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Differential invariants for a class of diffusion equations

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Знайдено повну групу еквівалентності класу (1+1)-вимірних еволюційних рівнянь другого порядку, яка виявилася нескінченновимірною. Методологію екваріантних рухомих реперів застосовано у регулярному випадку процедури нормалізації до побудови рухомого репера групи, пов'язаної з групою еквівалентності в контексті перетворень еквівалентності між рівняннями класу. За допомогою побудованого рухомого репера описано алгебру диференціальних інваріантів цієї групи через отримання мінімальної генеруючої множини диференціальних інваріантів і повної множини операторів інваріантного диференціювання.

We find the complete equivalence group of a class of (1+1)-dimensional second-order evolution equations, which is infinite-dimensional. The equivariant moving frame methodology is invoked to construct, in the regular case of the normalization procedure, a moving frame for a group related to the equivalence group in the context of equivalence transformations among equations of the class under consideration. Using the moving frame constructed, we describe the algebra of differential invariants of the former group by obtaining a minimum generating set of differential invariants and a complete set of independent operators of invariant differentiation.

1. Introduction. Invariants and differential invariants of transformation groups, in particular, point symmetry groups admitted by systems of differential equations have a wide range of applications and are therefore an intensively investigated subject. Differential invariants play a central role in the invariant parameterization problem [1, 2, 30] and in the problem of invariant discretization [3, 5, 7]. They are also used to construct invariant differential equations and invariant variational problems [22, 23], as well as in computer vision, integrable systems, classical invariant theory and the calculus of variations [6, 22, 24].

Rather recently, finding differential invariants in problems related to group classification became a research topic of interest. The idea is to compute the differential invariants not for the point symmetry group of a single system of differential equations but for the equivalence group admitted by a class of such systems. The primary motivation for such a survey is to study the equivalence of systems of differential equations. Exploring equivalence, it is possible to explicitly determine point transformations among systems from a class [28]. Such a mapping between two systems of differential equations is especially helpful if wide sets of exact solutions are known for one of the systems involved. These solutions then can be mapped to solutions of the equivalent system. Another case of particular interest is the mapping between nonlinear and linear elements of a class of systems of differential equations [19]. For the solution of the equivalence problem, finding differential invariants for the equivalence group is a main ingredient. There are a number of papers where some low-order differential invariants of the equivalence groups of various physically relevant classes of systems of differential equations were computed using the Lie infinitesimal method; see, e.g., [11, 12, 13, 14, 15, 17, 32, 33, 34, 35] and references therein.

In the present paper we will be concerned with differential invariants for a group[1] related to the equivalence group of the class of diffusion equations

$$u_t = u_{xx} + f(u, u_x) \tag{1}$$

in the context of equivalence transformations among equations of this class. This subject was originally considered in [32], using the infinitesimal method and restricting the order of differential invariants up to two. We revisit the construction of differential invariants for the class (1) from the very beginning, analyzing differential invariants of which group should be found. Then, we apply the method of equivariant moving frames in the formulation originally proposed and formulated by Fels and Olver [9, 10], which was later generalized to infinite-dimensional Lie (pseudo)groups in [6, 25, 26], and this is the setting that is needed

[1]In fact, this object and the "equivalence group" of the class (1) are Lie pseudogroups of locally defined point transformations. We use the term "group" for brevity since this does not lead to any confusion.

to study differential invariants for the class (1). The advantage of moving frames is that they allow for a canonical process of *invariantization*, which associates to each object, such as functions, differential functions, differential forms and total differentiation operators, its invariant counterpart. For the problem of finding differential invariants of a Lie transformation (pseudo)group, this property is especially convenient. The invariantization of the jet-space coordinate functions yields the so-called *normalized differential invariants*. The invariantized coordinate functions whose transformed counterparts were involved in the construction of the corresponding moving frame via the *normalization procedure* are equal to the respective constants chosen in the course of normalization. This is why these objects are called *phantom normalized differential invariants*. The *non-phantom normalized differential invariants* constitute a complete set of functionally independent differential invariants. As a further asset, the method of moving frames also permits to study the algebra of differential invariants by deriving relations, called *syzygies*, between invariant derivatives of non-phantom normalized differential invariants. Finding syzygies can aid in the establishment of a *minimum generating set* of differential invariants. See e.g. [6, 8, 22, 25, 26] for more details and an extensive discussion on the computation of differential invariants for both finite-dimensional Lie symmetry groups and for infinite-dimensional Lie (pseudo)groups using moving frames.

