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# Coupling of a swirl-type resonant sloshing and a mean rotational flow*

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Referring to experimental results by Prandtl (1949) and Hutton (1964) as well as more recent model tests by Royon-Lebeaud, Hopfinger & Cartellier (2007) and Reclari (2013), a Moiseev-type asymptotic almost periodic (steady-state) solution of a resonant sloshing problem is derived to show that a time-averaged rotational liquid flow, if occurs, becomes nonlinearly coupled with the dominant swirl-type wave component. The coupling appears as a necessary solvability condition and consists of nonlinear (differential) equations with respect to four amplitude parameters of the two lowest natural sloshing modes and the time-averaged velocity field, which is governed by a partial differential equation of the first order. Finding its unique solution requires to know the (Stokes) steady-streaming.

Із посиланням на експериментальні результати Прандтля (1949) та Хаттона (1964), а також на більш нові модельні випробування Ройон–Лебод, Хопфінгера & Картельєра (2007) та Рекларі (2013), виводится асимптотичний майже періодичний (усталений) розв'язок типу Моісєєва резонансної задачі про хлюпання рідини з метою показати, що усереднена кругова течія (якщо така виникає) є нелінійним чином пов'язаною з домінантною компонентою кругової хвилі. Цей зв'язок виникає як необхідна умова розв'язності задачі та приймає форму нелінійних диференціальних рівнянь відносно чотирьох амплітудних параметрів для двох перших натуральних форм коливань рідини та усередненого поля швидкостей, яке описується диференціальним рівнянням у частинних похідних першого порядку. Аби мати єдиний розв'язок останнього рівняння, треба знати усталену вторинну течію, обумовлену поверхневим шаром.

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## Introduction

Mentioning an anecdote about P.-S. Laplace, who, as the anecdote tells, used a glass of wine to show that resonantly forced swirl-type sloshing leads, contradictory to some hydrodynamic theorems, to a non-negligible rotational fluid-particles transport, Prandtl (1949) [10] conducted a series of dedicated model tests on resonant sloshing in an upright circular cylindrical tank, which performs an orbital (circular) horizontal motion with the forcing frequency close to the lowest natural sloshing frequency. The experimental observations confirmed the aforementioned phenomenon by Laplace. In 1964, Hutton [5] repeated Prandtl's experiments by using a longitudinally excited tank and the fact that the swirl-type stable steady-state sloshing is possible in a small vicinity of the primary resonant zone for that excitation type. The Hutton' paper reports visual observations and measurements of angular particle-transport velocity *versus* the vertical/horizontal probe position and the surface wave amplitude. Because this velocity was measured far from the tank walls/bottom and its nondimensional values had a clearly quadratic character with respect to the nondimensional wave amplitude, Hutton logically related the mean angular particle transport to the angular Stokes drift, which is also a second-order hydrodynamic phenomenon in the Lagrangian specification [4]. However, his idea was not supported by the experiments. The most attractive discrepancy between the measured and theoretical particle-transport velocity was that the Stokes drift exponentially decays to the bottom but the experimental values weakly depend on the vertical probe position. Royon-Lebeaud, Hopfinger & Cartellier [14] followed Hutton [5] by applying a harmonic horizontal excitation to the tank. They observed and documented the *Lagrange–Prandtl–Hutton* phenomenon writing down that "nonlinear (swirling) waves can transfer angular momentum to the whole liquid column that starts to rotate". Because, in contrast to the orbital tank excitation [10], the longitudinal forcing causes the swirl-type sloshing only in the primary resonance zone, the works [5] and [14] were not able to examine whether the angular mean transport is actually due to the resonant sloshing.

A growing interest to the liquid sloshing dynamics in laboratory tanks (incl. bioreactors) initialised a series of dedicated model tests whose particular focus was a mean (steady) rotational flow accompanying the swirl-type wave motions. Because the related applications deal with orbital tank excitations of various frequencies and amplitudes (radii of the orbits), namely, with the Prandtl' case, the corresponding novel experimental reports in [1,2,11,12,15] documented what happens in the primary resonance zone, away from any resonances (linear sloshing theory is applicable), and where the so-called secondary resonances matter [3]. All the authors discussed the steady-streaming flow due to the oscillatory Stokes boundary layers on the wetted tank walls/bottom and below the free surface but, since theoretical description of the steady-streaming looks an open problem in three-dimensional context [1, 13], the actually-done theoretical estimates were based on the angular Stokes drift prediction (the Lagrangian contribution to the mean flow). As matter of fact, they followed Hutton [5] and, therefore, a satisfactory agreement was found away from the resonance cases, but, as expected, the measured and theoretical results differ in the studied resonant zones.

