

Formality morphism as the mechanism of \star -product associativity: how it works

R. Buring[†], *A.V. Kiselev*[‡]

[†] *Johannes Gutenberg–Universität, Mainz, Germany*
E-mail: rburing@uni-mainz.de

[‡] *University of Groningen, The Netherlands*
E-mail: A.V.Kiselev@rug.nl

Морфізм $\mathcal{F} = \{\mathcal{F}_n, n \geq 1\}$, що забезпечує властивість формальності алгебр в деформаційному квантуванні за Концевичем, задано набором відображень тензорних степеней диференціальної градуїованої алгебри Лі (dgLa) мультивекторних полів в dgLa полідиференціальних операторів на скінченновимірних афінних многовидах. Хоча перший член \mathcal{F}_1 сам по собі не є морфізмом алгебр Лі, послідовність \mathcal{F} в цілому є L_∞ -морфізмом. На його основі будується відображення елементів Маурера–Картана, яке ставить у відповідність пуасоновим бівекторам деформації $\mu_A \mapsto \star_{A[[\hbar]]}$, що добудовують звичайне множення функцій до асоціативних некомутативних \star -добутків на просторі степеневих рядів по \hbar . При цьому асоціативність \star -добутків забезпечено — на мові графів Концевича, що представляють полідиференціальні оператори, — диференціальними наслідками тотожності Якобі. Мета роботи — проілюструвати цей алгебраїчний механізм для \star -добутків Концевича (зокрема, з гармонічними пропагаторами).

The formality morphism $\mathcal{F} = \{\mathcal{F}_n, n \geq 1\}$ in Kontsevich's deformation quantization is a collection of maps from tensor powers of the differential graded Lie algebra (dgLa) of multivector fields to the dgLa of polydifferential operators on finite-dimensional affine manifolds. Not a Lie algebra morphism by its term \mathcal{F}_1 alone, the entire set \mathcal{F} is an L_∞ -morphism instead. It induces a map of the Maurer–Cartan elements, taking Poisson bi-vectors to deformations $\mu_A \mapsto \star_{A[[\hbar]]}$ of the usual multiplication of functions into associative noncommutative \star -products of power series in \hbar . The associativity of \star -products is then realized, in terms of the Kontsevich graphs which encode polydifferential operators, by differential consequences of the Jacobi identity. The aim of this paper is to illustrate the work of this algebraic mechanism for the Kontsevich \star -products (in particular, with harmonic propagators).

1. Introduction. The Kontsevich formality morphism \mathcal{F} relates two differential graded Lie algebras (dgLa). Its domain of definition is the shifted-graded vector space $T_{\text{poly}}^{\downarrow[1]}(M^r)$ of multivectors on an affine real finite-dimensional manifold M^r ; the graded Lie algebra structure is the Schouten bracket $[\![,]\!]$ and the differential is set to (the bracket with) zero by definition. On the other hand, the target space of the formality morphism \mathcal{F} is the graded vector space $D_{\text{poly}}^{\downarrow[1]}(M^r)$ of polydifferential operators on M^r ; the graded Lie algebra structure is the Gerstenhaber bracket $[\cdot, \cdot]_G$ and the differential $d_H = [\mu_A, \cdot]$ is induced by using the multiplication μ_A in the algebra $A := C^\infty(M^r)$ of functions on M^r . It is readily seen that w.r.t. the above notation, Poisson bi-vectors \mathcal{P} satisfying the Jacobi identity $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ on M^r are the Maurer–Cartan elements (indeed, $(d \equiv 0)(\mathcal{P}) + \frac{1}{2}[\![\mathcal{P}, \mathcal{P}]\!] = 0$). Likewise, for a (non)commutative star-product $\star = \mu_{A[[\hbar]]} + \langle \text{tail} =: B \rangle$, which deforms the usual multiplication $\mu = \mu_{A[[\hbar]]}$ in $A[[\hbar]] = C^\infty(M^r) \otimes_{\mathbb{R}} \mathbb{R}[[\hbar]]$ by a tail B w.r.t. a formal parameter \hbar , the requirement that \star be associative again is the Maurer–Cartan equation,

$$[\mu, B]_G + \frac{1}{2}[B, B]_G = 0 \quad \iff \quad \frac{1}{2}[\mu + B, \mu + B]_G = 0.$$

Here, the leading order equality $[\mu, \mu]_G = 0$ expresses the given associativity of the product μ itself.

The Kontsevich formality mapping $\mathcal{F} = \{\mathcal{F}_n: T_{\text{poly}}^{\otimes n} \rightarrow D_{\text{poly}}, n \geq 1\}$ in [15, 16] is an L_∞ -morphism which induces a map that takes Maurer–Cartan elements \mathcal{P} , i.e., formal Poisson bi-vectors $\hat{\mathcal{P}} = \hbar\mathcal{P} + \bar{o}(\hbar)$ on M^r , to Maurer–Cartan elements¹, i.e., the tails B in solutions \star of the associativity equation on $A[[\hbar]]$.

The theory required to build the Kontsevich map \mathcal{F} is standard, well reflected in the literature (see [15, 16], as well as [9, 11] and references therein); a proper choice of signs is analysed in [2, 20]. The framework of homotopy Lie algebras and L_∞ -morphisms, introduced by Schlessinger–Stasheff [19], is available from [17], cf. [10] in the context of present paper.

So, the general fact of (existence of) factorization,

$$\text{Assoc}(\star)(\mathcal{P})(f, g, h) = \diamond(\mathcal{P}, [\![\mathcal{P}, \mathcal{P}]\!])(f, g, h), \quad f, g, h \in A[[\hbar]], \quad (1)$$

is known to the expert community. Indeed, this factorization is immediate from the construction of L_∞ -morphism in [16, Section 6.4].

¹In fact, the morphism \mathcal{F} is a quasi-isomorphism (see [16, Theorem 6.3]), inducing a bijection between the sets of gauge-equivalence classes of Maurer–Cartan elements.

We shall inspect how this mechanism works in practice, i.e., how precisely the \star -product is made associative in its perturbative expansion whenever the bi-vector \mathcal{P} is Poisson, thus satisfying the Jacobi identity $\text{Jac}(\mathcal{P}) := \frac{1}{2}[\mathcal{P}, \mathcal{P}] = 0$. To the same extent as our paper [6] justifies a similar factorization, $[\mathcal{P}, \mathcal{Q}(\mathcal{P})] = \diamond(\mathcal{P}, [\mathcal{P}, \mathcal{P}])$, of the Poisson cocycle condition for universal deformations $\hat{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ of Poisson structures,² we presently motivate the findings in [5] for $\star \bmod \bar{o}(\hbar^3)$, proceeding to the next order $\star \bmod \bar{o}(\hbar^4)$ from [7] (and higher orders, recently available from [3]).³ Let us emphasize that the theoretical constructions and algorithms (contained in the computer-assisted proof scheme under study and in the tools for graph weight calculation) would still work at arbitrarily high orders of expansion $\star \bmod \bar{o}(\hbar^k)$ as $k \rightarrow \infty$. Explicit factorization (1) up to $\bar{o}(\hbar^k)$ helps us build the star-product $\star \bmod \bar{o}(\hbar^k)$ by using a self-starting iterative process, because the Jacobi identity for \mathcal{P} is the only obstruction to the associativity of \star . Specifically, the Kontsevich weights of graphs on fewer vertices (yet with a number of edges such that they do not show up in the perturbative expansion of \star) dictate the coefficients of Leibniz orgraphs in operator \diamond at higher orders in \hbar . These weights in the r.h.s. of (1) constrain the higher-order weights of the Kontsevich orgraphs in the expansion of \star -product itself. This is important also in the context of a number-theoretic open problem about the (ir)rational value $(\text{const} \in \mathbb{Q} \setminus \{0\}) \cdot \zeta(3)^2 / \pi^6 + (\text{const} \in \mathbb{Q})$ of a graph weight at \hbar^7 in \star (see [12] and [3]).

Our paper is structured as follows. First, we fix notation and recall some basic facts from relevant theory. Secondly, we provide three examples which illustrate the work of formality morphism in solving Eq. (1). Specifically, we read the operators $\diamond_k = \diamond \bmod \bar{o}(\hbar^k)$ satisfying

$$\text{Assoc}(\star)(\mathcal{P})(f, g, h) \bmod \bar{o}(\hbar^k) = \diamond_k(\mathcal{P}, [\mathcal{P}, \mathcal{P}])(f, g, h) \quad (1')$$

at $k = 2, 3$, and 4. This corresponds to the expansions $\star \bmod \bar{o}(\hbar^k)$ in [16], [5], and [7], respectively. One can then continue with $k = 5, 6$; these expansions are in [3]. Independently, one can probe such factorizations using other stable formality morphisms: for instance, the ones

²Universal w.r.t. all Poisson brackets on all finite-dimensional affine manifolds, such infinitesimal deformations were pioneered in [15]; explicit examples of these flows $\hat{\mathcal{P}} = \mathcal{Q}(\mathcal{P})$ are given in [4, 6, 8].