The further organization of this paper is as follows. In Section 2 we compute the equivalence group and the equivalence algebra of the class (1). Section 3 is devoted to the selection of a group to be considered and a preliminary analysis of equivariant moving frames associated with this group. The structure of the algebra of differential invariants is determined in the main Section 4. This includes a description of a minimum generating set of differential invariants and a complete set of independent operators of invariant differentiation, which serve to exhaustively describe the set of differential invariants. Moreover, for each $k \in \mathbb{N}_0$ we explicitly present a functional basis of differential invariants of order not greater than k.

2. The equivalence group. The auxiliary system for the class (1), which is satisfied by the arbitrary element f, is $f_t = f_x = f_{u_t} = f_{u_{tt}} = f_{u_{tx}} = f_{u_{xx}} = 0$. By definition [27, 28, 29, 31], the (usual) equivalence group $G^{\sim}$ of the class (1) consists of the point transformations in the space with coordinates $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, f)$ that have the following properties:

- they are projectable to the space with the coordinates (t, x, u),
- their components for derivatives of u are found by prolongation using the chain rule, and
- they map every equation from the class (1) to an equation from the same class.

To begin finding the group $G^{\sim}$, we fix an arbitrary equation of the class (1), $u_t = u_{xx} + f(u, u_x)$, and aim to find point transformations in the space with coordinates (t, x, u),

$$\tilde{t} = T(t, x, u), \quad \tilde{x} = X(t, x, u), \quad \tilde{u} = U(t, x, u), \tag{2}$$

that transform the fixed equation to an equation of the same class,

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}} + \tilde{f}(\tilde{u}, \tilde{u}_{\tilde{x}}). \tag{3}$$

A preliminary simplification is obtained from noting that the class (1) is a subclass of the class of second-order (1+1)-dimensional semi-linear evolution equations. Any point transformation between two equations from the latter class satisfies the constraints $T_x = T_u = X_u = 0$, i.e., $\tilde{t} = T(t)$, $\tilde{x} = X(t, x)$, and $T_t X_x U_u \neq 0$. See [16, 18, 21] for further details. After taking into account the above constraints, the required transformed derivatives read

$$\tilde{u}_{\tilde{t}} = \frac{1}{T_t}\left(\mathrm{D}_t U - \frac{X_t}{X_x}\mathrm{D}_x U\right), \quad \tilde{u}_{\tilde{x}} = \frac{1}{X_x}\mathrm{D}_x U,$$
$$\tilde{u}_{\tilde{x}\tilde{x}} = \left(\frac{1}{X_x}\mathrm{D}_x\right)^2 U,$$

where D_t and D_x are the usual total derivative operators with respect to t and x, respectively. Substituting these expressions and $u_t = u_{xx} + f$ into Eq. (3), we split the resulting equation with respect to u_{xx} yielding $T_t = X_x^2$. The remaining equation is

$$f = \frac{T_t}{U_u}\tilde{f} - U_t + \frac{X_t}{X_x}(U_x + U_u u_x) + U_{xx} + 2U_{xu}u_x + U_{uu}u_x^2. \tag{4}$$

The differential consequences of Eq. (4) that are obtained by separate differentiations with respect to t and x can be split with respect to derivatives of $\tilde{f}$ since they are regarded as independent for equivalence transformations. This yields the equations

$$T_{tt} = X_{xt} = X_{tt} = U_t = U_x = 0.$$

The equation (4) itself gives the f-component of equivalence transformations.

The arbitrary element f in fact depends only on u and u_x. The space with coordinates (t,x,u,u_x,f) is preserved by all elements of $G^\sim$. This is why we can assume this space as the underlying space for $G^\sim$ and present merely the transformation components for its coordinates.

As a result, we have proved the following theorem.

Theorem 1. *The equivalence group $G^\sim$ of the class* (1) *is constituted by the transformations*

$$\begin{aligned}&\tilde t = C_1^2 t + C_0, \quad \tilde x = C_1 x + C_1 C_2 t + C_3, \quad \tilde u = \varphi(u),\\ &\tilde u_{\tilde x} = C_1^{-1}\varphi' u_x, \quad \tilde f = C_1^{-2}\left(\varphi' f - C_2\varphi' u_x - \varphi'' u_x^2\right),\end{aligned} \tag{5}$$

where $C_0, C_1, C_2, C_3 \in \mathbb{R}$, φ is an arbitrary smooth function of u and $C_1\varphi' \neq 0$.

