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Generic realizations of conformal and de Sitter algebras

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Отримано нові породжуючі реалізації конформної алгебри Лі та двох алгебр де Сіттера. Побудовано деформацію алгебри Пуанкаре до алгебр де Сіттера.

New generic realizations of conformal Lie algebra and two de Sitter algebras are obtained. Deformation of the Poincaré algebra to the de Sitter ones is constructed.

1. Introduction. Each well-established physical theory has its own certain fundamental invariance group and, therefore, realizations (representations by first-order differential operators) of their Lie algebras are effectively used for reduction, integration, differential invariants, etc., see e.g. [1, 2, 3, 5, 8].

In this work we consider three types of conformal groups: standard conformal group $\mathrm{C}(3,1)$ and two conformal groups of pseudoeuclidian spaces $\mathrm{C}(3,0)$ and $\mathrm{C}(2,1)$. For the respective Lie algebras $\mathfrak{c}(3,1)$, $\mathfrak{c}(3,0)$ and $\mathfrak{c}(2,1)$ we construct the maximal possible (generic) realizations using the algebraic approach proposed in [7]. Some covariant realizations of the conformal and de Sitter algebras are well known, but we first represent realizations in fifteen and ten essential variables respectively. Realizations in smaller number of variables can be obtained from the given ones by means of projection with respect a subalgebra.

The paper is arranged as follows. Fist we outline the algorithm for construction of realizations and define the conformal Lie algebra. Then we obtain it's generic realization and we do the same for the both de Sitter Lie algebras $\mathfrak{so}(4,1)$ and $\mathfrak{so}(3,2)$. And, finally, we include naturally the contraction parameters to de Sitter algebras in such a way, that contraction results are the Poincaré algebra.

2. Definitions and conventions. Let V be an n-dimensional vector space over the field of real numbers. Consider a Lie algebra $\mathfrak{g}$ on V spanned by a basis $\{e_1, e_2, \ldots, e_n\}$ with the structure constants $C_{ij}^k \in \mathbb{R}$, here and below $i, j, k = 1, 2, \ldots, n$. We denote an open domain of $\mathbb{R}^m$ as M and $\mathrm{Vect}(M)$ is the Lie algebra of smooth vector fields on M with the Lie product defined as commutator (i.e., the Lie algebra of first-order linear differential operators with analytical function coefficients).

A *realization of a Lie algebra* $\mathfrak{g}$ *in vector fields on* M is a homomorphism $R(\mathfrak{g}) = R\colon \mathfrak{g} \to \mathrm{Vect}(M)$. The realization is called *faithful* if $\ker R = \{0\}$ and *unfaithful* otherwise.

In Lie theory realizations are considered locally at some neighborhood $U_x \subset M \subset \mathbb{R}^m$ of a point $x \in M$ and in most of the cases without loss of generality the realization can be considered in a neighborhood of a zero point $x = 0$.

Denote local coordinates of a point $x \in M$ as $(x_1, \ldots, x_m)$, then in coordinate form a realization $R(\mathfrak{g})$ is performed by the images $\Xi_i(x)$ of the basis elements e_i of a general form

$$\Xi_i(x) = R(e_i) = \sum_{l=1}^{m} \xi_{il}(x_1, x_2, \ldots, x_m)\partial_l, \tag{1}$$

hereafter $\partial_l = \frac{\partial}{\partial x_l}$ and the coefficients $\xi_{il}(x_1, x_2, \ldots, x_m)$ are smooth (analytic) functions.

Let us fix a point $x \in M$ and let R_x be a realization of $\mathfrak{g}$ at this point. Consider the linear map $R_x\colon \mathfrak{g} \to \mathrm{Vect}(M)(x)$ that transforms a vector $v \in \mathfrak{g}$ to it's image $R(v(x))$ at x. The matrix that corresponds to this linear map is the n by m matrix ξ formed by the coefficients of the realization (1)

$$\xi(x) = \begin{pmatrix} \xi_{11}(x) & \xi_{12}(x) & \ldots & \xi_{1m}(x) \\ \xi_{21}(x) & \xi_{22}(x) & \ldots & \xi_{2m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n1}(x) & \xi_{n2}(x) & \ldots & \xi_{nm}(x) \end{pmatrix}.$$

The rank of the linear map R_x, or, equivalently, the rank of the matrix $\xi(x)$ at a point x is called a *rank of realization* R at point x and is denoted $\operatorname{rank} R_x$. The realization rank value possess the obvious inequality $0 \leq \operatorname{rank} R_x \leq n$, where n is the dimension of a Lie algebra $\mathfrak{g}$. The second inequality is dictated by the number of rows in matrix ξ, which is equal to the number of basis vector fields of $\mathfrak{g}$.

A realization R of a Lie algebra $\mathfrak{g}$ is called *transitive* if the action of the local Lie group corresponding to R is transitive. Or, equivalently (see [4]), a realization R of a Lie algebra $\mathfrak{g}$ is called *transitive* if rank $R_p = m$ for all $p \in M$.

For many practical applications it is necessary to decide if two given sets of first order differential operators (with the isomorphic commutation relations) can be transformed to each other or not. This task is rather complicated even in the case of small number of operators and variables.

Roughly speaking, two realizations are equivalent, if they can be transformed to the identical form by means of non-singular automorphic basis changes ($e_i \mapsto \tilde{e}_i$) and 1 to 1 changes of variables ($x_l \mapsto y_l = \varphi_l(x)$) with non-zero Jacobi determinant.