³Note that both the approaches – to noncommutative associative \star -products and deformations of Poisson structures – rely on the same calculus of oriented graphs by Kontsevich [13, 14, 15, 16].

which correspond to a different star-product, the weights in which are determined by a logarithmic propagator instead of the harmonic one (see [1, 18]).

2. Two differential graded Lie algebra structures. Let M^r be an r -dimensional affine real manifold (we set $\mathbb{k} = \mathbb{R}$ for simplicity). In the algebra $A := C^\infty(M^r)$ of smooth functions, denote by μ_A (or equivalently, by the dot \cdot) the usual commutative, associative, bi-linear multiplication. The space of formal power series in \hbar over A will be $A[[\hbar]]$ and the \hbar -linear multiplication in it is μ (instead of $\mu_{A[[\hbar]]}$). Consider two differential graded Lie algebra structures. First, we have that the shifted-graded space $T_{\text{poly}}^{\downarrow[1]}(M^r)$ of multivector fields on M^r is equipped with the shifted-graded skew-symmetric Schouten bracket $[[,]]$ (itself bi-linear by construction and satisfying the shifted-graded Jacobi identity); the differential is set to zero. Secondly, the vector space $D_{\text{poly}}^{\downarrow[1]}(M^r)$ of polydifferential operators (linear in each argument but not necessarily skew over the set of arguments or a derivation in any of them) is graded by using the number of arguments m : by definition, let $\text{deg}(\theta(m \text{ arguments})) := m - 1$. For instance, $\text{deg}(\mu_A) = 1$. The Lie algebra structure on $D_{\text{poly}}^{\downarrow[1]}(M^r)$ is the Gerstenhaber bracket $[\cdot, \cdot]_G$; for two homogeneous operators Φ_1 and Φ_2 it equals $[\Phi_1, \Phi_2]_G = \Phi_1 \circ \Phi_2 - (-1)^{\text{deg } \Phi_1 \cdot \text{deg } \Phi_2} \Phi_2 \circ \Phi_1$, where the directed, non-associative insertion product is, by definition

$$\begin{aligned}
 (\Phi_1 \circ \Phi_2)(a_0, \dots, a_{k_1+k_2}) &= \sum_{i=0}^{k_1} (-1)^{ik_2} \Phi_1(a_0 \otimes \dots \otimes a_{i-1} \\
 &\quad \otimes \Phi_2(a_i \otimes \dots \otimes a_{i+k_2}) \otimes a_{i+k_2+1} \otimes \dots \otimes a_{k_1+k_2}).
 \end{aligned}$$

In the above, $\Phi_i: A^{\otimes(k_i+1)} \rightarrow A$ so that $a_j \in A$.

Example 1. The associativity of the product μ_A in the algebra of functions $A = C^\infty(M^r)$ is the statement that

$$\begin{aligned}
 \mu_A^{(1)}(\mu_A^{(2)}(a_0, a_1), a_2) &+ (-1)^{(i=1) \cdot (\text{deg } \mu_A=1)} \mu_A^{(1)}(a_0, \mu_A^{(2)}(a_1, a_2)) \\
 &- (-1)^{(\text{deg } \mu_A^{(1)}=1) \cdot (\text{deg } \mu_A^{(2)}=1)} \{ \mu_A^{(1)}(\mu_A^{(1)}(a_0, a_1), a_2) \\
 &- \mu_A^{(2)}(a_0, \mu_A^{(1)}(a_1, a_2)) \} = 2\{(a_0 \cdot a_1) \cdot a_2 - a_0 \cdot (a_1 \cdot a_2)\} = 0.
 \end{aligned}$$

So, the associator $\text{Assoc}(\mu_A)(a_0, a_1, a_2) = \frac{1}{2}[\mu_A, \mu_A]_G(a_0, a_1, a_2) = 0$ for any $a_j \in A$.

Like $\llbracket \cdot, \cdot \rrbracket$, the Gerstenhaber bracket satisfies the shifted-graded Jacobi identity. The Hochschild differential on $D_{\text{poly}}^{\downarrow[1]}(M^r)$ is $d_H = [\mu_A, \cdot]_G$; indeed, its square vanishes, $d_H^2 = 0$, due to the Jacobi identity for $[\cdot, \cdot]_G$ into which one plugs the equality $[\mu_A, \mu_A]_G = 0$.

3. The Maurer–Cartan elements. In every differential graded Lie algebra with a Lie bracket $[\cdot, \cdot]$, the Maurer–Cartan (MC) elements are solutions of degree 1 for the Maurer–Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0, \quad (2)$$

where d is the differential (equal, we recall, to $d_H = [\mu_A, \cdot]_G$ on $D_{\text{poly}}^{\downarrow[1]}(M^r)$ and zero identically on $T_{\text{poly}}^{\downarrow[1]}(M^r)$). Likewise, the Lie algebra structure $[\cdot, \cdot]$ is the Gerstenhaber bracket $[\cdot, \cdot]_G$ and the Schouten bracket $\llbracket \cdot, \cdot \rrbracket$, respectively.)

Now tensor the degree-one parts of both dgLa structures with $\hbar \cdot \mathbb{k}[[\hbar]]$, i.e., with formal power series starting at \hbar^1 , and, preserving the notation (that is, extending the brackets and the differentials by \hbar -linearity), consider the same Maurer–Cartan equation (2). Let us study its formal power series solutions $\alpha = \hbar^1 \alpha_1 + \dots$.

So far, in the Poisson world we have that the Maurer–Cartan bivectors are formal Poisson structures $0 + \hbar \mathcal{P}_1 + \bar{o}(\hbar)$ satisfying (2), which is $\llbracket \hbar \mathcal{P}_1 + \bar{o}(\hbar), \hbar \mathcal{P}_1 + \bar{o}(\hbar) \rrbracket = 0$ with zero differential. In the world of associative structures, the Maurer–Cartan elements are the tails B in expansions $\star = \mu + B$, so that the associativity equation $[\star, \star]_G = 0$ reads (for $[\mu, \mu]_G = 0$)

$$[\mu, B]_G + \frac{1}{2}[B, B]_G = 0,$$

which is again (2).

4. The L_∞ -morphisms. Our goal is to have (and use) a morphism $T_{\text{poly}}^{\downarrow[1]}(M^r) \rightarrow D_{\text{poly}}^{\downarrow[1]}(M^r)$ which would induce a map that takes Maurer–Cartan elements in the Poisson world to Maurer–Cartan elements in the associative world.

The leading term \mathcal{F}_1 , i.e., the first approximation to the morphism which we consider, is the Hochschild–Kostant–Rosenberg (HKR) map (obviously, extended by linearity),

$$\mathcal{F}: \xi_1 \wedge \dots \wedge \xi_m \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma \xi_{\sigma(1)} \otimes \dots \otimes \xi_{\sigma(m)},$$

which takes a split multi-vector to a polydifferential operator (in fact, an m -vector). More explicitly, we have that

$$\begin{aligned} \mathcal{F}_1: (\xi_1 \wedge \cdots \wedge \xi_m) \\ \mapsto \left(a_1 \otimes \cdots \otimes a_m \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^\sigma \prod_{i=1}^m \xi_{\sigma(i)}(a_i) \right), \end{aligned} \quad (3)$$

here $a_j \in A := C^\infty(M^r)$. For zero-vectors $h \in A$, one has $\mathcal{F}_1: h \mapsto (1 \mapsto h)$.