The infinitesimal generators of one-parameter subgroups of $G^\sim$, which constitute the equivalence algebra $\mathfrak{g}^\sim$ of the class (1), can be derived from (5) by differentiation, cf. the proof of Corollary 11 in [20] or the proof of Corollary 6 in [4]. These generators coincide with those determined in [32]. As we will later need them for the description of the algebra of differential invariants of a group related to $G^\sim$ in the context of the $G^\sim$-equivalence among equations of the class (1), we present them here. The general element of $\mathfrak{g}^\sim$ is

$$Q = \tau\partial_t + \xi\partial_x + \phi\partial_u + \eta\partial_{u_x} + \theta\partial_f,$$

where the components are of the form

$$\begin{aligned}&\tau = 2c_1 t + c_0, \quad \xi = c_1 x + c_2 t + c_3, \quad \phi = \phi(u),\\ &\eta = (\phi' - c_1)u_x, \quad \theta = (\phi' - 2c_1)f - c_2 u_x - \phi'' u_x^2,\end{aligned}$$

in which c_0, c_1, c_2 and c_3 are arbitrary real constants, and ϕ is an arbitrary smooth function of u. In other words, the equivalence algebra $\mathfrak{g}^\sim$ of the class (1) is spanned by the vector fields

$$\begin{aligned}&\partial_t, \quad 2t\partial_t + x\partial_x - u_x\partial_{u_x} - 2f\partial_f, \quad t\partial_x - u_x\partial_f,\\ &\phi\partial_u + \phi' u_x\partial_{u_x} + (\phi' f - \phi'' u_x^2)\partial_f,\end{aligned}$$

where ϕ runs through the set of smooth functions of u.

3. Preliminary analysis of moving frames. Let us first clarify the space of independent and dependent variables to be used and the group to be considered. While formally the arbitrary element f is a smooth function on the second-order jet space with coordinates $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$, practically it explicitly depends only on u and u_x. This is why subsequently we will only consider the projection of the equivalence transformations to the space with coordinates (u, u_x, f). As a shorthand, we denote $v := u_x$ and $\tilde{v} := \tilde{u}_{\tilde{x}} = V(u, v) := C_1^{-1}\varphi'(u)v$. In other words, we will in fact study differential invariants of the projection G_1 of $G^\sim$ to the space with coordinates (u, v, f), where u and v are the independent variables and f is the dependent variable. The infinitesimal counterpart of G_1 is the projection $\mathfrak{g}_1$ of $\mathfrak{g}^\sim$ to the space with coordinates (u, v, f).

In order to describe the algebra of differential invariants of the group G_1, we now construct a moving frame for this group. Since it is infinite-dimensional, we have to use the machinery developed for Lie pseudo-groups, see [6, 25] for an extensive description of this subject.

The first step in the construction of the moving frame is the computation of the lifted horizontal coframe, the dual of which yields the implicit total differentiation operators $\mathrm{D}_{\tilde{u}}$ and $\mathrm{D}_{\tilde{v}}$. For the equivalence transformations (5), the lifted horizontal coframe is

$$\mathrm{d_h}\tilde{u} = (\mathrm{D}_u U)\,\mathrm{d}u + (\mathrm{D}_v U)\,\mathrm{d}v = \varphi'\,\mathrm{d}u,$$
$$\mathrm{d_h}\tilde{v} = (\mathrm{D}_u V)\,\mathrm{d}u + (\mathrm{D}_v V)\,\mathrm{d}v = \frac{\varphi''}{C_1}v\,\mathrm{d}u + \frac{\varphi'}{C_1}\,\mathrm{d}v.$$

Computing the dual, we derive that

$$\mathrm{D}_{\tilde{u}} = \frac{1}{\varphi'}\mathrm{D}_u - \frac{\varphi''}{(\varphi')^2}v\mathrm{D}_v, \quad \mathrm{D}_{\tilde{v}} = \frac{C_1}{\varphi'}\mathrm{D}_v \tag{6}$$

are the required implicit differentiation operators. Acting with them on the transformation component for f, we find that

$$\tilde{f}_{ij} = \frac{\partial^{i+j}\tilde{f}}{\partial\tilde{u}^i\partial\tilde{v}^j} = \mathrm{D}_{\tilde{u}}^i\mathrm{D}_{\tilde{v}}^j F,$$

where $i, j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and

$$\tilde{f}_{00} = \tilde{f} = F := \frac{1}{C_1^2}(\varphi' f - C_2\varphi' v - \varphi'' v^2)$$

is the f-component of equivalence transformations. In particular, the derivatives up to order 2 are exhausted by