Staying within the framework of the Eulerian specification of fluid flows, the present paper constructs a mathematical theory, which clarifies coupling the time-averaged (mean rotational) flow and the resonant swirling waves in an upright circular cylindrical tank when the forcing frequency is close to the lowest natural sloshing frequency (the primary resonance). Elliptical orbital horizontal tank motions are assumed, for which rotary and longitudinal excitations are two limit cases. The constructed theory assumes defined the Stokes steady-streaming at the wetted tank surface and the mean free surface and uses the Moiseev-type asymptotic technique [9]. The latter technique is normally employed for analysing the steady-state sloshing with irrotational flows when the forcing frequency is close to the lowest natural sloshing frequency and there are no secondary resonances [3]. The present study admits rotational flows of an incompressible inviscid liquid. According to the Moiseev asymptotic technique, the lowest-order asymptotic wave component is of the order $O(\epsilon^{1/3})$, where $O(\epsilon)$ characterises the forcing amplitude. This means that the lowest-order steady flow component $\boldsymbol{W}$ equals to $O(\epsilon^{2/3})$. Focusing, as in [3], on four amplitude parameters associated with the $O(\epsilon^{1/3})$ amplifications of the two lowest natural sloshing modes by the primary cosine- and sine- harmonic components and $\boldsymbol{W}$, derives four differential equations with respect to the four amplitude parameters (differentiation by the slow time) and a nonlinear partial differential equation of the first order with respect to $\boldsymbol{W}$. The latter equation is defined in the hydrostatic liquid domain and appear as a necessary solvability condition of the original free-surface problem. The coupling occurs, if and only if, a swirl-type wave is realised. The differential equation for $\boldsymbol{W}$ requires the corresponding inhomogeneous boundary condition, which should, most probably, follow from the Stokes boundary layer steady-streaming. Because this boundary condition is the only inhomogeneous quantity in the solvability condition with respect to $\boldsymbol{W}$, the constructed theory describes, in fact, a physical mechanism how and why the Stokes boundary layer and associated steady-streaming generate the mean rotational flow in bulk.

## 1 Statement

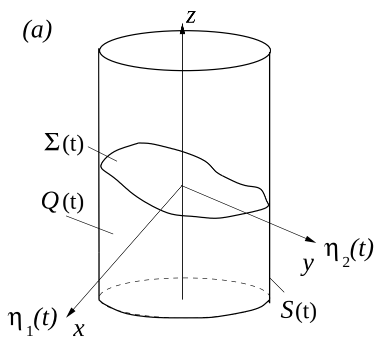


**Figure 1.** Panel (a): Upright circular cylindrical tank moves horizontally along an elliptic orbit so that the nondimensional tank translatory velocity is $\boldsymbol{v}_O(t) = (v_{O1}(t), v_{O2}(t), 0) = (\dot{\eta}_1(t), \dot{\eta}_2(t), 0)$, where $\eta_1(t) = \eta_{1a}\cos t$, $\eta_2(t) = \eta_{2a}\sin t$. Coordinate system is rigidly fixed with the tank so that the mean free surface $\Sigma_0$ belongs to $Oxy$ and the origin is in the centre of $\Sigma_0$. Panel (b): The hydrostatic liquid shape: $Q_0$ is the mean liquid domain, $\Sigma_0$ is the mean free surface, $V_0$ is the hydrostatically wetted walls, and $B_0$ is the bottom.

An upright circular cylindrical rigid tank is partially filled with an inviscid incompressible liquid; rotational flows are allowed. The tank moves horizontally and translatory along an elliptic orbit to excite a resonant liquid sloshing (the orbital frequency is close to the lowest natural sloshing frequency). The liquid sloshing dynamics is considered in the non-inertial tank-fixed coordinate system $Oxyz$ (cylindrical coordinates $(r, \theta, z)$) so that $Oxy$-plane coincides with the mean free surface $\Sigma_0$ and $Oz$ passes through the centre of $\Sigma_0$ ($V_0$ is the mean wetted tank walls, $Q_0$ is the mean liquid domain and $B_0$ is the tank bottom). Figure 1 introduces the main geometric notations including the translatory tank velocity $\boldsymbol{v}_O(t) = (\dot\eta_1(t), \dot\eta_2(t), 0) = (-\eta_{1a}\sin t, \eta_{2a}\cos t, 0)$, the time-depending liquid domain $Q(t)$, and the free surface $\Sigma(t)$. The nondimensional forcing amplitudes $\eta_{1a}$ and $\eta_{2a}$ are small, i.e.