Claim 1 ([16, Section 4.6.2]). *The leading term, map \mathcal{F}_1 , is not a Lie algebra morphism (which, if it were, would take the Schouten bracket of multivectors to the Gerstenhaber bracket of polydifferential operators).*

Proof (by counterexample). Take two bi-vectors; their Schouten bracket is a tri-vector, but the Gerstenhaber bracket of two bi-vectors is a differential operator which has homogeneous components of differential orders (2,1,1) and (1,1,2). And in general, those components do not vanish. \square

The construction of not a single map \mathcal{F}_1 but of an entire collection $\mathcal{F} = \{\mathcal{F}_n, n \geq 1\}$ of maps does nevertheless yield a well-defined mapping of the Maurer–Cartan elements from the two differential graded Lie algebras.⁴

Theorem 2 ([16, Main Theorem]). *There exists a collection of linear maps $\mathcal{F} = \{\mathcal{F}_n: T_{\text{poly}}^{\downarrow[1]}(M^r)^{\otimes n} \rightarrow D_{\text{poly}}^{\downarrow[1]}(M^r), n \geq 1\}$ such that \mathcal{F}_1 is the HKR map (3) and \mathcal{F} is an L_∞ -morphism of the two differential graded Lie algebras: $(T_{\text{poly}}^{\downarrow[1]}(M^r), [\cdot, \cdot], d = 0) \rightarrow (D_{\text{poly}}^{\downarrow[1]}(M^r), [\cdot, \cdot]_G, d_H = [\mu_A, \cdot]_G)$. Namely,*

- (1) each component \mathcal{F}_n is homogeneous of own grading $1 - n$,
- (2) each morphism \mathcal{F}_n is graded skew-symmetric, i.e.,

$$\mathcal{F}_n(\dots, \xi, \eta, \dots) = -(-1)^{\deg(\xi) \cdot \deg(\eta)} \mathcal{F}_n(\dots, \eta, \xi, \dots)$$

for ξ, η homogeneous,

⁴The name ‘formality’ for the collection \mathcal{F} of maps is motivated by Theorem 4.10 in [16] and by the main theorem in *loc. cit.*

(3) for each $n \geq 1$ and (homogeneous) multivectors $\xi_1, \dots, \xi_n \in T_{\text{poly}}^{\downarrow[1]}(M^r)$, we have that (cf. [11, Section 3.6])

$$\begin{aligned} & d_H(\mathcal{F}_n(\xi_1, \dots, \xi_n)) - (-1)^{n-1} \sum_{i=1}^n (-1)^u \mathcal{F}_n(\xi_1, \dots, d\xi_i, \dots, \xi_n) \\ & + \frac{1}{2} \sum_{\substack{p+q=n \\ p, q > 0}} \sum_{\sigma \in S_{p,q}} (-1)^{pn+t} [\mathcal{F}_p(\xi_{\sigma(1)}, \dots, \xi_{\sigma(p)}), \end{aligned} \quad (4)$$

$$\begin{aligned} & \mathcal{F}_q(\xi_{\sigma(p+1)}, \dots, \xi_{\sigma(n)})]_G \\ & = (-1)^n \sum_{i < j} (-1)^s \mathcal{F}_{n-1}([\xi_i, \xi_j], \xi_1, \dots, \widehat{\xi}_i, \dots, \widehat{\xi}_j, \dots, \xi_n). \end{aligned} \quad (5)$$

In the above formula, σ runs through the set of (p, q) -shuffles, i.e., all permutations $\sigma \in S_n$ such that $\sigma(1) < \dots < \sigma(p)$ and independently $\sigma(p+1) < \dots < \sigma(n)$; the exponents t and s are the numbers of transpositions of odd elements which we count when passing (t) from $(\mathcal{F}_p, \mathcal{F}_q, \xi_1, \dots, \xi_n)$ to $(\mathcal{F}_p, \xi_{\sigma(1)}, \dots, \xi_{\sigma(p)}, \mathcal{F}_q, \xi_{\sigma(p+1)}, \dots, \xi_{\sigma(n)})$, and (s) from (ξ_1, \dots, ξ_n) to $(\xi_i, \xi_j, \xi_1, \dots, \widehat{\xi}_1, \dots, \widehat{\xi}_j, \dots, \xi_n)$.⁵

Remark 1. Let $n := 1$, then equality (5) in Theorem 2 is

$$\begin{aligned} & d_H \circ \mathcal{F}_1 - (-1)^{1-1} \cdot (-1)^{u=0} \text{ from } (d, \xi_1) \rightarrow (d, \xi_1) F_1 \circ d = 0 \\ & \iff d_H \circ \mathcal{F}_1 = \mathcal{F}_1 \circ d, \end{aligned}$$

whence \mathcal{F}_1 is a morphism of complexes.

• Let $n := 2$, then for any homogeneous multivectors ξ_1 and ξ_2 ,

$$\begin{aligned} & \mathcal{F}_1([\xi_1, \xi_2]) - [\mathcal{F}_1(\xi_1), \mathcal{F}_1(\xi_2)]_G = d_H(\mathcal{F}_2(\xi_1, \xi_2)) \\ & + \mathcal{F}_2((d=0)(\xi_1), \xi_2) + (-1)^{\deg \xi_1} \mathcal{F}_2(\xi_1, (d=0)(\xi_2)), \end{aligned}$$

so that in our case \mathcal{F}_1 is “almost” a Lie algebra morphism but for the discrepancy which is controlled by the differential of the (value of the) succeeding map \mathcal{F}_2 in the sequence $\mathcal{F} = \{\mathcal{F}_n, n \geq 1\}$. Big formula (5) shows in precisely which sense this is also the case for higher homotopies \mathcal{F}_n , $n \geq 2$ in the L_∞ -morphism \mathcal{F} . Indeed, an L_∞ -morphism is a map between dgLas which, in every term, almost preserves the bracket up to a homotopy $d_H \circ \{\dots\}$ provided by the next term.

⁵The exponent u is not essential for us now because the differential d on $T_{\text{poly}}^{\downarrow[1]}(M^r)$ is set equal to zero identically, so that the entire term with u does not contribute (recall \mathcal{F}_n is linear).

Even though neither \mathcal{F}_1 nor the entire collection $\mathcal{F} = \{\mathcal{F}_n, n \geq 1\}$ is a dgLa morphism, their defining property (5) guarantees that \mathcal{F} gives us a well defined mapping of the Maurer–Cartan elements (which, we recall, are formal Poisson bi-vectors and tails B of associative (non)commutative multiplications $\star = \mu + B$ on $A[[\hbar]]$, respectively).

Corollary 3. *The natural \hbar -linear extension of \mathcal{F} , now acting on the space of formal power series in \hbar with coefficients in $T_{\text{poly}}^{\downarrow[1]}(M^r)$ and with zero free term by the rule*

$$\xi \mapsto \sum_{n \geq 1} \frac{1}{n!} \mathcal{F}_n(\xi, \dots, \xi),$$

takes the Maurer–Cartan elements $\tilde{\mathcal{P}} = \hbar\mathcal{P} + \bar{o}(\hbar)$ to the Maurer–Cartan elements $B = \sum_{n \geq 1} \frac{1}{n!} \mathcal{F}_n(\tilde{\mathcal{P}}, \dots, \tilde{\mathcal{P}}) = \hbar\tilde{\mathcal{P}} + \bar{o}(\hbar)$. (Note that the HKR map \mathcal{F}_1 , extended by \hbar -linearity, still is an identity mapping on multi-vectors, now viewed as special polydifferential operators.)

In plain terms, for a bivector \mathcal{P} itself Poisson, formal Poisson structures $\tilde{\mathcal{P}} = \hbar\mathcal{P} + \bar{o}(\hbar)$ satisfying $[[\tilde{\mathcal{P}}, \tilde{\mathcal{P}}]] = 0$ are mapped by \mathcal{F} to the tails $B = \hbar\mathcal{P} + \bar{o}(\hbar)$ such that $\star = \mu + B$ is associative and its leading order deformation term is a given Poisson structure \mathcal{P} .

Proof of Corollary 3. Let us presently consider the restricted case when $\tilde{\mathcal{P}} = \hbar\mathcal{P}$, without any higher order tail $\bar{o}(\hbar)$. The Maurer–Cartan equation in $D_{\text{poly}}^{\downarrow[1]}(M^r) \otimes \hbar\mathbb{k}[[\hbar]]$ is $[\mu, B]_G + \frac{1}{2}[B, B]_G = 0$, where

$$B = \sum_{n \geq 1} \frac{1}{n!} \mathcal{F}_n(\tilde{\mathcal{P}}, \dots, \tilde{\mathcal{P}})$$

and we let $\tilde{\mathcal{P}} = \hbar\mathcal{P}$, so that $B = \sum_{n \geq 1} \frac{\hbar^n}{n!} \mathcal{F}_n(\mathcal{P}, \dots, \mathcal{P})$. Let us plug this formal power series in the l.h.s. of the above equation. Equating the coefficients at powers \hbar^n and multiplying by $n!$, we obtain the expression

$$[\mu, \mathcal{F}_n(\mathcal{P}, \dots, \mathcal{P})]_G + \frac{1}{2} \sum_{\substack{p+q=n \\ p, q > 0}} \frac{n!}{p!q!} [\mathcal{F}_p(\mathcal{P}, \dots, \mathcal{P}), \mathcal{F}_q(\mathcal{P}, \dots, \mathcal{P})]_G.$$

It is readily seen that now the sum $\sum_{\sigma \in S_{p,q}}$ in (5) over the set of (p, q) -shuffles of $n = p+q$ identical copies of an object \mathcal{P} just counts the number

of ways to pick p copies going first in an ordered string of length n . To balance the signs, we note at once that by item (2) in Theorem 2, see above, $\mathcal{F}_p(\dots, \mathcal{P}^{(\alpha)}, \mathcal{P}^{(\alpha+1)}, \dots) = \mathcal{F}_p(\dots, \mathcal{P}^{(\alpha+1)}, \mathcal{P}^{(\alpha)}, \dots)$ because bi-vector's shifted degree is $+1$, so that no (p, q) -shuffles of $(\mathcal{P}, \dots, \mathcal{P})$ contribute with any sign factor. The only sign contribution that remains stems from the symbol \mathcal{F}_q of grading $1 - q$ transported along p copies of odd-degree bi-vector \mathcal{P} ; this yields $t = (1 - q) \cdot p$ and $(-1)^{pn+t} = (-1)^{p \cdot (p+q)} \cdot (-1)^{(1-q) \cdot p} = (-1)^{p \cdot (p+1)} = +$.