$$
\begin{aligned}
\tilde f_{10} &= \frac{1}{C_1^2\varphi'}\left(\varphi' f_u + \varphi''(f - vf_v) - \varphi''' v^2 + 2\frac{(\varphi'')^2}{\varphi'}v^2\right),\\
\tilde f_{01} &= \frac{1}{C_1\varphi'}\left(\varphi' f_v - C_2\varphi' - 2\varphi'' v\right),\\
\tilde f_{20} &= \frac{1}{C_1^2\varphi'}\left(f_{uu} - \frac{\varphi''}{\varphi'}(f_u - 2vf_{uv}) + \left(\frac{\varphi''}{\varphi'}\right)^2 v^2 f_{vv}\right.\\
&\qquad\left. + \left(\frac{\varphi''}{\varphi'}\right)'(f - vf_v) - (\varphi')^2\left(\frac{1}{\varphi'}\left(\frac{1}{\varphi'}\right)''\right)' v^2\right),\\
\tilde f_{11} &= \frac{1}{C_1\varphi'^2}\left(\varphi' f_{uv} - \varphi'' v f_{vv} - 2\varphi''' v + 4\frac{\varphi''^2}{\varphi'}v\right),\\
\tilde f_{02} &= \frac{1}{\varphi'^2}(\varphi' f_{vv} - 2\varphi'').
\end{aligned}
$$

There are a relative invariant and a relative conditional invariant which play a significant role in the following consideration. By taking the difference $\tilde f_{00} - \tilde v\tilde f_{01}$ we exclude the inessential constant C_2, which only arises in $\tilde f_{00}$ and $\tilde f_{01}$,

$$
\tilde f_{00} - \tilde v\tilde f_{01} = \frac{1}{C_1^2}\left(\varphi'(f - vf_v) + \varphi'' v^2\right).
$$

Combining further $2(\tilde f_{00} - \tilde v\tilde f_{01}) + \tilde v^2\tilde f_{02}$ to exclude φ'', we obtain

$$
\tilde W = \frac{1}{C_1^2}W, \quad \text{where} \quad
\begin{aligned}
W &= 2f - 2vf_v + v^2 f_{vv},\\
\tilde W &= 2\tilde f - 2\tilde v\tilde f_{\tilde v} + \tilde v^2\tilde f_{\tilde v\tilde v},
\end{aligned}
$$

i.e., W is a relative invariant of G_1. In other words, the condition $W = 0$ is preserved by any equivalence transformation in the class (1). Analogously, the combination $2\tilde f_{10} - v\tilde f_{11}$ gives

$$
\tilde S = \frac{1}{C_1^2}S + \frac{1}{C_1^2}\frac{\varphi''}{\varphi'}W, \quad \text{where} \quad
\begin{aligned}
S &= 2f_u - vf_{uv},\\
\tilde S &= 2\tilde f_{\tilde u} - \tilde v\tilde f_{\tilde u\tilde v}.
\end{aligned}
\tag{7}
$$

This means that S is a relative invariant of G_1 if the condition $W = 0$ is satisfied. Values of the differential functions W and S determine which normalization conditions should be chosen.

We next find appropriate normalization conditions, which form the basis for the construction of an equivariant moving frame. As φ arises only in U, we can set U to any value including zero. The value of V can be set to any constant excluding zero, and all these possibilities are equivalent. We find it convenient to put $V = 1$ and express $\varphi' = C_1/v$. The constraint $W = 0$ singles out the *singular* case for the moving frame construction, which has to be investigated separately. Within this singular case, there is the *ultra-singular* subcase associated with the constraint $S = 0$. Indeed, under the constraint $W = 0$ the equation (7) can be solved for C_1 if and only if $S \neq 0$.

4. Differential invariants for the regular case. In this paper, we only consider the *regular* case for moving frames of G_1, where $W \neq 0$. In this case, the following normalization conditions can be used to determine a complete moving frame

$$\begin{aligned}
&\tilde{u} = 0, \quad \tilde{v} = 1, \quad \tilde{f} = 1, \quad \tilde{f}_{01} = 0, \quad \tilde{f}_{02} = 0,\\
&\tilde{f}_{i0} = -\frac{v^2\varphi^{(i+2)}}{C_1{}^2(\varphi')^i} + \frac{1}{C_1{}^2}\sum_{i'=0}^{i}\binom{i}{i'}\frac{1}{(\varphi')^{i'}}\left(\frac{\varphi''}{(\varphi')^2}\right)^{i-i'} f_{i',i-i'}\\
&\qquad + \dots = 0,\ i \in \mathbb{N}.
\end{aligned} \tag{8}$$