$$\eta_{1a} \sim \eta_{2a} = O(\epsilon) \ll 1. \tag{1}$$

The *nondimensional* sloshing problem is formulated by adopting the characteristic linear size, time and mass

$$l_* = r_0/k, \quad t_* = 1/\sigma \text{ and } m_* = \rho l_*^3, \tag{2}$$

respectively, where $r_0$ is the tank radius, $\sigma$ is the forcing frequency, $\rho$ is the liquid density, and $k$ is the lowest positive root of the transcedental equation $J_1'(k) = 0$ ($J_1$ is the Bessel function of the first kind).

**Remark 1.1.** *Using the denominator $k$ in (2) (typical characteristic dimension equals to $r_0$ [3]) is needed to get analytically simpler expressions for the two lowest degenerating natural sloshing modes, which take now the form*

$$J_1(r)Z(z)\cos\theta \quad \textit{and} \quad J_1(r)Z(z)\cos\theta; \quad Z(z) = \cosh(z+h)/\sinh h, \tag{3}$$

*where $h$ is the nondimensional mean liquid depth; the natural sloshing frequencies are computed by*

$$\sigma_{Mi}^2 = \frac{g}{l_*} k_{Mi} \tanh\left(k_{Mi}\frac{h}{l_*}\right), \quad J_M'(k_{Mi}k) = 0; \quad M \ge 0, \; i \ge 1 \tag{4}$$

*(according to the chosen normalisation, $k_{11} = 1$; $g$ is the gravity acceleration).*

Within the framework of the *Eulerian specification*, the liquid flow is described by the velocity field $\boldsymbol{v}(r,\theta,z,t)$, the equation $z = \zeta(r,\theta,t)$ determines the free surface and the pressure field is defined by $p(r,\theta,z,t)$. These three unknowns should be found from the corresponding free-surface problem, which consists of the kinematic and dynamic parts. The *kinematic* part includes the continuity equation and the normal-velocity boundary conditions

$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{in} \quad Q(t), \tag{5a}$$

$$\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{v}_O\cdot\boldsymbol{n} \quad \text{on} \quad S(t), \tag{5b}$$

$$\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{v}_O \cdot \boldsymbol{n} + \dot{\zeta}/\sqrt{1 + (\nabla\zeta)^2} \quad \text{on} \quad \Sigma(t), \tag{5c}$$

(henceforth, the *dot* denotes the partial *time derivative*), where $\boldsymbol{n}$ is the outer normal. In addition, one should require the volume conservation condition

$$\int_{Q(t)} dQ = \text{const}. \tag{6}$$

The *dynamic* part includes the Euler equation and the dynamic (constant–pressure) condition. The nondimensional Euler equation in the (tank-fixed) non-inertial coordinate system takes the form [6]:

$$\dot{\boldsymbol{v}} - (\boldsymbol{v} - \boldsymbol{v}_O) \times \operatorname{rot} \boldsymbol{v} + \nabla \left[\tfrac{1}{2}|\boldsymbol{v}|^2 - \boldsymbol{v} \cdot \boldsymbol{v}_O + p + \bar{g}z\right] = \mathbf{0} \quad \text{in} \quad Q(t), \tag{7}$$

where $\bar{g} = g/(l_* \sigma^2)$ is the nondimensional gravity acceleration. The constant-pressure condition reads as

$$p = 0 \quad \text{on} \quad \Sigma(t). \tag{8}$$

**Remark 1.2.** *Because the vorticity* $\boldsymbol{\omega} = \operatorname{rot} \boldsymbol{v}$ *is not zero, one can introduce the vorticity equation written (after applying the rotor operation to the Euler equation (7)) in the form*

$$\dot{\boldsymbol{\omega}} = \operatorname{rot}\left[(\boldsymbol{v} - \boldsymbol{v}_O) \times \boldsymbol{\omega}\right] \quad in \quad Q(t). \tag{9}$$

*After solving (9),* $\boldsymbol{v}$ *may be restored by using the Biot-Savart law.*