The left-hand side of the Maurer–Cartan equation (2) is, by the above, expressed by the left-hand side of (5) which the L_∞ -morphism \mathcal{F} satisfies. In the right-hand side of (5), we now obtain (with, actually, whatever sign factors) the values of linear mappings \mathcal{F}_{n-1} at twice the Jacobiator $[[\tilde{\mathcal{P}}, \tilde{\mathcal{P}}]]$ as one of the arguments. All these values are therefore zero, which implies that the right-hand side of the Maurer–Cartan equation (2) vanishes, so that the tail B indeed is a Maurer–Cartan element in the Hochschild cochain complex (in other words, the star-product $\star = \mu + B$ is associative).

This completes the proof in the restricted case when $\tilde{\mathcal{P}} = \hbar\mathcal{P}$. Formal power series bi-vectors $\tilde{\mathcal{P}} = \hbar\mathcal{P} + \bar{o}(\hbar)$ refer to the same count of signs as above, yet the calculation of multiplicities at \hbar^n (for all possible lexicographically ordered p - and q -tuples of n arguments) is an extensive exercise in combinatorics. \square

Corollary 4. *Because the right-hand side of (2) in the above reasoning is determined by the right-hand side of (5), we read off an explicit formula of the operator \diamond that solves the factorization problem*

$$\text{Assoc}(\star)(\mathcal{P})(f, g, h) = \diamond(\mathcal{P}, [[\mathcal{P}, \mathcal{P}]]) (f, g, h), \quad f, g, h \in A[[\hbar]]. \quad (1)$$

Indeed, the operator is

$$\diamond = 2 \cdot \sum_{n \geq 1} \frac{\hbar^n}{n!} \cdot c_n \cdot \mathcal{F}_{n-1}([\mathcal{P}, \mathcal{P}], \mathcal{P}, \dots, \mathcal{P}). \quad (6)$$

But what are the coefficients $c_n \in \mathbb{R}$ equal to? Let us find it out.

5. Explicit construction of the formality morphism \mathcal{F} . The first explicit formula for the formality morphism \mathcal{F} which we study in this paper was discovered by Kontsevich in [16, Section 6.4], providing an expansion of every term \mathcal{F}_n using weighted decorated graphs:

$$\mathcal{F} = \left\{ \mathcal{F}_n = \sum_{m \geq 0} \sum_{\Gamma \in G_{n,m}} W_\Gamma \cdot \mathcal{U}_\Gamma \right\}.$$

Here Γ belongs to the set $G_{n,m}$ of oriented graphs on n internal vertices (i.e., arrowtails), m sinks (from which no arrows start), and $2n+m-2 \geq 0$ edges, such that at every internal vertex there is an ordering of outgoing edges. By decorating each edge with a summation index that runs from 1 to r , by viewing each edge as a derivation $\partial/\partial x^\alpha$ of the arrowhead vertex content, by placing n multivectors from an ordered tuple of arguments of \mathcal{F}_n into the respective vertices, now taking the sum over all indices of the resulting products of the content of vertices, and skew-symmetrizing over the n -tuple of (shifted-)graded multivectors, we realize each graph at hand as a polydifferential operator $T_{\text{poly}}^{\downarrow[1]}(M^r)^{\otimes n} \rightarrow D_{\text{poly}}^{\downarrow[1]}(M^r)$ whose arguments are multivectors. Note that the value $\mathcal{F}_n(\xi_1, \dots, \xi_n)$ itself is, by construction, a differential operator w.r.t. the contents of sinks of the graph Γ . All of this is discussed in detail in [13, 14, 15, 16] or [4, 5, 7].

The formula for the harmonic weights $W_\Gamma \in \mathbb{R}$ is given in [16, Section 6.2]; it is

$$W_\Gamma = \left(\prod_{k=1}^n \frac{1}{\#\text{Star}(k)!} \right) \cdot \frac{1}{(2\pi)^{2n+m-2}} \int_{\bar{C}_{n,m}^+} \bigwedge_{e \in E_\Gamma} d\phi_e,$$

where $\#\text{Star}(k)$ is the number of edges *starting* from vertex k , $d\phi_e$ is the “harmonic angle” differential 1-form associated to the edge e , and the integration domain $\bar{C}_{n,m}^+$ is the connected component of $\bar{C}_{n,m}$ which is the closure of configurations where points q_j , $1 \leq j \leq m$ on \mathbb{R} are placed in increasing order: $q_1 < \dots < q_m$. For convenience, let us also define

$$w_\Gamma = \left(\prod_{k=1}^n \#\text{Star}(k)! \right) \cdot W_\Gamma.$$

The convenience is that by summing over labelled graphs Γ , we actually sum over the equivalence classes $[\Gamma]$ (i.e., over unlabeled graphs) with multiplicities $(w_\Gamma/W_\Gamma) \cdot n!/\#\text{Aut}(\Gamma)$. The division by the volume $\#\text{Aut}(\Gamma)$ of the symmetry group eliminates the repetitions of graphs which differ only by a labeling of vertices but, modulo such, do not differ by the labeling of ordered edge tuples (issued from the vertices which are matched by a symmetry).

Let us remember that the integrand in the formula of W_Γ is defined in terms of the harmonic propagator; other propagators (e.g., logarithmic, or other members of the family interpolating between harmonic and

logarithmic [1, 18]) would give other formality morphisms. A path integral realization of the \star -product itself and of the components \mathcal{F}_n in the formality morphism is proposed in [10].

To calculate the graph weights W_Γ in practice, we employ methods which were outlined in [7], as well as [12, Appendix E] (about the cyclic weight relations), and [3] that puts those real values in the context of Riemann multiple zeta functions and polylogarithms.⁶ Examples of such decorated oriented graphs Γ and their weights W_Γ will be given in the next section.

5.1. Sum over equivalence classes. The sum in Kontsevich's formula is over *labeled* graphs: internal vertices are numbered from 1 to n , and the edges starting from each internal vertex k are numbered from 1 to $\#\text{Star}(k)$. Under a re-labeling $\sigma: \Gamma \mapsto \Gamma^\sigma$ of internal vertices and edges it is seen from the definitions that the operator \mathcal{U}_Γ and the weight W_Γ enjoy the same skew-symmetry property (as remarked in [16, Section 6.5]), whence $W_\Gamma \cdot \mathcal{U}_\Gamma = W_{\Gamma^\sigma} \cdot \mathcal{U}_{\Gamma^\sigma}$. It follows that the sum over labeled graphs can be replaced by a sum over equivalence classes $[\Gamma]$ of graphs, modulo labeling of internal vertices and edges. For this it remains to count the size of an equivalence class: the edges can be labeled in $\prod_{k=1}^n \#\text{Star}(k)!$ ways, while the n internal vertices can be labeled in $n!/\#\text{Aut}(\Gamma)$ ways.

Example 2. The double wedge on two ground vertices has only *one* possible labeling of vertices, due to the automorphism that interchanges the wedges.

We denote by $M_\Gamma = (\prod_{k=1}^n \#\text{Star}(k)!) \cdot n!/\#\text{Aut}(\Gamma)$ the *multiplicity* of the graph Γ , and let $\bar{G}_{n,m}$ be the set of equivalence classes $[\Gamma]$ modulo labeling of $\Gamma \in G_{n,m}$. The formula for the formality morphism can then be rewritten as

$$\mathcal{F} = \left\{ \mathcal{F}_n = \sum_{m \geq 0} \sum_{[\Gamma] \in \bar{G}_{n,m}} M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma \right\};$$

here the Γ in $M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma$ is *any* representative of $[\Gamma]$. Any ambiguity in signs (due to the choice of representative) in the latter two factors is cancelled in their product. Note that the factor $(\prod_{k=1}^n \#\text{Star}(k)!)$ in M_Γ kills the corresponding factor in W_Γ , as remarked in [16, Section 6.5].

