_{00},\tilde b_0)}{\partial(a_2,\dots,a_r,a_{01},a_{00},b_0)} \neq 0$.

To extract an effective generalized equivalence group from the generalized equivalence group, we set $\bar c_2 := c_2/a_{01}$, $\bar c_3 := -c_3/a_{01}$ and get rid of the dependence of other $\bar c$'s on the arbitrary elements.

Proposition 19. *An effective generalized equivalence group* $\hat{G}^{\sim}_{\mathrm{II},1}$ *of the class* $\mathcal{F}_{\mathrm{II},1}$ *is constituted by the point transformations of the form*

$$
\begin{aligned}
&\tilde{t} = c_1 t + \frac{c_2}{a_{01}}, \quad \tilde{x} = X^1(t)x, \quad \tilde{u} = \frac{X^1(t)}{c_1}\left(u + \frac{c_4 a_{01}}{2}e^{a_{01}t/2}x\right),\\
&\tilde{a}_j = \frac{a_j}{c_1}, \quad \tilde{a}_{01} = \frac{a_{01}}{c_1}, \quad \tilde{a}_{00} = \frac{1}{c_1}(a_{00}+c_3),\\
&\tilde{b}_0 = \frac{1}{4c_1^2}\left(4b_0 + (a_{01}-2a_{00})c_3 - c_3^2\right),
\end{aligned}
$$

where $X^1(t) := \exp\left(-\frac{c_3}{a_{01}} + c_4\exp(\frac{a_{01}t}{2})\right)$ *and c's are arbitrary constants with* $c_1 \neq 0$.

The effective generalized equivalence group $\hat{G}^{\sim}_{\mathrm{II},1}$ is not a normal subgroup of $\bar{G}^{\sim}_{\mathrm{II},1}$, which is readily seen after writing the time-transformation out. Therefore, it is not unique as an effective generalized equivalence group as conjugate subgroups in $\bar{G}^{\sim}_{\mathrm{II},1}$ are also effective generalized equivalence groups. Thus, the existence of a class of differential equations with unique nontrivial (proper) effective generalized equivalence group is still a question.

III. A class of differential equations of the form

$$
\begin{aligned}
u_t + uu_x = \sum_{j=2}^{r} a_j e^{\alpha x}u_j + (a_{01}e^{\alpha x} + a_{00})u + b_2 e^{2\alpha x}\\
+ b_1 e^{\alpha x} + b_0 \quad \text{with } \alpha a_r \neq 0
\end{aligned}
$$

admits additional admissible transformations if and only if

$$
b_0 = -\frac{a_{00}^2 + a_{00}}{2\alpha} \qquad \text{and} \qquad b_1 = -\frac{a_{00}a_{01}}{\alpha}.
$$

Proposition 20. *The class* $\mathcal{F}_{\mathrm{III}}$ *is normalized in the generalized sense. Its generalized equivalence group is constituted by the point transformations of the form*

$$
\begin{aligned}
&\tilde{t} = \bar{T}, \quad \tilde{x} = \bar{c}_5 x - \frac{\bar{c}_5}{\alpha}\ln|\bar{c}_4\bar{T}_t|, \quad \tilde{u} = \frac{\bar{c}_5}{\bar{T}_t}\left(u - \frac{\bar{T}_{tt}}{\alpha\bar{T}_t}\right),\\
&\tilde{\alpha} = \frac{\alpha}{\bar{c}_5}, \quad \tilde{a}_j = \bar{c}_4\bar{c}_5^j a_j, \quad \tilde{a}_{01} = \bar{c}_4 a_{01}, \quad \tilde{a}_{00} = \bar{c}_3, \quad \tilde{b}_2 = \bar{c}_4^2\bar{c}_5 b_2,
\end{aligned}
$$

where the function T of t and the arbitrary elements θ is defined by

$$\bar{T}(t,\theta)=\frac{1}{\bar{c}_3}\ln\left|\bar{c}_3\left(\bar{c}_1\frac{e^{a_{00}t}-1}{a_{00}}+\bar{c}_2\right)+1\right|,$$

and takes the following values at the singular points

$$\begin{aligned}
&\bar{T}(t,\theta)=\bar{c}_1\frac{e^{a_{00}t}-1}{a_{00}}+\bar{c}_2 && \textit{if}\quad \bar{c}_3=0 \ \textit{and}\ a_{00}\neq 0,\\
&\bar{T}(t,\theta)=\frac{1}{\bar{c}_3}\ln|\bar{c}_3(\bar{c}_1t+\bar{c}_2)| && \textit{if}\quad a_{00}=0 \ \textit{and}\ \bar{c}_3\neq 0,\\
&\bar{T}(t,\theta)=\bar{c}_1t+\bar{c}_2 && \textit{if}\quad (a_{00},\bar{c}_3)=(0,0),
\end{aligned}$$