## 2 Almost steady-state asymptotic solution

**The Moiseev-type asymptotic solution** [3, 9] implies that, if the forcing amplitude has the order $O(\epsilon)$ (1), the dominant wave amplitude response is associated with the lowest (primary excited) natural sloshing modes (3) of the order $O(\epsilon^{1/3})$. In addition, the Moiseev detuning condition

$$\Lambda = \frac{\sigma_{11}^2}{\sigma^2} - 1 = O(\epsilon^{2/3}) \tag{10}$$

is required, which expresses the closeness of the forcing frequency to the lowest natural sloshing frequency $\sigma_{11}$. Employing the multi-timing procedure also introduces the “quick” $2\pi$ periodic oscillations by $t$ and the slow time variable [7, 8]

$$\tau = \tfrac{1}{2}\epsilon^{2/3} t. \tag{11}$$

*Asymptotic almost-periodic solution* of the free-surface problem (5)–(8) is posed in terms of $O(\epsilon^{1/3})$ where each summand is a function of spatial variables, $\tau$, and it is $2\pi$-periodic by $t$. The velocity field

$$\boldsymbol{v} = \boldsymbol{v}_{1/3}(r,\theta,z,t;\tau) + \boldsymbol{v}_{2/3}(r,\theta,z,t;\tau) + \boldsymbol{v}_{3/3}(r,\theta,z,t;\tau) + \ldots$$
$$= \nabla\phi_{1/3} + \left[\nabla\phi_{2/3} + \boldsymbol{w}_{2/3}\right] + \left[\boldsymbol{v}_O + \nabla\phi_{3/3} + \boldsymbol{w}_{3/3}\right] + \ldots \quad (12)$$

has the lowest-order asymptotic quantity identical to that for irrotational flows but the lowest-order rotational flow component emerges in the second-order approximation. The free-surface elevations and the pressure are posed as

$$\zeta(r,\theta,t;\tau) = \zeta_{1/3} + \zeta_{2/3} + \zeta_{3/3} + \ldots, \quad (13a)$$
$$p(r,\theta,z,t;\tau) = -\bar{g}z + p_{1/3} + p_{2/3} + p_{3/3} + \ldots, \quad (13b)$$

where $p_{0/3} = -\bar{g}z$ is the hydrostatic pressure.

**The lowest-order asymptotic component** can be taken from [3]. By introducing the four slowly-varying amplitude parameters $a(\tau) \sim \bar{a}(\tau) \sim \bar{b}(\tau) \sim b(\tau) = O(\epsilon^{1/3})$ at the primary harmonics by $t$, the component becomes expressed by the two lowest natural sloshing modes and, accounting for specific normalisation (2), it takes the form

$$\phi_{1/3}(r,\theta,z,t;\tau) = Z(z)J_1(r)\left[\cos\theta\,(-a(\tau)\sin t + \bar{a}(\tau)\cos t)\right.$$
$$\left.+ \sin\theta\,(-\bar{b}(\tau)\sin t + b(\tau)\cos t)\right], \quad (14a)$$

$$\zeta_{1/3}(r,\theta,t;\tau) = J_1(r)\left[\cos\theta\,(a(\tau)\cos t + \bar{a}(\tau)\sin t)\right.$$
$$\left.+ \sin\theta\,(\bar{b}(\tau)\cos t + b(\tau)\sin t)\right], \quad (14b)$$

$$p_{1/3}(r,\theta,z,t;\tau) = -\dot{\phi}_{1/3} = Z(z)J_1(r)\left[\cos\theta\,(a(\tau)\cos t + \bar{a}(\tau)\sin t)\right.$$
$$\left.+ \sin\theta\,(\bar{b}(\tau)\cos t + b(\tau)\sin t)\right]; \quad (14c)$$

$$\boldsymbol{v}_{1/3} = \nabla\phi_{1/3} = \left[-a(\tau)\sin t + \bar{a}(\tau)\cos t\right]\,\boldsymbol{v}_a(r,\theta,z)$$
$$+ \left[-\bar{b}(\tau)\sin t + b(\tau)\cos t\right]\,\boldsymbol{v}_b(r,\theta,z), \quad (14d)$$

where

$$\boldsymbol{v}_a(r,\theta,z) = \left( J_1'(r)\cos\theta Z(z), -\frac{J_1(r)}{r}\sin\theta Z(z), J_1(r)\cos\theta Z'(z) \right),$$
$$\boldsymbol{v}_b(r,\theta,z) = \left( J'(r)\sin\theta Z(z), \frac{J_1(r)}{r}\cos\theta Z(z), J_1(r)\sin\theta Z'(z) \right) \tag{15}$$

are considered in the cylindrical coordinate frame whose three unit vectors are $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{z}}$, respectively.