Let us have a diffeomorphism of M such that for the corresponding $x, y \in M$ we have $y_1 = \varphi_1(x_1, \ldots, x_m)$, $y_2 = \varphi_2(x_1, \ldots, x_m)$, $\ldots$, $y_m = \varphi_m(x_1, \ldots, x_m)$. Then the realization of the form (1) transforms to the following:

$$\tilde{R}(e_i) = \sum_{l=1}^{m} \tilde{\xi}_{il}(y)\partial_{y_l} = \sum_{l=1}^{m} \left(\sum_{l'=1}^{m} \tilde{\xi}_{il'}(x) \frac{\partial \varphi_l(x)}{\partial x_{l'}} \right) \partial_{y_l}.$$

Note, that the coefficients $\tilde{\xi}_{il}(y)$ are written in terms of y using the inverse transformation φ^{-1}.

It is obvious that application of transformations from $\operatorname{Aut}(\mathfrak{g})$ to the realization R does not change the rank of R, and none of diffeomorphisms of M can change the realization rank either. Therefore the equivalent realizations have the same ranks.

Let a realization $R(x)\colon \mathfrak{g} \to \operatorname{Vect}(M)$ has a rank $r = \operatorname{rank} R < m$ at a regular point $x \in M$, where $m = \dim M$. Then there exists a locally equivalent realization $\tilde{R}(y)\colon \mathfrak{g} \to \operatorname{Vect}(M)$ at a regular point $y \in M$ such that the coefficients of basic vector fields $\tilde{\xi}_{il}(y) = 0$ for all $i = 1, \ldots, n$, $l = r+1, \ldots, m$. To prove this let us construct the desired diffeomorphism. Since the realization rank is equal to r it is known from the theory of invariants [9] that there are $m - r$ functionally independent invariants $J_1(x_1, \ldots, x_m), \ldots, J_{m-r}(x_1, \ldots, x_m)$ of the realization R. The diffeomorphism of the form $y_a = x_a$, $a = 1, \ldots, r$; $y_{r+b} = J_b$,

$b = 1, \ldots, m-r$ gives the following zero coefficients of the realization $\tilde{R}$: $\tilde{\xi}_{i(r+b)}(y) = R(e_i)(J_b) = 0$ for all $i = 1, \ldots, n$, $b = 1, \ldots, m-r$.

The above variables $y_1, \ldots, y_r$ are called *essential* and the rest of non-zero variables from $y_{r+1}, \ldots, y_m$ are called *additional*.

Example 1. Consider two-dimensional abelian Lie algebra $2A_1$. It is well-known that the basis elements of this algebra can be realized by two operators of translations

$$R_1(e_1) = \partial_1, \quad R_1(e_2) = \partial_2.$$

It was shown in [10] that there are exactly two inequivalent realizations of $2A_1$, and the second one is

$$R_2(e_1) = \partial_1, \quad R_2(e_2) = x_2\partial_1.$$

In these cases $\operatorname{rank} R_1 = 2$ and $\operatorname{rank} R_2 = 1$.

Consider the formal sum of these realizations $R_3 = R_1 + R_2$ (R_1 for the variables (x_1, x_2) and R_2 for the variables (x_3, x_4)), namely

$$R_3(e_1) = \partial_1 + \partial_3, \quad R_3(e_2) = \partial_2 + x_4\partial_3.$$

As far as $[\partial_1 + \partial_3, \partial_2 + x_4\partial_3] = 0$, then R_3 do realize the Lie algebra $2A_1$ in the space of four variables (x_1, x_2, x_3, x_4) and $\operatorname{rank} R_3 = 2$, what means that the number of essential variables is equal to 2.

Indeed, the diffeomorphism φ given by the non-singular functions

$$\varphi_1(x_1, \ldots, x_4) = x_1, \quad \varphi_2(x_1, \ldots, x_4) = x_2,$$
$$\varphi_3(x_1, \ldots, x_4) = x_1 - x_3 + x_2x_4, \quad \varphi_4(x_1, \ldots, x_4) = x_4$$

transforms the realization R_3 to the equivalent realization R_1 in 2 essential variables.

In case of transitive realizations all variables are essential and, since $\operatorname{rank} R \leq n$, any transitive realization of a Lie algebra is realized in not more then n variables.

A recent paper [7] establishes the one-to-one correspondence between inequivalent transitive realizations of a Lie algebra $\mathfrak{g}$ and Int-inequivalent subalgebras of $\mathfrak{g}$. Moreover, this relation was extended to the non-transitive case as well, see [4].

The coefficients $\xi_k^i(x)$ of the generic realization

$$\Xi_i = \sum_{k=1}^{n} \xi_k^i(x) \frac{\partial}{\partial x_k}, \quad i = 1, 2, \ldots, n,$$

can be recovered from the left-invariant differential one-forms

$$\Omega^i = \sum_{l=1}^{n} \omega_i^l(x) \mathrm{d}x_l$$

using the duality $\omega_i^l(x)\xi_k^i(x) = \delta_k^l$ and the coefficients $\omega_i^l(x)$ of the differential one-forms are constructed as follows:

$$\omega_i^l(x) = \left(A^{(1)}\left(x^1\right)A^{(2)}\left(x^2\right)\cdots A^{(i-1)}\left(x^{i-1}\right)\right)_i^l,$$

where $i = 2, 3, \ldots, n$, $l = 1, 2, \ldots, n$, $\omega_1^l = \delta_1^l$, and the matrices $A^{(p)}$, $p = 1, 2, \ldots, n$, are the exponential solutions of the system

$$\dot{A}^{(p)}(t) = -\operatorname{ad}_{e_p} A^{(p)}(t), \quad A^{(p)}(0) = I.$$

All the rest of transitive realizations of a fixed Lie algebra are constructed by means of projection of the generic realization using the known set of $\mathrm{Aut}(\mathfrak{g})$-inequivalent subalgebras and the following rule.