⁶It is the values w_Γ instead of W_Γ which are calculated by software [3].

5.2. The coefficient of a graph in the \star -product. The \star -product associated with a Poisson structure \mathcal{P} is given by Corollary 3:

$$\begin{aligned} \star &= \mu + \sum_{n \geq 1} \frac{\hbar^n}{n!} \mathcal{F}_n(\mathcal{P}, \dots, \mathcal{P}) \\ &= \mu + \sum_{n \geq 1} \frac{\hbar^n}{n!} \sum_{[\Gamma] \in \bar{G}_{n,2}} M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma(\mathcal{P}, \dots, \mathcal{P}). \end{aligned}$$

For a graph $\Gamma \in G_{n,2}$ such that each internal vertex has two outgoing edges (these are the only graphs that contribute, because we insert bi-vectors) we have $M_\Gamma = 2^n \cdot n! / \#\text{Aut}(\Gamma)$. In total, the coefficient of $\mathcal{U}_\Gamma(\mathcal{P}, \dots, \mathcal{P})$ at \hbar^n is $2^n / \#\text{Aut}(\Gamma) \cdot W_\Gamma = w_\Gamma / \#\text{Aut}(\Gamma)$. The skew-symmetrization *without prefactor* of bi-vector coefficients in $\mathcal{U}_\Gamma(\mathcal{P}, \dots, \mathcal{P})$ provides an extra factor 2^n .

Example 3 (at \hbar^1). The coefficient of the wedge graph is $1/2$ and the operator is $2\mathcal{P}$, hence we recover \mathcal{P} .

5.3. The coefficient of a Leibniz graph in the associator. The factorizing operator \diamond for $\text{Assoc}(\star)$ is given by Corollary 4:

$$\begin{aligned} \diamond &= 2 \cdot \sum_{n \geq 1} \frac{\hbar^n}{n!} \cdot c_n \cdot \mathcal{F}_{n-1}([\mathcal{P}, \mathcal{P}], \mathcal{P}, \dots, \mathcal{P}) \\ &= 2 \cdot \sum_{n \geq 1} \frac{\hbar^n}{n!} \cdot c_n \cdot \sum_{[\Gamma] \in \bar{G}_{n-1,3}} M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma([\mathcal{P}, \mathcal{P}], \mathcal{P}, \dots, \mathcal{P}). \end{aligned}$$

For a graph $\Gamma \in G_{n-1,3}$ where one internal vertex has three outgoing edges and the rest have two, we have $M_\Gamma = 3! \cdot 2^{n-2} \cdot (n-1)! / \#\text{Aut}(\Gamma)$. In total, the coefficient of $\mathcal{U}_\Gamma([\mathcal{P}, \mathcal{P}], \mathcal{P}, \dots, \mathcal{P})$ at \hbar^n is

$$\left[2 \cdot \frac{1}{n!} \cdot c_n \cdot 3! \cdot 2^{n-2} \cdot (n-1)! \right] \cdot \frac{W_\Gamma}{\#\text{Aut}(\Gamma)} = \left[2 \cdot \frac{c_n}{n} \right] \cdot \frac{w_\Gamma}{\#\text{Aut}(\Gamma)}.$$

The skew-symmetrization *without prefactor* of bi- and tri-vector coefficients in the operator $\mathcal{U}_\Gamma([\mathcal{P}, \mathcal{P}], \mathcal{P}, \dots, \mathcal{P})$ provides an extra factor $3! \cdot 2^{n-2}$.

Example 4 (at \hbar^2). The coefficient of the tripod graph is $c_2 \cdot \frac{1}{3!}$ and the operator is $3! \cdot [\mathcal{P}, \mathcal{P}]$, hence we recover $c_2[\mathcal{P}, \mathcal{P}] = \frac{2}{3} \text{Jac}(\mathcal{P})$. (The right-hand side is known from the associator, e.g., from [5].) This yields $c_2 = 1/3$. In addition, we see that the HKR map \mathcal{F}_1 acts here by the identity on $[\mathcal{P}, \mathcal{P}]$.

In the next section, we shall find that at \hbar^n , the coefficients of our Leibniz graphs (with $\text{Jac}(\mathcal{P})$ inserted instead of $[[\mathcal{P}, \mathcal{P}]]$) are

$$\frac{[[\mathcal{P}, \mathcal{P}]]}{\text{Jac}(\mathcal{P})} \cdot \left[3! \cdot 2^{n-2} \right] \cdot \left[2 \cdot \frac{c_n}{n} \right] \cdot \frac{w_\Gamma}{\#\text{Aut}(\Gamma)} = 2^n \cdot \frac{w_\Gamma}{\#\text{Aut}(\Gamma)},$$

so $3! \cdot 2^n \cdot \frac{c_n}{n} = 2^n$. We deduce that $c_n = n/3! = n/6$ in all our experiments.

Conjecture. *For all $n \geq 2$, the coefficients in (6) are $c_n = n/3! = n/6$ (hence, the coefficients of markers Γ for equivalence classes $[\Gamma]$ of the Leibniz graphs in (6) are $2^n \cdot w_\Gamma / \#\text{Aut}(\Gamma)$), although it still remains to be explained how exactly this follows from the L_∞ condition (5).*

6. Examples. Let \mathcal{P} be a Poisson bi-vector on an affine manifold M^r . We inspect the associativity of the star-product $\star = \mu + \sum_{n \geq 1} \frac{\hbar^n}{n!} \mathcal{F}_n(\mathcal{P}, \dots, \mathcal{P})$ given by Corollary 3 by illustrating the work of the factorization mechanism from Corollary 4. The powers of deformation parameter \hbar provide a natural filtration $\hbar^2 \cdot \mathbf{A}^{(2)} + \hbar^3 \cdot \mathbf{A}^{(3)} + \hbar^4 \cdot \mathbf{A}^{(4)} + \bar{o}(\hbar^4)$ so that we verify the vanishing of $\text{Assoc}(\star)(\mathcal{P})(\cdot, \cdot, \cdot) \bmod \bar{o}(\hbar^4)$ for $\star \bmod \bar{o}(\hbar^4)$ order by order.

At \hbar^0 there is nothing to do (indeed, the usual multiplication is associative). All contribution to the associator of \star at \hbar^1 cancels out because the leading deformation term $\hbar\mathcal{P}$ in the star-product $\star = \mu + \hbar\mathcal{P} + \bar{o}(\hbar)$ is a bi-derivation. The order \hbar^2 was discussed in Example 4 in Section 5.3.

Remark 2. In all our reasoning at any order $\hbar^{n \geq 2}$, the Jacobiator in Leibniz graphs is expanded (w.r.t. the three cyclic permutations of its arguments) into the Kontsevich graphs, built of wedges, in such a way that the internal edge, connecting two Poisson bi-vectors in $\text{Jac}(\mathcal{P})$, is proclaimed Left by construction. Specifically, the algorithm to expand each Leibniz graphs is as follows:

1. Split the trivalent vertex with ordered targets (a, b, c) into two wedges: the first wedge stands on a and b (in that order), and the second wedge stands on the first wedge-top and c (in that order), so that the internal edge of the Jacobiator is marked Left, preceding the Right edge towards c .
2. Re-direct the edges (if any) which had the tri-valent vertex as their target, to one of the wedge-tops; take the sum over all possible combinations (this is the iterated Leibniz rule).

3. Take the sum over cyclic permutations of the targets of the edges which (initially) have (a, b, c) as their targets (this is the expansion of the Jacobiator).

6.1. The order \hbar^3 . To factorize the next order expansion of the associator, $\text{Assoc}(\star)(\mathcal{P}) \bmod \bar{o}(\hbar^3) = \hbar^2 \cdot \mathbf{A}^{(2)} + \hbar^3 \cdot \mathbf{A}^{(3)} + \bar{o}(\hbar^3)$, at \hbar^3 in the operator \diamond in the right-hand side of (1), we use graphs on $n - 1 = 2$ vertices, $m = 3$ sinks, and $2(n - 1) + m - 2 = 5$ edges.

At \hbar^3 , two internal vertices in the Leibniz graphs in the r.h.s. of factorization (1) are manifestly different: one vertex, containing the bi-vector \mathcal{P} , is a source of two outgoing edges, and the other, with $[[\mathcal{P}, \mathcal{P}]$, of three. Therefore, the automorphism groups of such Leibniz graphs (under relabellings of internal vertices of the same valency but with the sinks fixed) can only be trivial, i.e., one-element. (This will not necessarily be the case of Leibniz graphs on $(n - 2) + 1$ internal vertices at $\hbar^{\geq 4}$: compare Examples 8 vs 9 on p. 39 below, where the weight of a graph is divided further by the size of its automorphism group.)