$\bar{c}$'s are smooth functions of θ with $\bar{c}_1\bar{c}_4\bar{c}_5\frac{\partial(\bar{\alpha},\tilde{a}_2,\dots,\tilde{a}_r,\tilde{a}_{00},\tilde{a}_{01},\tilde{b}_2)}{\partial(\alpha,a_2,\dots,a_r,a_{00},a_{01},b_2)}\neq 0.$

To find an effective generalized equivalence group of the class $\mathcal{F}_{\rm III}$ we resort to the following heuristic speculation. The arbitrary element $\tilde{a}_{00}$ may take any real value. Thus, it sufficient to parameterize $\tilde{a}_{00}$ to be $a_{00}+c_3$, $c_3\in\mathbb{R}$. We preserve the number of initial conditions parameterizing T and guaranteeing the necessary domain for values of $\tilde{a}_{00}$. To satisfy another condition of an effective generalized equivalence group we drop any dependence of remaining $\bar{c}$'s on the arbitrary elements. In fact, we chose a correct parameterization for them already in the theorem.

Proposition 21. *An effective generalized equivalence group $\hat{G}^{\sim}_{\rm III}$ of the class $\mathcal{F}_{\rm III}$ is constituted by the point transformations of the form*

$$\tilde{t}=T,\quad \tilde{x}=c_5x-\frac{c_5}{\alpha}\ln|c_4T_t|,\quad \tilde{u}=\frac{c_5}{T_t}\left(u-\frac{T_{tt}}{\alpha T_t}\right),\quad \tilde{\alpha}=\frac{\alpha}{c_5},$$
$$\tilde{a}_j=c_4c_5^ja_j,\quad \tilde{a}_{01}=c_4a_{01},\quad \tilde{a}_{00}=a_{00}+c_3,\quad \tilde{b}_2=c_4^2c_5b_2,$$

where the function T is equal to

$$T(t)=\frac{1}{a_{00}+c_3}\ln\left|(a_{00}+c_3)\left(c_1\frac{e^{a_{00}t}-1}{a_{00}}+c_2\right)+1\right|,$$

and takes the following values at the singular points

$$\begin{aligned}
&T(t)=c_1\frac{e^{a_{00}t}-1}{a_{00}}+c_2 && \textit{if}\quad c_3=-a_{00}\neq 0,\\
&T(t)=\frac{1}{c_3}\ln|c_3(c_1t+c_2)| && \textit{if}\quad a_{00}=0 \ \textit{and}\ c_3\neq 0,
\end{aligned}$$

$$T(t) = c_1 t + c_2 \qquad\qquad if \quad (a_{00}, c_3) = (0, 0),$$

and c's are arbitrary constants with $c_1 c_4 c_5 \neq 0$.

Guided by the same logic as for the class $\mathcal{F}_{\mathrm{II},1}$, we can show nonuniqueness of effective generalized equivalence groups for $\mathcal{F}_{\mathrm{III}}$ as well.

IV. Finally we discuss the last subclass $\mathcal{F}_{\mathrm{IV}}$ of $\mathcal{F}$ admitting additional admissible transformations. It consists of equations

$$u_t + uu_x = \sum_{j=2}^{r} a_j u_j + a_0 u + b_1 x + b_0.$$

Since the arbitrary elements a_j are scaled under the action of the equivalence group of the class, it is reasonable to single out two subclasses of the class under question: $\mathcal{F}_{\mathrm{IV},0}$ with $a_j = 0$ for all $j = 2, \ldots, r-1$, and complementary to it the subclass $\mathcal{F}_{\mathrm{IV},1}$ with at least one a_j nonzero.