Let $\mathfrak{h} = \langle e_{m+1}, \ldots, e_n \rangle$ be a subalgebra of $\mathfrak{g} = \langle e_1, \ldots, e_n \rangle$ with a complementary space $\{e_1, \ldots, e_m\}$, then, using the above approach and the shortcut $\partial_i = \frac{\partial}{\partial x_i}$, we will obtain the realization of basis elements in the form

$$R(e_i) = \xi_i^1(x_1, x_2, \ldots, x_m)\partial_1 + \cdots + \xi_i^m(x_1, x_2, \ldots, x_m)\partial_m$$
$$+ \xi_i^{m+1}(x_1, x_2, \ldots, x_n)\partial_{m+1} + \cdots + \xi_i^n(x_1, x_2, \ldots, x_n)\partial_n.$$

The realization projected on the coordinates $x_1, x_2, \ldots, x_m$ is well defined and has the form

$$\mathrm{pr}_{\mathfrak{h}} R(e_i) = \xi_i^1(x_1, x_2, \ldots, x_m)\partial_1 + \cdots + \xi_i^m(x_1, x_2, \ldots, x_m)\partial_m.$$

The subalgebra that corresponds to the given realization is the kernel of its linear map at the origin of coordinates. In other words at the point $x = 0 \in \mathbb{R}^m$ the realization vectors that form a basis of corresponding subalgebra are identically equal to zero.

Example 2. Consider the realizations

$$R_1\colon\ e_1=\partial_1,\quad e_2=x_2\partial_1,\quad e_3=x_1\partial_1+2x_2\partial_2,$$
$$R_2\colon\ e_1=\partial_1,\quad e_2=x_1\partial_1-x_2\partial_2,\quad e_3=\partial_2.$$

At the origin of coordinates $x=0$ their basis vectors have the form

$$R_1(x=0)\colon\ e_1=\partial_1,\quad e_2=0,\quad e_3=0,$$
$$R_2(x=0)\colon\ e_1=\partial_1,\quad e_2=0,\quad e_3=\partial_2.$$

Therefore the realization R_1 corresponds to the subalgebra $\langle e_2,e_3\rangle$ and R_2 corresponds to $\langle e_2\rangle$.

The structure of realizations constructed by means of the algebraic method reminds a tree diagram, namely: a realization corresponding to a subalgebra $\mathfrak{h}_1$ can be constructed by means of projection from a realization corresponding to a subalgebra $\mathfrak{h}_2$ if $\mathfrak{h}_2\subset\mathfrak{h}_1$.

Note that all inequivalent realizations of a fixed Lie algebra can be obtained by the above method, as far as any realization corresponds to a quotient group G/H that acts effectively on some subspace M, where H is a subgroup that corresponds to some subalgebra $\mathfrak{h}$.

In this paper we use the above method to construct the realizations of three conformal Lie algebras in maximal possible number of essential variables, that is we construct realizations that correspond to zero subalgebras.

3. Conformal Lie algebra. First of all we consider a conformal group and it's 15-dimensional Lie algebra $\mathfrak{c}(3,1)$. The conformal Lie group $\mathrm{C}(3,1)=\mathrm{SO}(4,2)=\mathrm{SU}(2,2)$ of the Minkowski space is the maximal invariance group of the Maxwell equations in the flat space-time. This group in many aspects unite all physical groups. It is generated by 10 Poincaré generators P_μ, $J_{\mu\nu}$, dilatation generator D and generators of special conformal transformations K_μ, hereafter $\mu,\nu=1,2,\ldots,4$. The non-zero commutation relations of the Lie algebra are

$$[J_{\mu\nu},J_{\rho\sigma}]=g_{\mu\rho}J_{\nu\sigma}-g_{\nu\rho}J_{\mu\sigma}+g_{\mu\sigma}J_{\rho\nu}-g_{\nu\sigma}J_{\rho\mu}, \tag{2}$$
$$[J_{\mu\nu},P_\rho]=g_{\mu\rho}P_\nu-g_{\nu\rho}P_\mu, \tag{3}$$
$$[J_{\mu\nu},K_\rho]=g_{\mu\rho}K_\nu-g_{\nu\rho}K_\mu, \tag{4}$$
$$[P_\mu,K_\nu]=2(g_{\mu\nu}D+J_{\mu\nu}), \tag{5}$$
$$[P_\mu,D]=P_\mu, \tag{6}$$

$$[K_\mu, D] = -K_\mu. \tag{7}$$

Here $g_{\mu\nu}$ is the metric tensor of the Minkowski space $g_{11} = g_{22} = g_{33} = -g_{44} = 1$.

It is possible to consider conformal groups $\mathrm{C}(p, q)$ of the pseudoeuclidian spaces with metric tensors

$$g_{11} = g_{22} = \dots = g_{pp} = -g_{p+1,p+1} = \dots = -g_{p+q,p+q} = 1 \tag{8}$$

and $\mu, \nu = 1, \dots, p + q = n$.