The coefficient of \hbar^3 in the factorizing operator \diamond ,

$$\text{coeff}(\diamond, \hbar^3) = 2 \cdot \frac{1}{3!} \cdot c_3 \cdot \sum_{[\Gamma] \in \bar{G}_{2,3}} M_\Gamma \cdot W_\Gamma \cdot \mathcal{U}_\Gamma([[P, P], P, \dots, P]),$$

expands into a sum of ≤ 24 admissible oriented graphs. Indeed, there are six essentially different oriented graph topologies, filtered by the number of sinks on which the tri-vector $[[P, P]]$ and bi-vector P stand; the ordering of sinks in the associator then yields $3 + 3 + 3 \times 2 + 3 \times 2 + 3 = 24$ oriented graphs. (None of them is a zero orgraph.) As we recall from [5], only thirteen of them actually occur with nonzero coefficients in the term $\mathbf{A}^{(3)} \sim \hbar^3$ in $\text{Assoc}(\star)(\mathcal{P})$, the remaining eleven have zero weights.⁷ The weights of 15 relevant oriented Leibniz graphs from [5] are listed in Table 1.⁸

Here we let by definition

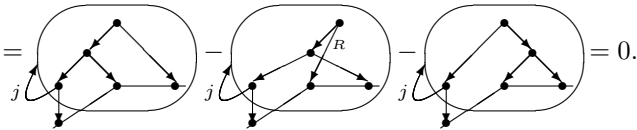
$$I_f := \partial_j(\text{Jac}(\mathcal{P})(\mathcal{P}^{ij}, g, h)) \partial_i f$$

⁷Yet, these seemingly ‘unnecessary’ graphs can contribute to the cyclic weight relations (see [12, Appendix E]): zero values of some of such graph weights can simplify the system of linear relations between nonzero weights.

⁸To get the values, one uses the software [3] by Banks–Panzer–Pym or, independently, exact symbolic or approximate numeric methods from [7], also taking into account the cyclic weight relations from [12, Appendix E].

Table 1. Weights w_Γ of oriented Leibniz graphs Γ in coeff (\diamond, \hbar^3) .

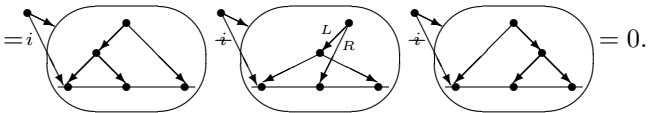
$(S_f)_{221}$	$=$	$[01; 012]$	$\frac{1}{12}$	$(S_g)_{122}$	$=$	$[12; 012]$	$\frac{1}{12}$
$(I_f)_{112}$	$=$	$[02; 312]$	$\frac{1}{48}$	$(I_g)_{112}$	$=$	$[12; 032]$	$\frac{1}{48}$
$(S_f)_{211}$	$=$	$[04; 012]$	$\frac{1}{24}$	$(I_g)_{211}$	$=$	$[10; 032]$	$\frac{-1}{48}$
$(I_f)_{111}$	$=$	$[04; 312]$	$\frac{1}{48}$	$(I_h)_{111}$	$=$	$[24; 013]$	$\frac{-1}{48}$
$(S_g)_{111}$	$=$	$[14; 012]$	0	$(I_f)_{121}$	$=$	$[01; 312]$	$\frac{1}{24}$
$(S_h)_{212}$	$=$	$[20; 012]$	$\frac{-1}{12}$				
$(S_h)_{112}$	$=$	$[24; 012]$	$\frac{-1}{24}$				
$(I_h)_{211}$	$=$	$[20; 013]$	$\frac{-1}{48}$				
$(I_g)_{111}$	$=$	$[14; 032]$	0				
$(I_h)_{121}$	$=$	$[21; 013]$	$\frac{-1}{24}$				



Likewise, $I_g := \partial_j(\text{Jac}(\mathcal{P})(f, \mathcal{P}^{ij}, h)) \cdot \partial_i g$ and $I_h := \partial_j(\text{Jac}(\mathcal{P})(f, g, \mathcal{P}^{ij}) \cdot \partial_i h$, respectively.⁹

We also set

$$S_f := \mathcal{P}^{ij} \partial_j \text{Jac}(\mathcal{P})(\partial_i f, g, h)$$



Similarly, we let $S_g := \mathcal{P}^{ij} \partial_j \text{Jac}(\mathcal{P})(f, \partial_i g, h) = 0$ and define $S_h := \mathcal{P}^{ij} \partial_j \text{Jac}(\mathcal{P})(f, g, \partial_i h) = 0$. Note that after all the Leibniz rules are reworked, each of the six graphs I_f, \dots, S_h – with the Jacobiator $\text{Jac}(\mathcal{P}) = \frac{1}{2}[[\mathcal{P}, \mathcal{P}]]$ at the tri-valent vertex – splits into several homogeneous components, like $(I_f)_{111}$ or $(S_h)_{212}$; taken alone, each of the components encodes a zero polydifferential operator of respective orders.

⁹In [5], the indices i and j were interchanged in the definitions of both I_g and I_h (compare the expression of I_f); that typo is now corrected in the above formulae.

Claim 5. *Multiplied by a common factor $(\llbracket \mathcal{P}, \mathcal{P} \rrbracket / \text{Jac}(\mathcal{P})) \cdot 2^{k-1} = 2 \cdot 4 = 8$, the Leibniz graph weights from Table 1 at \hbar^3 fully reproduce the factorization which was found in the main Claim in [5], namely:*

$$\begin{aligned} \mathbf{A}_{221}^{(3)} &= \frac{2}{3}(S_f)_{221}, & \mathbf{A}_{122}^{(3)} &= \frac{2}{3}(S_g)_{122}, & \mathbf{A}_{212}^{(3)} &= -\frac{2}{3}(S_h)_{212}, \\ \mathbf{A}_{111}^{(3)} &= \frac{1}{6}(I_f - I_h)_{111}, & \mathbf{A}_{112}^{(3)} &= \left(\frac{1}{6}I_f + \frac{1}{6}I_g - \frac{1}{3}S_h\right)_{112}, \\ \mathbf{A}_{121}^{(3)} &= \frac{1}{3}(I_f - I_h)_{121}, & \mathbf{A}_{211}^{(3)} &= \left(\frac{1}{3}S_f - \frac{1}{6}I_g - \frac{1}{6}I_h\right)_{211}. \end{aligned}$$

Otherwise speaking, the sum of these Leibniz oriented graphs with these weights (times $2 \cdot 4 = 8$), when expanded into the sum of 39 weighted Kontsevich graphs (built only of wedges), equals identically the \hbar^3 -proportional term in the associator $\text{Assoc}(\star)(\mathcal{P})(f, g, h)$.

Proof scheme. The encodings of weighted Kontsevich-graph expansions of the homogeneous components of the weighted Leibniz graphs I_f, \dots, S_h , which show up in the associator at \hbar^3 and which are processed according to the algorithm in Remark 2, are listed in Appendix A. Reducing that collection modulo skew symmetry at internal vertices, we reproduce, as desired, the entire term $\mathbf{A}^{(3)}$ in the expansion $\hbar^2 \cdot \mathbf{A}^{(2)} + \hbar^3 \cdot \mathbf{A}^{(3)} + \bar{o}(\hbar^3)$ of the associator $\text{Assoc}(\star)(\mathcal{P}) \bmod \bar{o}(\hbar^3)$. \square

Three examples, corresponding to the leftmost column of equalities in Claim 5, illustrate this scheme at order \hbar^3 . The three cases differ in that for $\mathbf{A}_{221}^{(3)}$ in Example 5, there is just one Leibniz graph without any arrows acting on the Jacobiator vertex. In the other Example 6 for $\mathbf{A}_{121}^{(3)}$, there are two Leibniz graphs still without Leibniz-rule actions on the Jacobiators in them, so that we aim to show how similar terms are collected.¹⁰ Finally, in Example 7 about $\mathbf{A}_{111}^{(3)}$ there are two Leibniz graphs with one Leibniz rule action per either graph: an arrow targets the two internal vertices in the Jacobiator.