Proposition 22. *The class $\mathcal{F}_{\mathrm{IV},1}$ is normalized in the generalized sense. Its generalized equivalence group is constituted by the point transformations of the form*

$$\begin{aligned}
&\tilde{t} = \bar{T}^1 t + \bar{T}^0, \quad \tilde{x} = \bar{X}^1 x + \bar{X}^0, \quad \tilde{u} = \frac{\bar{X}^1}{\bar{T}^1} u + \frac{\bar{X}^0_t}{\bar{T}^1},\\
&\tilde{a}_j = \frac{(\bar{X}^1)^r}{\bar{T}^1} a_j, \quad a_0 = \frac{a_0}{\bar{T}^1}, \quad \tilde{b}_1 = \frac{b_1}{(\bar{T}^1)^2},\\
&\tilde{b}_0 = \frac{1}{(\bar{T}^1)^2}\left(\bar{X}^1 b_0 + \bar{c}_3\right),
\end{aligned}$$

where

$$\bar{X}^0(t,\theta) = \begin{cases}
\bar{c}_1 e^{\lambda_1 t} + \bar{c}_2 e^{\lambda_2 t} + \bar{c}_3 & if \quad \lambda_1 \neq 0, \quad D > 0,\\
\bar{c}_1 t + \bar{c}_2 e^{\lambda_2 t} + \bar{c}_3 & if \quad \lambda_1 = 0, \quad D > 0,\\
\bar{c}_1 e^{b_1 t/2} + \bar{c}_2 t e^{b_1 t/2} + \bar{c}_3 & if \quad b_1 \neq 0, \quad D = 0,\\
\bar{c}_1 t^2 + \bar{c}_2 t + \bar{c}_3 & if \quad b_1 = 0, \quad D = 0,\\
e^{b_1 t/2}\big(\bar{c}_1 \sin(\sqrt{-D}t) & \\
\quad + \bar{c}_2 \cos(\sqrt{-D}t)\big) + \bar{c}_3 & if \quad D < 0,
\end{cases}$$

where $D = b_1^2 + 4a_0$ and $\lambda_{1,2} = (b_1 \pm \sqrt{D})/2$ with $|\lambda_1| < |\lambda_2|$, $\bar{X}^1$, $\bar{T}^0$, $\bar{T}^1$ and $\bar{c}$'s run through the set smooth functions of the arbitrary elements $\theta = (a_j, a_0, b_1, b_0)$ with $\bar{X}^1 \bar{T}^1 \frac{\partial(\tilde{a}_2, \ldots, \tilde{a}_r, \tilde{a}_0, \tilde{b}_1, \tilde{b}_0)}{\partial(a_2, \ldots, a_r, a_0, b_1, b_0)} \neq 0$.

Here the function $X^0(t)$ is a solution of the ordinary differential equation $X^0_{ttt} - b_1 X^0_{tt} - a_0 X^0_t = 0$ and thus it smoothly depends on the parameters b_1, a_0 and all the initial conditions.

The equivalence groupoid of the class $\mathcal{F}_{\mathrm{IV},0}$ depends essentially on the order r of equations therein. So we consider both the cases separately. First assume that $r > 2$ and denote the class of such equations $\mathcal{F}^{r>2}_{\mathrm{IV},0}$. This class admits additional admissible transformations if and only if $b_1 = a_0^2(r-1)/(r-2)^2$, so we reduce a tuple of the arbitrary elements thereof by the element b_1.

Proposition 23. *The class $\mathcal{F}^{r>2}_{\mathrm{IV},0}$ is normalized in the generalized sense. Its generalized equivalence group is constituted by the point transformations of the form*

$$\tilde t = \bar T, \quad \tilde x = \bar X^1 x + \bar X^0, \quad \tilde u = \frac{\bar X^1}{\bar T_t} u + \frac{\bar X^1_t}{\bar T_t} x + \frac{\bar X^0_t}{\bar T_t},$$
$$\tilde a_r = \frac{(\bar X^1)^r}{\bar T_t} a_r, \quad \tilde a_0 = \bar c_3, \quad \tilde b_0 = \bar c_5,$$

where the pair of smooth functions $\big(\bar T, \bar X^0\big)$ of t and the arbitrary elements θ equal to