Consider the group $\mathrm{SO}(p + 1, q + 1) = \mathrm{span}\{I_{ab}\}$, $I_{ab} = -I_{ba}$ with the commutators

$$[I_{ab}, I_{cd}] = g_{ac}I_{bd} - g_{bc}I_{ad} + g_{ad}I_{cb} - g_{bd}I_{ca},$$

where g_{ab} are from (8) and $g_{n+1,n+1} = -g_{n+2,n+2} = 1$. Then matching

$$J_{\mu\nu} = I_{\mu\nu}, \quad P_\mu = I_{\mu,n+1} - I_{\mu,n+2}, \quad K_\mu = I_{\mu,n+1} + I_{\mu,n+2},$$
$$D = I_{n+1,n+2}$$

we get the isomorphism $\mathrm{C}(p, q) \simeq \mathrm{SO}(p + 1, q + 1)$. Therefore a number of well-known groups (like de Sitter groups) are conformal groups of pseudoeuclidian spaces. Consider the well-known realization of the conformal group

$$P_\mu = \partial_\mu, \quad J_{\mu\nu} = x_\nu\partial_\mu - x_\mu\partial_\nu, \quad D = x_\nu\partial_\nu,$$
$$K_\mu = 2x_\mu x_\nu\partial_\nu - x^2\partial_\mu;$$

hereafter the summation with respect to the repeated indices is implied and $x^2 = x_1^2 + \dots + x_n^2$.

Let us define the subalgebra that corresponds to the given realization. To do this we study the realization at the point $x = (0, 0, 0, 0)$ and see that the kernel of this linear map coincides with the subalgebra $\mathrm{span}\{J_{\mu\nu}, D, K_\mu\}$. Indeed, this is proven by the construction and projection of the generic realization of $\mathfrak{c}(3, 1)$ with the following complementary part $\{P_\mu, J_{\mu\nu}, K_\mu, D\}$ taken in the lexicographical order. To make formula more readable we have introduced the shortcuts:

$$\sin x_i = \mathrm{s}_i, \quad \cos x_i = \mathrm{c}_i, \quad \tan x_i = \mathrm{t}_i, \quad \sinh x_i = \mathrm{sh}_i,$$
$$\cosh x_i = \mathrm{ch}_i, \quad \tanh x_i = \mathrm{th}_i, \quad i = 1, 10.$$

$R_{\text{generic}}(\mathfrak{c}(3,1))$:

$$P_1 = \partial_1, \quad P_2 = \partial_2, \quad P_3 = \partial_3, \quad P_4 = \partial_4,$$
$$J_{12} = x_2\partial_1 - x_1\partial_{x_2} + \partial_5,$$
$$J_{13} = x_3\partial_1 - x_1\partial_3 - \mathrm{th}_6\mathrm{s}_5\partial_5 + \mathrm{c}_5\partial_6 + 2\frac{\mathrm{s}_5}{\mathrm{ch}_6}\partial_8,$$
$$J_{14} = -x_4\partial_1 - x_1\partial_4 + 2\frac{\mathrm{t}_7\mathrm{s}_5}{\mathrm{ch}_6}\partial_5 + \mathrm{t}_7\mathrm{s}_6\mathrm{c}_5\partial_6 + \mathrm{c}_5\mathrm{c}_6\partial_7$$
$$+ \frac{\mathrm{s}_9\mathrm{s}_6\mathrm{c}_5\mathrm{c}_8 + \mathrm{th}_6\mathrm{s}_7\mathrm{c}_9\mathrm{s}_5 + \mathrm{s}_9\mathrm{s}_8\mathrm{s}_5}{\mathrm{c}_7\mathrm{c}_9}\partial_8$$
$$- \frac{\mathrm{c}_5\mathrm{s}_6\mathrm{s}_8 - \mathrm{s}_5\mathrm{c}_8}{\mathrm{c}_7}\partial_9 + \frac{\mathrm{s}_5\mathrm{s}_8 + \mathrm{c}_5\mathrm{s}_6\mathrm{c}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_{10},$$
$$J_{23} = x_3\partial_2 - x_2\partial_3 + -\mathrm{th}_6\mathrm{c}_5\partial_5 - \mathrm{s}_5\partial_6 + \frac{\mathrm{c}_5}{\mathrm{ch}_6}\partial_8,$$
$$J_{24} = -x_4\partial_2 - x_2\partial_4 + \frac{\mathrm{t}_7\mathrm{c}_5}{\mathrm{c}_7\mathrm{ch}_6}\partial_5 - \mathrm{t}_7\mathrm{s}_5\mathrm{s}_6\partial_6 - \mathrm{s}_5\mathrm{c}_6\partial_7$$
$$+ \frac{\mathrm{th}_6\mathrm{s}_7\mathrm{c}_9\mathrm{c}_5 - \mathrm{s}_9\mathrm{s}_6\mathrm{c}_8\mathrm{s}_5 + \mathrm{s}_9\mathrm{c}_5\mathrm{s}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_8$$
$$+ \frac{\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8}{\mathrm{c}_7}\partial_9 - \frac{\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 - \mathrm{c}_5\mathrm{s}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_{10},$$
$$J_{34} = -x_4\partial_3 - x_3\partial_4 + \mathrm{c}_6\mathrm{t}_7\partial_6 - \mathrm{s}_6\partial_7 + \frac{\mathrm{t}_9\mathrm{c}_8\mathrm{c}_6}{\mathrm{c}_7}\partial_8$$
$$- \frac{\mathrm{s}_8\mathrm{c}_6}{\mathrm{c}_7}\partial_9 + \frac{\mathrm{c}_6\mathrm{c}_8}{\mathrm{c}_7\mathrm{c}_9}\partial_{10},$$
$$K_1 = \left(x_1^2 - x_2^2 - x_3^2 + x_4^2\right)\partial_1 + 2x_1x_2\partial_2 + 2x_1x_3\partial_3 + 2x_1x_4\partial_4$$
$$- 2\left(x_2 + \frac{x_4\mathrm{t}_7\mathrm{s}_5}{\mathrm{ch}_6} - x_3\mathrm{th}_6\mathrm{s}_5\right)\partial_5$$
$$- 2(x_4\mathrm{t}_7\mathrm{s}_6 + x_3)\mathrm{c}_5\partial_6 - 2x_4\mathrm{c}_5\mathrm{c}_6\partial_7$$
$$- 2\left(\frac{x_4\mathrm{t}_9\mathrm{s}_6\mathrm{c}_5\mathrm{c}_8}{\mathrm{c}_7} + x_4\mathrm{t}_7\mathrm{th}_6\mathrm{s}_5 + \frac{x_3\mathrm{s}_5}{\mathrm{ch}_6} + \frac{x_4\mathrm{t}_9\mathrm{s}_5\mathrm{s}_8}{\mathrm{c}_7}\right)\partial_8$$
$$+ 2\frac{(\mathrm{c}_5\mathrm{s}_6\mathrm{s}_8 - \mathrm{s}_5\mathrm{c}_8)x_4}{\mathrm{c}_7}\partial_9$$
$$- 2\frac{(\mathrm{s}_5\mathrm{s}_8 + \mathrm{c}_5\mathrm{s}_6\mathrm{c}_8)x_4}{\mathrm{c}_7\mathrm{c}_9}\partial_{10} - (2x_1x_{11} - \mathrm{ch}_7\mathrm{c}_5\mathrm{c}_6)\partial_{11}$$
$$+ (\mathrm{sh}_7\mathrm{sh}_9\mathrm{c}_5\mathrm{c}_6 + \mathrm{ch}_9(\mathrm{s}_5\mathrm{c}_8 - \mathrm{s}_6\mathrm{s}_8\mathrm{c}_5) - 2x_1x_{12})\partial_{12}$$
$$+ (\mathrm{sh}_9\mathrm{sh}_{10}(\mathrm{s}_5\mathrm{c}_8 - \mathrm{c}_5\mathrm{s}_6\mathrm{s}_8) + \mathrm{ch}_{10}(\mathrm{s}_6\mathrm{c}_8\mathrm{c}_5 + \mathrm{s}_5\mathrm{s}_8)$$