Example 5. Take the Leibniz graph $(S_f)_{221} = [01; 012]$. Its weight is $1/12$. Multiplying the Leibniz graph by 8 times its weight and expanding

¹⁰To collect and compare the Kontsevich orgraphs (built of wedges, i.e., ordered edge pairs issued from internal vertices), we can bring every such graph to its normal form, that is, represent it using the *minimal* base- $(\# \text{ sinks} + \# \text{ internal vertices})$ number, encoding the graph as the list of ordered pairs of target vertices, by running over all the relabellings of internal vertices. (The labelling of ordered sinks is always $0 \prec 1 \prec \dots \prec m - 1$.)

the Jacobiator (there are no Leibniz rules to expand) yields the sum of three Kontsevich graphs: $\frac{2}{3}([01; 01; 42] + [01; 12; 40] + [01; 20; 41])$. This is identically equal to the differential order $(2, 2, 1)$ homogeneous part $A_{221}^{(3)}$ of $\text{Assoc}(\star)(\mathcal{P})$ at \hbar^3 . For instance, these terms are listed in [7, Appendix D].

Example 6. Take the Leibniz graphs $(I_f)_{121} = [01; 312]$ and $(I_h)_{121} = [21; 013]$. Their weights are $1/24$ and $-1/24$, respectively; multiply them by 8. Expanding the Jacobiator in the linear combination $\frac{1}{3}(I_f - I_h)_{121}$ yields the sum of Kontsevich graphs $\frac{1}{3}([01; 31; 42] + [01; 12; 43] + [01; 23; 41] - [21; 01; 43] - [21; 13; 40] - [21; 30; 41])$. The two Leibniz graphs have a Kontsevich graph in common: $[01; 12; 43] = [21; 01; 43]$ (recall that internal vertex labels can be permuted at no cost and the swap $L \rightleftharpoons R$ at a wedge costs a minus sign). This gives one cancellation; the remaining four terms equal $A_{121}^{(3)}$ as listed in [7, Appendix D].

Example 7. Take the Leibniz graphs $(I_f)_{111} = [04; 312]$ and $(I_h)_{111} = [24; 013]$. Their weights are $1/48$ and $-1/48$, respectively; multiply them by 8. Expanding the Jacobiator and the Leibniz rule in the linear combination $\frac{1}{6}(I_f - I_h)_{111}$ yields the sum of Kontsevich graphs:

$$\begin{aligned} & \frac{1}{6}([04; 31; 42] + [04; 12; 43] + [04; 23; 41] + [05; 31; 42] \\ & + [05; 12; 43] + [05; 23; 41] - [24; 01; 43] - [24; 13; 40] \\ & - [24; 30; 41] - [25; 01; 43] - [25; 13; 40] - [25; 30; 41]). \end{aligned}$$

Two pairs of graphs cancel; namely $[05; 31; 42] = [25; 30; 41]$ and the pair $[05; 23; 41] = [25; 13; 40]$. The remaining eight terms equal $A_{111}^{(3)}$ as listed in [7, Appendix D].

6.2. The order \hbar^4 . Let us proceed with the term $A^{(4)}$ at \hbar^4 in $\text{Assoc}(\star)(\mathcal{P})(\cdot, \cdot, \cdot) \bmod \bar{o}(\hbar^4)$. The numbers of Kontsevich oriented graphs in the star-product expansion grow as fast as

$$\begin{aligned} \star &= \hbar^0 \cdot (\# \text{graphs} = 1) + \hbar^1 \cdot (\# = 1) + \hbar^2 \cdot (\# = 4) + \hbar^3 \cdot (\# = 13) \\ &+ \hbar^4 \cdot (\# = 247) + \hbar^5 \cdot (\# = 2356) + \hbar^6 \cdot (\# = 66041) + \bar{o}(\hbar^6); \end{aligned}$$

here we report the count of all nonzero-weight Kontsevich oriented graphs. Counting them modulo automorphisms (which may also swap the sinks), Banks, Panzer, and Pym obtain the numbers $(\hbar^0 : 1, \hbar^1 : 1,$

$\hbar^2 : 3, \hbar^3 : 8, \hbar^4 : 133, \hbar^5 : 1209, \hbar^6 : 33268$). This shows that at orders $\hbar^{k \geq 4}$, the use of graph-processing software is indispensable in the task of verifying factorization (1) using weighted graph expansion (6) of the operator \diamond .

Specifically, the number of Kontsevich oriented graphs at \hbar^k in the left-hand side of the factorization problem $\text{Assoc}(\star)(\mathcal{P})(\cdot, \cdot, \cdot) = \diamond(\mathcal{P}, \llbracket \mathcal{P}, \mathcal{P} \rrbracket)(\cdot, \cdot, \cdot)$, and the number of Leibniz graphs which assemble with nonzero coefficients to a solution \diamond in the right-hand side is presented in Table 2. At \hbar^4 , the expansion of $\text{Assoc}(\star)(\mathcal{P}) \bmod \bar{o}(\hbar^4)$ requires 241

Table 2. Number of graphs in either side of the factorization.

k	2	3	4	5	6
LHS: # K. orgraphs	3 (Jac)	39	740	12464	290305
RHS: # L. orgraphs, coeff $\neq 0$	1 (Jac)	13	241	?	?
Reference	§5.3, [16]	§6.1, [5]	§6.2, [7]	⏟ [3]	

nonzero coefficients of Leibniz graphs on 3 sinks, $2 = n - 1$ internal vertices for bi-vectors \mathcal{P} and one internal vertex for the tri-vector $\llbracket \mathcal{P}, \mathcal{P} \rrbracket$, and therefore, $2(n - 1) + 3 = 2n + 3 - 2 = 7$ oriented edges.

Remark 3. Again, this set of Leibniz graphs is well structured. Indeed, it is a disjoint union of homogeneous differential operators arranged according to their differential orders w.r.t. the sinks, e.g., $(1, 1, 1), (2, 1, 1), (1, 2, 1), (1, 1, 2)$, etc., up to $(3, 3, 1)$.

Example 8. The Leibniz graph $L_{331} := [01; 01; 012]$ of differential orders $(3, 3, 1)$ has the weight $1/24$ according to [3]. Multiplied by a universal (for all graphs at \hbar^4) factor $2^4 = 16$ and the factor $1/(\# \text{Aut}(L_{331})) = 1/2$ due to this graph’s symmetry ($3 \rightleftharpoons 4$), it expands to $\frac{1}{3}([01; 01; 01; 52] + [01; 01; 12; 50] + [01; 01; 20; 51])$ by the definition of Jacobi’s identity. This sum of three weighted Kontsevich orgraphs reproduces exactly $A_{331}^{(4)}$, which is known from [7, Table 8 in Appendix D].

Example 9. The Leibniz graph $L_{322} := [01; 02; 012]$ of differential orders $(3, 2, 2)$ has the weight $1/24$ according to [3]. Multiplied now by a universal (for all graphs at \hbar^4) factor $2^4 = 16$ and the factor $1/(\# \text{Aut}(L_{322})) = 1$, it expands to $\frac{2}{3}([01; 02; 01; 52] + [01; 02; 12; 50] + [01; 02; 20; 51])$. This sum reproduces $A_{322}^{(4)}$ (again, see [7, Table 8 in Appendix D]).

Example 10. Consider at the differential order $(1, 3, 2)$ at \hbar^4 the three Leibniz graphs $L_{132}^{(1)} := [12; 13; 012]$, $L_{132}^{(2)} := [12; 12; 014]$, and $L_{132}^{(3)} := [12; 01; 412]$. They have no symmetries, i.e., their automorphism groups are one-element, and their weights are $W(L_{132}^{(1)}) = 1/72$, $W(L_{132}^{(2)}) = 1/48$, and $W(L_{132}^{(3)}) = 1/48$, respectively. Pre-multiplied by their weights and universal factor $2^4 = 16$, these Leibniz graphs expand to

$$\begin{aligned} & \frac{2}{9} ([12; 13; 01; 52] + [12; 13; 12; 50] + [12; 13; 20; 51]) \\ & + \frac{1}{3} ([12; 12; 01; 54] + [12; 12; 14; 50] + [12; 12; 40; 51]) \\ & + \frac{1}{3} ([12; 01; 41; 52] + [12; 01; 12; 54] + [12; 01; 24; 51]). \end{aligned}$$

There is one cancellation, since $[12; 01; 12; 54] = -[12; 12; 01; 54]$. The remaining seven terms reproduce exactly $A_{132}^{(4)}$; that component is known from [7, Table 8 in Appendix D]. Actually, there was another Leibniz graph at this homogeneity order, $L_{132}^{(4)} := [12; 15; 012]$, but its weight is zero and hence it does not contribute. (Indeed, we get an independent verification of this by having already balanced the entire homogeneous component at differential orders $(1, 3, 2)$ in the associator.)