$$(\bar c_1 t + \bar c_2,\ \bar c_7 t^2 + \bar c_6 t + \bar c_5) \quad \textit{if} \quad a_0 = 0 \ \textit{and}\ \bar c_3 = 0,$$

$$\left(\frac{1}{\bar c_3}\ln|\bar c_3(\bar c_1 t + \bar c_2)|,\ \frac{\bar c_5 r^2}{\bar c_3^2(r-1)} + \frac{\bar c_6 t + \bar c_7}{|t + \bar c_2/(\bar c_1\bar c_3)|^{1/r}} \right.$$
$$\left. - \frac{\bar c_3^2 b_0 \bar X^1}{2}\left(t + \frac{\bar c_2}{\bar c_1\bar c_3}\right)^2\right) \ \textit{if}\ a_0 = 0 \ \textit{and}\ \bar c_3 \neq 0,$$

$$\left(\frac{r-2}{\bar c_3 r}\ln\left|\frac{1}{\bar c_1}e^{\frac{a_0 r t}{r-2}} + \frac{\bar c_2}{\bar c_1}\right|,\ \frac{\bar c_5(r-2)^2}{\bar c_3^2(r-1)} + \frac{\left(\bar c_6 e^{\frac{a_0 r t}{r-2}} + \bar c_7\right)}{\left|\bar c_2/\bar c_1 + e^{\frac{a_0 r t}{r-2}}\right|^{1/r}}\right.$$
$$\left. + \frac{(r-2)^2 b_0}{(r-1)a_0^2}\bar X^1\right) \quad \textit{if} \quad a_0\bar c_3 \neq 0,$$

$$\left(\bar c_1 e^{\frac{a_0 r t}{r-2}} + \bar c_2,\ \frac{\bar c_5 \bar c_1^2}{2} e^{\frac{2a_0 r t}{r-2}} + \bar c_6 e^{\frac{a_0 r t}{r-2}} + \bar c_7 - \frac{(r-2)^2 b_0}{(r-1)a_0^2}\bar X^1\right)$$
$$\textit{if} \quad a_0 \neq 0 \ \textit{and}\ \bar c_3 = 0,$$

where $\bar c$'s are arbitrary smooth functions of θ with $\bar c_4\bar T_t\frac{\partial(\tilde a_r,\tilde a_0,\tilde b_0)}{\partial(a_r,a_0,b_0)}\neq 0$ as well as $\bar X^1(t,\theta)=\varepsilon(\bar c_4\bar T_t)^{1/r}$ with $\varepsilon=\pm1$ and $\bar c_4\bar T_t>0$ if r is even and $\varepsilon=1$ otherwise.

The function $\bar X^0$ in the second pair in the second set gives a general solution of the linear inhomogeneous equation on $X^0(t)$,

$$\bar c_5=b_0\frac{X^1}{{T_t}^2}+\frac{1}{T_t}\left(\frac{X^0_t}{T_t}\right)_t-\bar c_3\frac{X^0_t}{T_t}-\frac{\bar c_3^2(r-1)}{(r-2)^2}X^0,$$

parameterized by the function T in the set and the corresponding $X^1(t)$. Any particular solution of this equation seems impossible to be found with standard techniques. Here instead, we used a method used for the class $\mathcal F_{\rm I,00}$ with gauging the arbitrary elements a_0 and b_0 to 0 first and composing equivalence transformations thereafter.

Due to the above condition on the arbitrary elements b_1 and a_0, the class $\mathcal F^{r=2}_{\rm IV,0}$ admits additional admissible transformations if and only if $a_0=0$. Abusing notations we denote the subclass singled out by this condition again by $\mathcal F^{r=2}_{\rm IV,0}$.

Proposition 24. *The point transformation of the form*

$$\tilde t=T(t),\quad \tilde x=X^1(t)x+X^0(t),\quad \tilde u=\frac{X^1}{T_t}u+\frac{X^1_t}{T_t}x+\frac{X^0_t}{T_t}$$

connects the source and target equations in the class $\mathcal F^{r=2}_{\rm IV,0}$ if and only if $(X^1)^2/T_t=\mathrm{const}\neq0$, the parameter function T runs through the solution set of the system

$$\left(\frac{T_{tt}}{T_t}\right)_t-\frac12\left(\frac{T_{tt}}{T_t}\right)^2=2\tilde b_1T_t^2-2b_1,$$

and the parameter function X^0 of t satisfies the equation

$$\frac{1}{T_t}\left(\frac{X^0_t}{T_t}\right)_t-\tilde b_1X^0=\tilde b_0-b_0\frac{X^1}{{T_t}^2}.$$