$$
\begin{aligned}
&+ \mathrm{sh}_7\mathrm{ch}_9\mathrm{sh}_{10}\mathrm{c}_5\mathrm{c}_6 - 2x_1x_{13})\partial_{13}\\
&+ (\mathrm{sh}_9\mathrm{ch}_{10}(\mathrm{s}_5\mathrm{c}_8 + \mathrm{c}_5\mathrm{s}_6\mathrm{s}_8) + \mathrm{sh}_{10}(\mathrm{s}_6\mathrm{c}_8\mathrm{c}_5 + \mathrm{s}_5\mathrm{s}_8)\\
&+ \mathrm{sh}_7\mathrm{ch}_9\mathrm{ch}_{10}\mathrm{c}_5\mathrm{c}_6 - 2x_1x_{14})\partial_{14} + 2x_1\partial_{15},\\
K_2 = {}& 2x_1x_2\partial_1 + \big(-{x_1}^2 + {x_2}^2 - {x_3}^2 + {x_4}^2\big)\partial_{x_2} + 2x_2x_3\partial_3\\
&+ 2x_2x_4\partial_4 + 2\left(\frac{x_4\mathrm{c}_5\mathrm{t}_7}{\mathrm{ch}_6} - x_1 - x_3\mathrm{c}_5\mathrm{th}_6\right)\partial_5\\
&+ 2(x_3 + x_4\mathrm{t}_7\mathrm{s}_6)\mathrm{s}_5\partial_6 + 2x_4\mathrm{s}_5\mathrm{c}_6\partial_7\\
&+ 2\left(\frac{x_4\mathrm{t}_9}{\mathrm{c}_7}(\mathrm{s}_6\mathrm{c}_8\mathrm{s}_5 - \mathrm{c}_5\mathrm{s}_8) - x_4\mathrm{th}_7\mathrm{c}_5 - \frac{x_3\mathrm{c}_5}{\mathrm{ch}_6}\right)\partial_8\\
&- 2\frac{(\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8)x_4}{\mathrm{c}_7}\partial_9 + 2\frac{(\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8 - \mathrm{c}_5\mathrm{s}_8)x_4}{\mathrm{c}_7\mathrm{c}_9}\partial_{10}\\
&- (2x_2x_{11} + \mathrm{ch}_7\mathrm{s}_5\mathrm{c}_6)\partial_{11} + \big(\mathrm{ch}_9(\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{s}_8)\\
&- \mathrm{sh}_7\mathrm{sh}_9\mathrm{s}_5\mathrm{c}_6 - 2x_2x_{12}\big)\partial_{12} + \big(\mathrm{sh}_9\mathrm{ch}_{10}\mathrm{c}_5\mathrm{c}_8\\
&+ \mathrm{ch}_{10}\mathrm{c}_5\mathrm{s}_8 - \mathrm{sh}_7\mathrm{sh}_{10}\mathrm{ch}_9\mathrm{s}_5\mathrm{c}_6 - \mathrm{ch}_{10}\mathrm{s}_5\mathrm{s}_6\mathrm{c}_8\\
&+ \mathrm{sh}_9\mathrm{sh}_{10}\mathrm{s}_5\mathrm{s}_6\mathrm{s}_8 - 2x_2x_{13}\big)\partial_{13}\\
&+ \big(\mathrm{sh}_9\mathrm{ch}_{10}(\mathrm{c}_5\mathrm{c}_8 + \mathrm{s}_5\mathrm{s}_6\mathrm{c}_8) + \mathrm{sh}_{10}(\mathrm{s}_8\mathrm{c}_5 - \mathrm{s}_5\mathrm{s}_6\mathrm{c}_8)\\
&- \mathrm{sh}_7\mathrm{ch}_9\mathrm{ch}_{10}\mathrm{s}_5\mathrm{c}_6 - 2x_2x_{14}\big)\partial_{14} + 2x_2\partial_{15},\\
K_3 = {}& 2x_1x_3\partial_1 + 2x_2x_3\partial_{x_2} + \big(-x_1^2 - x_2^2 + x_3^2 + x_4^2\big)\partial_3 + 2x_3x_4\partial_4\\
&- 2\mathrm{th}_6(x_1\mathrm{s}_5 + x_2\mathrm{c}_5)\partial_5 - 2(x_4\mathrm{t}_7\mathrm{c}_6 - x_1\mathrm{c}_5 + x_2\mathrm{s}_5)\partial_6\\
&+ 2x_4\mathrm{s}_6\partial_7 + \left(\frac{x_1\mathrm{s}_5 + x_2\mathrm{c}_5}{\mathrm{ch}_6} - 2\frac{x_4\mathrm{t}_9\mathrm{c}_6\mathrm{c}_8}{\mathrm{c}_7}\right)\partial_8\\
&+ 2\frac{\mathrm{s}_8\mathrm{c}_6x_4}{\mathrm{c}_7}\partial_9 - 2\frac{\mathrm{c}_6\mathrm{c}_8x_4}{\mathrm{c}_7\mathrm{c}_9}\partial_{10} - (\mathrm{ch}_7\mathrm{s}_6 + 2x_3x_{11})\partial_{11}\\
&- (\mathrm{ch}_9\mathrm{c}_6\mathrm{s}_8 + 2x_3x_{12} + \mathrm{sh}_7\mathrm{sh}_9\mathrm{s}_6)\partial_{12}\\
&+ \big(\mathrm{ch}_{10}\mathrm{c}_6\mathrm{c}_8 - \mathrm{sh}_9\mathrm{sh}_{10}\mathrm{c}_6\mathrm{s}_8 - \mathrm{sh}_7\mathrm{ch}_9\mathrm{sh}_{10}\mathrm{s}_6\\
&- 2x_3x_{13}\big)\partial_{13} + \big(\mathrm{ch}_{10}\mathrm{c}_6\mathrm{c}_8 - \mathrm{sh}_9\mathrm{ch}_{10}\mathrm{c}_6\mathrm{s}_8\\
&- \mathrm{sh}_7\mathrm{ch}_9\mathrm{ch}_{10}\mathrm{s}_6 - 2x_3x_{14}\big)\partial_{14} + 2x_3\partial_{15},\\
K_4 = {}& -2x_1x_4\partial_1 - 2x_2x_4\partial_{x_2} - 2x_4x_3\partial_3 - \big(x_1^2 + x_2^2 + x_3^2 + x_4^2\big)\partial_4\\
&+ 2\frac{\mathrm{t}_7(x_1\mathrm{s}_5 + x_2\mathrm{c}_5)}{\mathrm{ch}_6}\partial_5 + 2\mathrm{t}_7(x_1\mathrm{s}_6\mathrm{c}_5 - x_2\mathrm{s}_6\mathrm{s}_5 + x_3\mathrm{c}_6)\partial_6\\
&- 2(x_2\mathrm{c}_6\mathrm{s}_5 - x_1\mathrm{c}_6\mathrm{c}_5 + x_3\mathrm{s}_6)\partial_7
\end{aligned}
$$