Intermediate conclusion. We have experimentally found the constants c_k in Corollary 4 which balance the Kontsevich graph expansion of the \hbar^k -term $A^{(k)}$ in the associator against an expansion of the respective term at \hbar^k in the r.h.s. of (1) using the weighted Leibniz graphs. Namely, we conjecture $c_k = k/6$ in Section 5.3. The origin of these constants, in particular how they arise from the sum over $i < j$ in the L_∞ condition (5) (perhaps, in combination with different normalizations of the objects which we consider) still remains to be explained, similar to the reasoning in [2, 20] where the signs are fixed. Note that both in the associator, which is quadratic w.r.t. the weights of Kontsevich graphs in \star , and in the operator \diamond , which is linear in the Kontsevich weights of Leibniz graphs, the weight values are provided simultaneously, by using identical techniques (for instance, from [3]). Indeed, the weights are provided by the integral formula which is universal with respect to all the graphs under study [16].

A. Encodings of weighted Kontsevich-graph expansions for (p, q, r) -homogeneous components $(I_f, \dots, S_h)_{pqr}$.

2/3 (S_f)_221}

3 3 1 0 1 0 1 4 2 2/3

3 3 1 0 1 1 2 4 0 2/3
3 3 1 0 1 2 0 4 1 2/3
2/3 (S_g)_{122}
3 3 1 1 2 0 1 4 2 2/3
3 3 1 1 2 1 2 4 0 2/3
3 3 1 1 2 2 0 4 1 2/3
-2/3 (S_h)_{212}
3 3 1 2 0 0 1 4 2 -2/3
3 3 1 2 0 1 2 4 0 -2/3
3 3 1 2 0 2 0 4 1 -2/3
1/6 (I_f)_{111}
3 3 1 0 4 3 1 4 2 1/6
3 3 1 0 4 1 2 4 3 1/6
3 3 1 0 4 2 3 4 1 1/6
3 3 1 0 5 3 1 4 2 1/6
3 3 1 0 5 1 2 4 3 1/6
3 3 1 0 5 2 3 4 1 1/6
-1/6 (I_h)_{111}
3 3 1 2 4 0 1 4 3 -1/6
3 3 1 2 4 1 3 4 0 -1/6
3 3 1 2 4 3 0 4 1 -1/6
3 3 1 2 5 0 1 4 3 -1/6
3 3 1 2 5 1 3 4 0 -1/6
3 3 1 2 5 3 0 4 1 -1/6
1/6 (I_f)_{112}
3 3 1 0 2 3 1 4 2 1/6
3 3 1 0 2 1 2 4 3 1/6
3 3 1 0 2 2 3 4 1 1/6
1/6 (I_g)_{112}
3 3 1 1 2 0 3 4 2 1/6
3 3 1 1 2 3 2 4 0 1/6
3 3 1 1 2 2 0 4 3 1/6
-1/3 (S_h)_{112}
3 3 1 2 4 0 1 4 2 -1/3
3 3 1 2 4 1 2 4 0 -1/3
3 3 1 2 4 2 0 4 1 -1/3
3 3 1 2 5 0 1 4 2 -1/3
3 3 1 2 5 1 2 4 0 -1/3
3 3 1 2 5 2 0 4 1 -1/3

```

# 1/3 (I_f)_{121}
3 3 1 0 1 3 1 4 2 1/3
3 3 1 0 1 1 2 4 3 1/3
3 3 1 0 1 2 3 4 1 1/3
# -1/3 (I_h)_{121}
3 3 1 2 1 0 1 4 3 -1/3
3 3 1 2 1 1 3 4 0 -1/3
3 3 1 2 1 3 0 4 1 -1/3
# 1/3 (S_f)_{211}
3 3 1 0 4 0 1 4 2 1/3
3 3 1 0 4 1 2 4 0 1/3
3 3 1 0 4 2 0 4 1 1/3
3 3 1 0 5 0 1 4 2 1/3
3 3 1 0 5 1 2 4 0 1/3
3 3 1 0 5 2 0 4 1 1/3
# -1/6 (I_g)_{211}
3 3 1 1 0 0 3 4 2 -1/6
3 3 1 1 0 3 2 4 0 -1/6
3 3 1 1 0 2 0 4 3 -1/6
# -1/6 (I_h)_{211}
3 3 1 2 0 0 1 4 3 -1/6
3 3 1 2 0 1 3 4 0 -1/6
3 3 1 2 0 3 0 4 1 -1/6

```

Acknowledgements. The first author thanks the Organisers of international workshop ‘Symmetries and Integrability of Equations of Mathematical Physics’ (December 22–24, 2018, Kyiv) for helpful discussions and warm atmosphere during the meeting. A part of this research was done while RB was visiting at RUG and AVK was visiting at JGU Mainz (supported by IM JGU via project 5020 and JBI RUG project 106552). The research of AVK is supported by the IHÉS (partially, by the Nokia Fund).

- [1] Alekseev A., Rossi C.A., Torossian C., Willwacher T., Logarithms and deformation quantization, *Invent. Math.* **206** (2016), 1–28, arXiv:1401.3200.
- [2] Arnal D., Manchon D., Masmoudi M., Choix des signes pour la formalité de M. Kontsevich. *Pacific J. Math.* **203** (2002), 23–66, arXiv:q-alg/0003003.
- [3] Banks P., Panzer E., Pym B., Multiple zeta values in deformation quantization, arXiv:1812.11649.

-
- [4] Bouisaghouane A., Buring R., Kiselev A., The Kontsevich tetrahedral flow revisited, *J. Geom. Phys.* **119** (2017), 272–285, arXiv:1608.01710.
- [5] Buring R., Kiselev A.V., On the Kontsevich \star -product associativity mechanism, *PEPAN Lett.* **14** (2017), 403–407, arXiv:1602.09036.
- [6] Buring R., Kiselev A.V., The orientation morphism: from graph cocycles to deformations of Poisson structures, *J. Phys. Conf. Ser.* **1194** (2019), 012017, 10 pp., arXiv:1811.07878.
- [7] Buring R., Kiselev A.V., The expansion $\star \bmod \bar{o}(\hbar^4)$ and computer-assisted proof schemes in the Kontsevich deformation quantization, *Exp. Math.*, to appear, arXiv:1702.00681.
- [8] Buring R., Kiselev A.V., Rutten N.J., Poisson brackets symmetry from the pentagon-wheel cocycle in the graph complex, *Physics Particles Nuclei* **49** (2018), 924–928, arXiv:1712.05259.
- [9] Cattaneo A., Formality and star products. in Poisson Geometry, Deformation Quantisation and Group Representations, *London Math. Soc., Lect. Note Ser.*, Vol 323, Cambridge University Press, Cambridge, 2005, 79–144.
- [10] Cattaneo A.S., Felder G., A path integral approach to the Kontsevich quantization formula, *Comm. Math. Phys.* **212** (2000), 591–611, arXiv:q-alg/9902090.
- [11] Cattaneo A., Keller B., Torossian C., Bruguières A., Déformation, quantification, théorie de Lie, *Panoramas et Synthèses*, Vol. 20, Soc. Math. de France, Paris, 2005.
- [12] Felder G., Willwacher T., On the (ir)rationality of Kontsevich weights, *Int. Math. Res. Not.* **2010** (2010), 701–716, arXiv:0808.2762.
- [13] Kontsevich M., Feynman diagrams and low-dimensional topology, *First Europ. Congr. of Math.*, Vol. 2 (Paris, 1992), *Progr. Math.*, Vol. 120, Birkhäuser, Basel, 1994, 97–121.
- [14] Kontsevich M., Homological algebra of mirror symmetry, *Proc. Intern. Congr. Math.*, Vol. 1, (Zürich, 1994), Birkhäuser, Basel, 1995, 120–139.
- [15] Kontsevich M., Formality conjecture, in Deformation Theory and Symplectic Geometry (Ascona, 1996), Editors D. Sternheimer, J. Rawnsley and S. Gutt, *Math. Phys. Stud.*, Vol. 20, Kluwer Acad. Publ., Dordrecht, 1997, 139–156.
- [16] Kontsevich M., Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* **66** (2003), 157–216. arXiv:q-alg/9709040.
- [17] Lada T., Stasheff J., Introduction to sh Lie algebras for physicists, *Internat. J. Theoret. Phys.* **32** (1993), 1087–1103, arXiv:hep-th/9209099.
- [18] Rossi C.A., Willwacher T., P. Etingof’s conjecture about Drinfeld associators, arXiv:1404.2047.
- [19] Schlessinger M., Stasheff J., The Lie algebra structure of tangent cohomology and deformation theory, *J. Pure Appl. Alg.* **38** (1985), 313–322.
- [20] Willwacher T., Calaque D., Formality of cyclic cochains, *Adv. Math.* **231** (2012), 624–650, arXiv:0806.4095.