In the expression for $\tilde{f}_{i0}$, we presented only the summands with the highest-order derivatives of φ and f, which are $\varphi^{(i+2)}$ and $f_{i',i-i'}$, $i' = 0, \dots, i$, respectively. We solve the first five equations with respect to C_1, C_2, φ, φ' and φ'' and substitute the obtained expressions into the other equations. For each fixed $i \in \mathbb{N}$, we solve the modified equation $\tilde{f}_{i0} = 0$ in view of the similar equations with lower values of i and thus find an expression for $\varphi^{(i+2)}$, the explicit form of which is essential for further consideration only for $i = 3$. This yields the following complete moving frame:

$$\begin{aligned}
&C_1 = \frac{W}{2v}, \quad C_2 = f_v - vf_{vv},\\
&\varphi = 0, \quad \varphi' = \frac{W}{2v^2}, \quad \varphi'' = \frac{W}{4v^2}f_{vv},\\
&\varphi''' = \frac{W}{4v^4}\Big(2f_u + (f - vf_v + v^2 f_{vv})f_{vv}\Big),\\
&\varphi^{(i+2)} = \frac{W}{2v^1}\sum_{i'=0}^{i}\binom{i}{i'}\left(\frac{v^2}{W}\right)^{i-i'} f_{i',i-i'} + \cdots, \quad i = 2, 3, \dots .
\end{aligned} \tag{9}$$

In the expression for $\varphi^{(i+2)}$, we presented only the summands with the highest-order derivatives $f_{i',i-i'}$, $i'=0,\dots,i$. The invariantization $I^{ij}=\iota(f_{ij})$ of the derivatives f_{ij} of $\tilde f$ that are not involved in the normalization conditions (8) gives rise to a complete set of functionally independent differential invariants of G_1. The lowest-order non-phantom normalized differential invariant is I^{11}, and it reads

$$I^{11}=-2v^2\frac{4f_u-2vf_{uv}+(2f-2vf_v+v^2f_{vv})f_{vv}}{(2f-2vf_v+v^2f_{vv})^2}.$$

This differential invariant is of second order. For each tuple (i,j) with $i+j\geqslant 3$ and $j\neq 0$, the maximal orders of derivatives of f and φ appearing in the expression for $\tilde f_{ij}$ are $i+j$ and $i+2$, respectively. This is why the maximal order of derivatives of f in the expression for $\tilde f_{ij}$ cannot be lowered in the course of the invariantization, i.e., the order of the normalized differential invariant I^{ij} is $i+j$. Therefore, there are precisely $\frac12 k(k+1)-2$ functionally independent differential G_1-invariants of order not greater than $k\geqslant 2$. They are given by the functions I^{11} and I^{ij} with $3\leqslant i+j\leqslant k$ and $j\neq 0$.

Apart from finding the complete set of functionally independent differential invariants of G_1 for each fixed order by successively invariantizing all the derivatives f_{ij}, the moving frame (9) can be used to determine the operators of invariant differentiation. They are found upon invariantizing the operators of total differentiation (6) and read

$$\mathrm{D}^{\mathrm{i}}_u=\frac{2v^2}{2f-2vf_v+v^2f_{vv}}\left(\mathrm{D}_u-\frac12 vf_{vv}\mathrm{D}_v\right),\quad \mathrm{D}^{\mathrm{i}}_v=v\mathrm{D}_v. \tag{10}$$

We now aim to investigate the structure of the algebra of differential invariants of G_1. The starting point for this investigation is the universal recurrence relation, which relates the differentiated invariantized differential functions or differential forms with the invariantization of the respective differentiated objects. This universal recurrence relation reads [25]

$$\mathrm{d}\iota(\Omega)=\iota\big(\mathrm{d}\Omega+Q^{(\infty)}(\Omega)\big). \tag{11}$$

The first step in our study is the evaluation of (11) for the independent variables u and v and the derivatives f_{ij}, $i,j\in\mathbb N_0$,

$$\mathrm{d_h}\iota(u)=\omega^1+\iota(\phi),\quad \mathrm{d_h}\iota(v)=\omega^2+\iota(\eta),$$

$$
\begin{aligned}
\mathrm{d_h} I^{ij} &= \mathrm{d_h}\iota(f_{ij}) = \iota(f_{i+1,j}\mathrm{d}u + f_{i,j+1}\mathrm{d}v + \theta^{ij})\\
&= I^{i+1,j}\omega^1 + I^{i,j+1}\omega^2 + \iota(\theta^{ij}),
\end{aligned}
$$