$$+ 2\left(\frac{t_9}{c_7}(x_2 s_5 s_6 c_8 - x_1 s_6 c_5 c_8 - x_3 c_6 c_8 - x_1 s_5 s_8 \right.$$
$$\left. - x_2 s_8 c_5) - th_6 t_7 (x_1 s_5 + x_2 c_5)\right)\partial_8$$
$$- \frac{2}{c_7}(x_2 - s_5 s_6 s_8 + x_1 c_5 s_6 s_8 + x_3 c_6 s_8 - x_1 s_5 c_8$$
$$- x_2 c_5 c_8)\partial_9 + \frac{2}{c_7 c_9}(x_1 c_5 s_6 c_8 - x_2 s_5 s_6 c_8 + x_3 c_6 c_8$$
$$+ x_1 s_5 s_8 + x_2 c_5 s_8)\partial_{10} + (2x_4 x_{11} + sh_7)\partial_{11}$$
$$+ (2x_4 x_{12} + ch_7 sh_9)\partial_{12} + (2x_4 x_{13} + ch_7 ch_9 sh_{10})\partial_{13}$$
$$+ (2x_4 x_{14} + ch_7 ch_9 ch_{10})\partial_{14} - 2x_4 \partial_{15},$$
$$D = x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - x_{11}\partial_{11} - x_{12}\partial_{12} - x_{13}\partial_{13}$$
$$- x_{14}\partial_{14} + \partial_{15}.$$