The last equation is linear inhomogeneous with respect to $X^0(T)$ for a given $T(t)$, while the differential equation on T is integrated in quadratures as an autonomous equation on $\ln|T_t|$ with standard techniques. Nonetheless, using the similar trick as was used for the class $\mathcal F_{\rm I,00}$, one can do better. More precisely, we gauge the arbitrary elements b_0 and b_1 to zeros by the point transformation of the form

$$\tilde{t}=T(t),\quad \tilde{x}=\sqrt{T_t}x+X^0(t),\quad \tilde{u}=\frac{u}{\sqrt{T_t}}+\frac{T_{tt}x}{2(T_t)^{3/2}}+\frac{X^0_t}{T_t},$$

where

$$\big(T,X^0\big)=\big(e^{2\sqrt{b_1}t},\ 4b_0(2\sqrt{b_1})^{3/2}e^{\sqrt{b_1}t}\big)\qquad \text{if}\quad b_1>0;$$
$$\big(T,X^0\big)=\left(\tan(\sqrt{-b_1}t),\ \frac{-b_0(-b_1)^{3/4}}{\cos\sqrt{-b_1}t}\right)\quad \text{if}\quad b_1<0,$$

obtaining the subclass $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ of $\mathcal{F}^{r=2}_{\mathrm{IV},0}$. Thereafter we present the equivalence groupoid of $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ by composing an equivalence transformation within the subclass $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ with point transformations mapping equations in the superclass to equations in the subclass and vice versa.

Proposition 25. *The class $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ is normalized in the usual sense. Its usual equivalence group is constituted by point transformations of the form*

$$\tilde{t}=T(t),\quad \tilde{x}=X^1(t)x+X^0,\quad \tilde{u}=\frac{X^1}{T_t}u+\frac{X^1_t}{T_t}x,\quad \tilde{a}_2=c_4a_2,$$

where $X^1(t)=\varepsilon(c_4\delta)^{1/2}/(c_3t+c_0)$, $T=(c_1t+c_2)/(c_3t+c_0)$, X^0 and c's are arbitrary constants with $\delta=c_1c_0-c_2c_3\neq 0$, c_0, c_1, c_2 and c_3 being defined up to a nonzero constant, $c_4\delta>0$ and $\varepsilon=\pm1$.

The point transformation $\mathcal{T}_{\tilde{T},\tilde{X}^0}$ which maps an equation in $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ to an equation in $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ is of the same form as above,

$$t=\tilde{T}(\tilde{t}),\quad x=\sqrt{|\tilde{T}_{\tilde{t}}|}\tilde{x}+\tilde{X}^0(\tilde{t}),\quad u=\frac{\tilde{u}}{\sqrt{|\tilde{T}_{\tilde{t}}|}}+\frac{\tilde{T}_{\tilde{t}\tilde{t}}\tilde{x}}{2|\tilde{T}_{\tilde{t}}|^{3/2}}+\frac{\tilde{X}^0_{\tilde{t}}}{\tilde{T}_{\tilde{t}}},$$

where $\tilde{T}(T(t))=t$ and $\tilde{X}^0(\tilde{t})=-(X^0/T_t)(\tilde{T}(\tilde{t}))$, that is,

$$\big(\tilde{T},\tilde{X}^0\big)=\left(\frac{\ln|\tilde{t}|}{2\sqrt{\tilde{b}_1}},\ \frac{\tilde{b}_0}{\tilde{b}_1}\right)\quad \text{if}\quad \tilde{b}_1>0;$$
$$\big(\tilde{T},\tilde{X}^0\big)=\left(\frac{\arctan(\tilde{t})}{\sqrt{-\tilde{b}_1}},\ \frac{\tilde{b}_0}{\tilde{b}_1}\right)\quad \text{if}\quad \tilde{b}_1<0.$$