where $\omega^1 = \iota(\mathrm{d}u)$, $\omega^2 = \iota(\mathrm{d}v)$, and

$$
\begin{aligned}
\theta^{ij} &= \mathrm{D}_u^i\mathrm{D}_v^j(\theta - \phi f_{10} - \eta f_{01}) + \phi f_{i+1,j} + \eta f_{i,j+1}\\
&= (j-2)c_1 f_{ij} - (j-1)\sum_{i'=0}^{i}\binom{i}{i'}\phi^{(i'+1)} f_{i-i',j}\\
&\quad - \sum_{i'=1}^{i}\binom{i}{i'}\Big(\phi^{(i')} f_{i-i'+1,j} + v\phi^{(i'+1)} f_{i-i',j+1}\Big)\\
&\quad - c_2\delta_{0i}(\delta_{0j}v + \delta_{1j}) - \phi^{(i+2)}(\delta_{0j}v^2 + 2\delta_{1j}v + 2\delta_{2j})
\end{aligned}
$$

is the f_{ij}-component of the infinite prolongation of the vector field $\phi\partial_u + \eta\partial_v + \theta\partial_f$. Here δ_{ij} is the Kronecker delta. The respective recurrence relations then split into two kinds, the first being the so-called phantom recurrence relations. For a well-defined moving frame cross-section, they can be uniquely solved for the invariantized Maurer–Cartan forms, which arise due to the presence of the correction term $\iota\big(Q^{(\infty)}(\Omega)\big)$ in (11). Then, plugging these invariantized Maurer–Cartan forms into the second kind of recurrence relations, the non-phantom ones, gives a complete description of the relation between the normalized and differentiated differential invariants, see [6, 25] for more details. For the chosen cross-section (8), the phantom recurrence relations read

$$
\begin{aligned}
&0 = \mathrm{d_h}\iota(u) = \omega^1 + \iota(\phi) = \omega^1 + \hat\phi,\\
&0 = \mathrm{d_h}\iota(v) = \omega^2 + \iota(\eta) = \omega^2 + \hat\phi' - \hat c_1,\\
&0 = \mathrm{d_h} I^{00} = \iota(\theta) = \hat\phi' - 2\hat c_1 - \hat c_2 - \hat\phi'',\\
&0 = \mathrm{d_h} I^{01} = I^{11}\omega^1 + \iota(\theta^{01}) = I^{11}\omega^1 - \hat c_2 - 2\hat\phi'',\\
&0 = \mathrm{d_h} I^{02} = I^{12}\omega^1 + I^{03}\omega^2 + \iota(\theta^{02}) = I^{12}\omega^1 + I^{03}\omega^2 - 2\hat\phi'',\\
&0 = \mathrm{d_h} I^{i0} = I^{i1}\omega^2 + \iota(\theta^{i0})\\
&\quad = I^{i1}\omega^2 + \hat\phi^{(i+1)} - \hat\phi^{(i+2)} - \sum_{i'=1}^{i-1}\binom{i}{i'} I^{i-i',1}\hat\phi^{(i'+1)}, \quad i\in\mathbb{N},
\end{aligned}
$$

where the forms $\hat c_1$, $\hat c_2$ and $\hat\phi^{(i)}$, $i\in\mathbb{N}_0$, are the invariantizations of the parameters c_1, c_2 and $\phi^{(i)}$ of the infinitely prolonged general element of the algebra $\mathfrak{g}_1$, respectively, $\hat c_1 = \iota(c_1)$, $\hat c_2 = \iota(c_2)$ and $\hat\phi^{(i)} = \iota(\phi^{(i)})$.

More rigorously, here the parameters c_1, c_2 and $\phi^{(i)}$, $i\in\mathbb{N}_0$, are interpreted as the coordinate functions on the infinite prolongation of $\mathfrak{g}_1$. Recall that under the prolongation we consider u and v to be the independent variables and f to be the dependent variable. In other words, these coefficients are first-order differential forms in the jet space $\mathrm{J}^\infty(u,v|\,f)$. Hence their invariantizations are also forms, which are called *invariantized Maurer–Cartan forms.*

The above system can be solved to yield the following invariantized Maurer–Cartan forms

$$\begin{aligned}
&\hat c_1=\left(\frac12 I^{12}-I^{11}\right)\omega^1+\left(\frac12 I^{03}-1\right)\omega^2,\\
&\hat c_2=(I^{11}-I^{12})\omega^1-I^{03}\omega^2,\\
&\hat\phi=-\omega^1,\quad \hat\phi'=\left(\frac12 I^{12}-I^{11}\right)\omega^1+\left(\frac12 I^{03}-2\right)\omega^2,\\
&\hat\phi''=\frac12 I^{12}\omega^1+\frac12 I^{03}\omega^2,\\
&\hat\phi^{(i+2)}=\hat\phi^{(i+1)}-\sum_{i'=1}^{i-1}\binom{i}{i'}I^{i-i',1}\hat\phi^{(i'+1)}+I^{i1}\omega^2,\quad i\in\mathbb{N}.
\end{aligned}\tag{12}$$