4. De Sitter Lie algebras. Consider de Sitter groups $SO(4,1)$ and SO(3,2) that are the groups of isometry transformations of pseudoeuclidean spaces with metric forms $x_1^2 + x_2^2 + x_3^2 - x_4^2 + x_5^2$ and $x_1^2 + x_2^2 + x_3^2 - x_4^2 - x_5^2$ respectively. Them are the movement groups of 4-dimensional Riemann spaces of a constant curvature (de Sitter spaces). Both de Sitter spaces describe the expanding Universe, where the radial velocities of galaxies are approximately proportional to distances from any space point. For the de Sitter Lie algebras we can use the isomorphisms $\mathfrak{c}(3,0) \sim \mathfrak{so}(4,1)$ and $\mathfrak{c}(2,1) \sim \mathfrak{so}(3,2)$ with the conformal commutation relations (2)–(7) for the metric tensors $g_{11} = g_{22} = g_{33} = 1$ and $g_{11} = g_{22} = -g_{33} = 1$ respectively. Then, constructing the generic realization by the method given in second section (with the complementary part $\{P_\mu, J_{\mu\nu}, K_\mu, D\}$ taken in the lexicographical order), we have got two following realizations. Note that it is possible to construct one realization for both de Sitter algebras (putting the parameter to the commutation relations that changes the tensor sign), but this essentially complicates calculations and appearance of realizations.
$R_{\text{generic}}(\mathfrak{c}(3,0))$:

$$P_1 = \partial_1, \quad P_2 = \partial_2, \quad P_3 = \partial_3, \quad J_{12} = x_2\partial_1 - x_1\partial_2 + \partial_4,$$
$$J_{13} = x_3\partial_1 - x_1\partial_3 - th_5 s_4\partial_4 + c_4\partial_5 + \frac{s_4}{ch_5}\partial_6,$$
$$J_{23} = x_3\partial_2 - x_2\partial_3 - th_5 c_4\partial_4 - s_4\partial_5 + \frac{c_4}{ch_5}\partial_6,$$

$$
\begin{aligned}
K_1 &= \left(x_1^2 - x_2^2 - x_3^2\right)\partial_1 + 2x_1x_2\partial_2 + 2x_1x_3\partial_3 \\
&\quad + 2(x_3\mathrm{s}_4\mathrm{th}_5 - x_2)\partial_4 - 2x_3\mathrm{c}_4\partial_5 - 2\frac{x_3\mathrm{s}_4}{\mathrm{ch}_5}\partial_6 \\
&\quad + (\mathrm{c}_4\mathrm{c}_5 - 2x_1x_7)\partial_7 + (\mathrm{s}_4\mathrm{ch}_6 - \mathrm{c}_4\mathrm{sh}_5\mathrm{sh}_6 - 2x_1x_8)\partial_8 \\
&\quad + (\mathrm{c}_4\mathrm{s}_5\mathrm{c}_6 + \mathrm{s}_4\mathrm{s}_6 - 2x_1x_9)\partial_9 + 2x_1\partial_{10}, \\
K_2 &= 2x_1x_2\partial_1 + \left(x_2^2 - x_1^2 - x_3^2\right)\partial_2 + 2x_2x_3\partial_3 \\
&\quad + 2(x_1 + x_3\mathrm{c}_4\mathrm{th}_5)\partial_4 + 2x_3\mathrm{s}_4\partial_5 - 2\frac{x_3\mathrm{c}_4}{\mathrm{ch}_5}\partial_6 \\
&\quad - (\mathrm{s}_4\mathrm{c}_5 + 2x_2x_7)\partial_7 + (\mathrm{s}_4\mathrm{s}_5\mathrm{s}_6 + \mathrm{c}_4\mathrm{c}_6 - 2x_8x_2)\partial_8 \\
&\quad + (\mathrm{c}_4\mathrm{s}_6 - \mathrm{s}_4\mathrm{s}_5\mathrm{c}_6 - 2x_2x_9)\partial_9 + 2x_2\partial_{10}, \\
K_3 &= 2x_1x_3\partial_1 + 2x_2x_3\partial_2 + \left(x_3^2 - x_1^2 - x_2^2\right)\partial_3 \\
&\quad - 2(x_1\mathrm{s}_4 + x_2\mathrm{c}_4)\mathrm{th}_5\partial_4 + 2(x_1\mathrm{c}_4 - x_2\mathrm{s}_4)\partial_5 \\
&\quad + 2\frac{x_1\mathrm{s}_4 + x_2\mathrm{c}_4}{\mathrm{ch}_5}\partial_6 - (\mathrm{s}_5 + 2x_3x_7)\partial_7 \\
&\quad - (\mathrm{c}_5\mathrm{s}_6 + 2x_8x_3)\partial_8 + (\mathrm{c}_5\mathrm{c}_6 - 2x_9x_3)\partial_9 + 2x_3\partial_{10}, \\
D &= x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_7\partial_7 - x_8\partial_8 - x_9\partial_9 + \partial_{10}.
\end{aligned}
$$