Proposition 26. *The class $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ is normalized in the generalized sense. Its effective generalized equivalence group is constituted by point transformations of the form*

$$\tilde t = P^2(T(P^1(t))),$$

$$\tilde x = \sqrt{P^2_{\bar t}P^1_t}\,X^1(\hat t)x + \sqrt{P^2_{\bar t}}\big(X^1R^1 + X^0\big) + R^2,$$

$$\tilde u = \frac{X^1 u}{\sqrt{P^2_{\bar t}P^1_t}\,T_{\hat t}} + \left(\frac{X^1P^1_{tt}}{2T_{\hat t}\sqrt{P^2_{\bar t}(P^1_t)^3}} + \frac{X^1_{\hat t}\sqrt{P^1_t}}{\sqrt{P^2_{\bar t}}\,T_{\hat t}} + \frac{P^2_{\bar t\bar t}X^1}{2(P^2_{\bar t})^2\sqrt{P^1_t}}\right)x$$

$$+\frac{X^1R^1_t}{T_{\hat t}\sqrt{P^2_{\bar t}}P^1_t} + \frac{X^1_{\hat t}R^1}{T_{\hat t}} + \frac{P^2_{\bar t\bar t}}{2(P^2_{\bar t})^{3/2}}\big(X^1R^1 + X^0\big) + \frac{R^2_{\bar t}}{P^2_{\bar t}},$$

$$\tilde a_2 = c_4a_2,\quad \tilde b_1 = \frac{1}{(P^2_{\bar t})^2}\left(\frac{1}{(P^1_t)^2}\left(b_1 + \frac12\left(\frac{P^1_{tt}}{P^1_t}\right)_t - \frac14\left(\frac{P^1_{tt}}{P^1_t}\right)^2\right)\right.$$
$$\left. + \frac12\left(\frac{P^2_{\bar t\bar t}}{P^2_{\bar t}}\right)_{\bar t} - \frac14\left(\frac{P^2_{\bar t\bar t}}{P^2_{\bar t}}\right)^2\right),$$

$$\tilde b_0 = \frac{1}{(P^2_{\bar t})^{3/2}}\left(\frac{b_0}{(P^1_t)^{3/2}} + \frac{1}{P^1_t}\left(\frac{R^1_t}{P^1_t}\right)_t\right) + \frac{1}{P^2_{\bar t}}\left(\frac{R^2_{\bar t}}{P^2_{\bar t}}\right)_{\bar t} - \tilde b_1R^2,$$

where $\hat t = P^1(t)$, $\bar t = T(\hat t)$, $\tilde t = P^2(\bar t)$, $X^1(\hat t) = \varepsilon(c_4T_{\hat t})^{1/2}$, $T = (c_1\hat t + c_2)/(c_3\hat t + c_0)$ with $\delta = c_1c_0 - c_2c_3 \neq 0$;

$$(P^1(t), R^1(t)) = \begin{cases} (t,\ -b_0t^2/2) & \text{if } b_1 = 0,\\ (\tan(\sqrt{-b_1}t),\ -b_0(-b_1)^{3/4}/\cos(\sqrt{-b_1}t)) & \text{if } b_1 < 0,\\ (e^{2\sqrt{b_1}t},\ 4b_0(2\sqrt{b_1})^{-3/2}e^{\sqrt{b_1}t}) & \text{if } b_1 > 0;\end{cases}$$

X^0 and c's are arbitrary constants and the pair of smooth functions $(P^2(\bar t), R^2(\bar t))$ runs through the set

$$\left\{\left(\bar t,\ \frac{c_6\bar t^2}{2}\right),\ \left(\frac{\ln|\bar t|}{2c_5},\ \frac{c_6}{c_5^2}\right),\ \left(\frac{\arctan\bar t}{c_5},\ -\frac{c_6}{c_5^2}\right)\right\},$$

with c_i, $i = 0,\dots,3$, being defined up to a nonzero constant, $c_4\delta > 0$, $P^2_{\bar t} > 0$ and $\varepsilon = \pm1$.

In the notation of Proposition 26, the point transformation $\mathcal{T}_{P^2,R^2}$ maps an equation in $\mathcal{F}^{r=2}_{\mathrm{IV},00}$ to an equation in $\mathcal{F}^{r=2}_{\mathrm{IV},0}$ with arbitrary-element tuples (b_0,b_1) equal to $(c_6,0)$, (c_6,c_5^2) and $(c_6,-c_5^2)$, respectively.

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