The explicit expression for the invariantized form $\hat\phi^{(i+2)}$, $i\in\mathbb{N}$, as a combination of ω^1 and ω^2 with coefficients being polynomials of normalized differential invariants is obtained by expanding the above expression when successively going over the values of i. In particular,

$$\begin{aligned}
&\hat\phi'''=\frac12 I^{12}\omega^1+\left(I^{11}+\frac12 I^{03}\right)\omega^2,\\
&\hat\phi^{(4)}=\left(\frac12 I^{12}-I^{11}I^{12}\right)\omega^1+\left(I^{11}+\frac12 I^{03}+I^{21}-I^{11}I^{03}\right)\omega^2.
\end{aligned}$$

For $i\geqslant 3$, the greatest value of $i'+j'$ for the normalized differential invariants $I^{i'j'}$ that are involved in $\hat\phi^{(i+2)}$ is $i+1$, and $I^{i1}\omega^2$ is the only summand with this value.

The non-phantom recurrence relations are

$$\begin{aligned}
\mathrm{d_h}I^{11}&=I^{21}\omega^1+I^{12}\omega^2+\iota(\theta^{11})\\
&=\big(I^{21}+2(I^{11})^2-I^{11}I^{12}-I^{12}\big)\omega^1\\
&\quad+\big(I^{12}-I^{11}I^{03}+I^{11}-I^{03}\big)\omega^2,
\end{aligned}$$

$$\mathrm{d_h} I^{ij} = I^{i+1,j}\omega^1 + I^{i,j+1}\omega^2 + \iota(\theta^{ij}), \quad i+j \geqslant 3,\ j \neq 0,$$

with

$$\begin{aligned}\iota(\theta^{ij}) = (j-2)I^{ij}\hat{c}_1 - (j-1)\sum_{i'=0}^{i}\binom{i}{i'}I^{i-i',j}\hat{\phi}^{(i'+1)} \\ - \sum_{i'=1}^{i}\binom{i}{i'}\Big(I^{i-i'+1,j}\hat{\phi}^{(i')} + I^{i-i',j+1}\hat{\phi}^{(i'+1)}\Big) \\ - \delta_{0i}(\delta_{0j}+\delta_{1j})\hat{c}_2 - (\delta_{0j}+2\delta_{1j}+2\delta_{2j})\hat{\phi}^{(i+2)}.\end{aligned}$$

The first non-phantom recurrence relation splits into

$$\begin{aligned}\mathrm{D}^{\mathrm{i}}_u I^{11} &= I^{21} + 2(I^{11})^2 - I^{11}I^{12} - I^{12},\\ \mathrm{D}^{\mathrm{i}}_v I^{11} &= I^{12} - I^{11}I^{03} + I^{11} - I^{03}.\end{aligned}$$

Therefore, the normalized differential invariants I^{12} and I^{21} are expressed in terms of invariant derivatives of I^{11} and I^{03},

$$\begin{aligned}I^{12} &= \mathrm{D}^{\mathrm{i}}_v I^{11} + I^{11}I^{03} - I^{11} + I^{03},\\ I^{21} &= \mathrm{D}^{\mathrm{i}}_u I^{11} - 2(I^{11})^2 \\ &\quad + (I^{11}+1)(\mathrm{D}^{\mathrm{i}}_v I^{11} + I^{11}I^{03} - I^{11} + I^{03}).\end{aligned} \tag{13}$$

In view of the above discussion on the invariantize forms $\hat{\phi}^{(i')}$, $i \in \mathbb{N}$, the expression for $\iota(\theta^{ij})$ with $i+j \geqslant 3$ and $j \neq 0$ implies that the greatest value of $i'+j'$ for $I^{i'j'}$ involved in $\iota(\theta^{ij})$ is $i+j$. Hence splitting the recurrence relation with $\mathrm{d_h}I^{ij}$ leads to expressions for $I^{i+1,j}$ and $I^{i,j+1}$ in terms of invariant derivatives of $I^{i'j'}$ with $i'+j' \leqslant i+j$. For example, from the non-phantom recurrence relation

$$\begin{aligned}\mathrm{d_h} I^{03} &= I^{13}\omega^1 + I^{04}\omega^2 + \iota(\theta^{03})\\ &= \left(I^{13} + I^{11}I^{03} - \frac{I^{12}I^{03}}{2}\right)\omega^1 + \left(I^{04} + I^{03} - \frac{(I^{03})^2}{2}\right)\omega^2\end{aligned}$$

we derive

$$\begin{aligned}\mathrm{D}^{\mathrm{i}}_u I^{03} &= I^{13} + I^{11}I^{03} - \frac{1}{2}I^{12}I^{03},\\ \mathrm{D}^{\mathrm{i}}_v I^{03} &= I^{04} - \frac{1}{2}(I^{03})^2 + I^{03}.\end{aligned}$$