$R_{\mathrm{generic}}(\mathfrak{c}(2,1))$:

$$
\begin{aligned}
&P_1 = \partial_1, \quad P_2 = \partial_2, \quad P_3 = \partial_3, \quad J_{12} = x_2\partial_1 - x_1\partial_2 + \partial_4, \\
&J_{13} = -x_3\partial_1 - x_1\partial_3 + \mathrm{s}_4\mathrm{t}_5\partial_4 + \mathrm{c}_4\partial_5 + \frac{\mathrm{s}_4}{\mathrm{c}_5}\partial_6, \\
&J_{23} = -x_3\partial_2 - x_2\partial_3 + \mathrm{c}_4\mathrm{t}_5\partial_4 - \mathrm{s}_4\partial_5 + \frac{\mathrm{c}_4}{\mathrm{c}_5}\partial_6, \\
&K_1 = \left(x_1^2 - x_2^2 + x_3^2\right)\partial_1 + 2x_1x_2\partial_2 + 2x_1x_3\partial_3 \\
&\qquad - 2(x_2 + x_3\mathrm{s}_4\mathrm{t}_5)\partial_4 - 2x_3\mathrm{c}_4\partial_5 - 2\frac{x_3\mathrm{s}_4}{\mathrm{c}_5}\partial_6 \\
&\qquad + (\mathrm{c}_4\mathrm{ch}_5 - 2x_1x_7)\partial_7 + (\mathrm{s}_4\mathrm{ch}_6 + \mathrm{c}_4\mathrm{sh}_5\mathrm{sh}_6 - 2x_1x_8)\partial_8 \\
&\qquad + (\mathrm{s}_4\mathrm{sh}_6 + \mathrm{c}_4\mathrm{sh}_5\mathrm{ch}_6 - 2x_1x_9)\partial_9 + 2x_1\partial_{10}, \\
&K_2 = 2x_1x_2\partial_1 + \left(x_2^2 + x_3^2 - x_1^2\right)\partial_2 + 2x_2x_3\partial_3 \\
&\qquad - 2(x_3\mathrm{c}_4\mathrm{t}_5 - x_1)\partial_4 + 2x_3\mathrm{s}_4\partial_5 - 2\frac{x_3\mathrm{c}_4}{\mathrm{c}_5}\partial_6 \\
&\qquad - (2x_2x_7 + \mathrm{s}_4\mathrm{ch}_5)\partial_7 + (\mathrm{c}_4\mathrm{ch}_6 - \mathrm{s}_4\mathrm{sh}_5\mathrm{sh}_6 - 2x_2x_8)\partial_8 \\
&\qquad + (\mathrm{c}_4\mathrm{sh}_6 - \mathrm{s}_4\mathrm{sh}_5\mathrm{ch}_6 - 2x_2x_9)\partial_9 + 2x_2\partial_{10}, \\
&K_3 = -2x_1x_3\partial_1 - 2x_2x_3\partial_2 - \left(x_1^2 + x_2^2 + x_3^2\right)\partial_3
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(\mathrm{t}_5(x_1\mathrm{s}_4 + x_2\mathrm{c}_4)\partial_4 + 2(x_1\mathrm{c}_4 - x_2\mathrm{s}_4)\partial_5 \\
&+ 2\frac{x_1\mathrm{s}_4 + x_2\mathrm{c}_4}{\mathrm{c}_5}\partial_6 + (2x_3x_7 + \mathrm{sh}_5)\partial_7 \\
&+ (\mathrm{ch}_5\mathrm{sh}_6 + 2x_3x_8)\partial_8 + (\mathrm{ch}_5\mathrm{ch}_6 + 2x_3x_9)\partial_9 - 2x_3\partial_{10},
\end{aligned}
$$

$$
D = x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_7\partial_7 - x_8\partial_8 - x_9\partial_9 + \partial_{10}.
$$

5. Connection to the Poincaré Lie algebra. Classical Poincaré algebra $\mathfrak{p}(1,3)$ is ten-dimensional and formed by the operators $\{P_\mu,\ J_{\mu\nu}\}$ with the commutation relations (2) and (3). Extending this set of commutation relations by the following ones

$$
[P_\nu, P_\mu] = \tau J_{\mu\nu}, \quad \tau \in \mathbb{R} \tag{9}
$$

we get the well-defined 10-dimensional Lie algebra which is the deformation $\mathfrak{p}^\tau(1,3)$ of $\mathfrak{p}(1,3)$ to the both de Sitter algebras at the same time. Indeed, for $\tau = 0$ $\mathfrak{p}^\tau(1,3)$ coincides with the Poincaré algebra, for $\tau \geq 0$ $\mathfrak{p}^\tau(1,3) \sim \mathfrak{so}(4,1)$ and for $\tau \leq 0$ $\mathfrak{p}^\tau(1,3) \sim \mathfrak{so}(3,2)$. So, one can construct uniform realizations for the both de Sitter and Poincaré algebras applying the algebraic method to the structure constants from the deformed relations (2), (3) and (9). The inverse connection between de Sitter and Poincaré algebras is provided by standard Inönü–Wigner contraction [6] with respect to the six-dimensional subalgebra $\mathfrak{so}(3,1)$.

The result of the paper can be used for construction of differential invariants and respective invariant differential equations [9].

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