This implies by induction, where the expressions (13) for I^{12} and I^{21} give the base case, that any non-phantom normalized differential invariant can be expressed in terms of invariant derivatives of I^{11} and I^{03}.

To find a minimum generating set of differential invariants for the projected group G_1, we should additionally check whether I^{03} can be expressed in terms of invariant derivatives of I^{11}. We use (11) to compute the commutator between the operators of invariant differentiation. This is done upon evaluating (11) for the basis horizontal forms $\mathrm{d}u$ and $\mathrm{d}v$,

$$\begin{aligned}
\mathrm{d_h}\iota(\mathrm{d}u) &= \iota(\phi'\mathrm{d}u) = \iota(\phi') \wedge \iota(\mathrm{d}u) \\
&= \left(2 - \frac{1}{2}I^{03}\right)\omega^1 \wedge \omega^2 = -Y^1_{12}\,\omega^1 \wedge \omega^2, \\
\mathrm{d_h}\iota(\mathrm{d}v) &= \iota\big(\phi'' v\mathrm{d}u + (\phi' - c_1)\mathrm{d}v\big) \\
&= \iota(\phi'' v) \wedge \iota(\mathrm{d}u) = -\frac{1}{2}I^{03}\omega^1 \wedge \omega^2 = -Y^2_{12}\,\omega^1 \wedge \omega^2.
\end{aligned}$$

The commutation relation then evaluates as

$$[\mathrm{D}^{\mathrm{i}}_u, \mathrm{D}^{\mathrm{i}}_v] = Y^1_{12}\mathrm{D}^{\mathrm{i}}_u + Y^2_{12}\mathrm{D}^{\mathrm{i}}_v = \left(\frac{1}{2}I^{03} - 2\right)\mathrm{D}^{\mathrm{i}}_u + \frac{1}{2}I^{03}\mathrm{D}^{\mathrm{i}}_v,$$

see [25] for details of the technique applied. Evaluating $[\mathrm{D}^{\mathrm{i}}_u, \mathrm{D}^{\mathrm{i}}_v]I^{11}$, we can derive the following expression for I^{03}:

$$I^{03} := \frac{2v^3 f_{vvv}}{2f - 2vf_v + v^2 f_{vv}} = 2\frac{2\mathrm{D}^{\mathrm{i}}_u I^{11} + [\mathrm{D}^{\mathrm{i}}_u, \mathrm{D}^{\mathrm{i}}_v]I^{11}}{\mathrm{D}^{\mathrm{i}}_u I^{11} + \mathrm{D}^{\mathrm{i}}_v I^{11}}.$$

As a result, we have proved the following theorem.

Theorem 2. *The algebra of differential invariants of the group G_1, which is the projection of the equivalence group $G^\sim$ of the class of diffusion equations* (1) *to the space with coordinates (u, v, f), is generated by the single differential invariant*

$$I^{11} = -2v^2\frac{4f_u - 2vf_{uv} + (2f - 2vf_v + v^2 f_{vv})f_{vv}}{(2f - 2vf_v + v^2 f_{vv})^2}$$

along with the two operators of invariant differentiation

$$\mathrm{D}^{\mathrm{i}}_u = \frac{2v^2}{2f - 2vf_v + v^2 f_{vv}}\left(\mathrm{D}_u - \frac{1}{2}vf_{vv}\mathrm{D}_v\right), \quad \mathrm{D}^{\mathrm{i}}_v = v\mathrm{D}_v.$$

All other differential invariants are functions of I^{11} and invariant derivatives thereof.

Corollary 1. *A functional basis of differential invariants of order not greater than $k \in \mathbb{N}_0$ in terms of invariant derivatives of non-phantom normalized differential invariants is exhausted by*

$$\big(\mathrm{D}^{\mathrm{i}}_u\big)^i\big(\mathrm{D}^{\mathrm{i}}_v\big)^j I^{11},\ i+j\leqslant k-2,\quad \big(\mathrm{D}^{\mathrm{i}}_v\big)^{j'} I^{03},\ j'\leqslant k-3.$$

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