The lowest-order solution (14) satisfies (5)–(7) within to higher-order terms. Inserting (14) into the dynamic boundary condition (8) leads to

$$(Z(0) - \bar{g}) \left[ \cos\theta(a(\tau)\cos t + \bar{a}(\tau)\sin t) + \sin\theta(\bar{b}(\tau)\cos t + b(\tau)\sin t) \right] = 0,$$

which is, globally, of the order $O(\epsilon^{/3/3})$ due to the Moiseev asymptotic detuning (10), i.e.

$$Z(0) - \bar{g} = Z(0)\left[1 - \frac{g}{l_* Z(0)}\frac{1}{\sigma^2}\right] = Z(0)\left[1 - \frac{\sigma_{11}^2}{\sigma^2}\right] = O(\epsilon^{2/3}). \tag{16}$$

**Classification of the resonant surface wave regimes** may be done by using the lowest-order approximation of the free surface motions (14b). As explained in [3], one classifies either *standing* or *swirling* wave type associated with

$$ab - \bar{a}\bar{b} \equiv 0 \text{ and } ab - \bar{a}\bar{b} \not\equiv 0, \tag{17}$$

respectively. Swirling (rotary) wave implies a progressive angular wave motions. Both wave types are extensively discussed in [3] and Ch. 9 of [4].

**The mean (time-averaged) flow.** According to (12), (13), the rotational flow velocity field component $\boldsymbol{w}(r,\theta,z,t;\tau)$ and the vortex $\boldsymbol{\omega}(r,\theta,z,t;\tau)$ are of the order $O(\epsilon^{2/3})$ so that

$$\boldsymbol{w} = \boldsymbol{w}_{2/3} + \boldsymbol{w}_{3/3} + \ldots, \quad \boldsymbol{\omega} = \operatorname{rot}\boldsymbol{v} = \operatorname{rot}\boldsymbol{w} = \boldsymbol{\omega}_{2/3} + \boldsymbol{\omega}_{3/3} + \ldots.$$

Furthermore, irrotational wave motions in bulk (associated with $\nabla\phi_{k/3}$) cannot generate a mean flow and, therefore, one can relate the mean (time-averaged) velocity field to $\boldsymbol{w}_{2/3}$, i.e.

$$\boldsymbol{W}(r,\theta,z;\tau) = \boldsymbol{w}_{2/3} = \langle \boldsymbol{v} \rangle_t, \ \boldsymbol{\Omega}(r,\theta,z;\tau) = \boldsymbol{\omega}_{2/3} = \langle \boldsymbol{\omega} \rangle_t = \operatorname{rot}\boldsymbol{w}_{2/3}. \tag{18}$$

Because $\boldsymbol{w}$ can be restored from $\boldsymbol{\omega}$ by using the Biot–Savart law, our forthcoming focus will be on the vortex $\boldsymbol{\omega}$. Using the vorticity equation (9) in the (3/3)-approximation,

$$\dot{\boldsymbol{\omega}}_{3/3} = \operatorname{rot}\, \left[\boldsymbol{v}_{1/3} \times \boldsymbol{\Omega}\right], \tag{19}$$

where $\boldsymbol{v}_{1/3}$ is given by (14d) and $\left\langle \boldsymbol{\omega}_{3/3} \right\rangle_t = \left\langle \dot{\boldsymbol{\omega}}_{3/3} \right\rangle_t = \mathbf{0}$ (according to definition (18)), gives the solution

$$\begin{aligned}\boldsymbol{\omega}_{3/3}(r,\theta,z,t;\tau) &= (a(\tau)\cos t + \bar{a}(\tau)\sin t)\operatorname{rot}\left[\boldsymbol{v}_a(r,\theta,z) \times \boldsymbol{\Omega}(r,\theta,z;\tau)\right] \\ &+ \left(\bar{b}(\tau)\cos t + b(\tau)\sin t\right)\operatorname{rot}\left[\boldsymbol{v}_b(r,\theta,z) \times \boldsymbol{\Omega}(r,\theta,z;\tau)\right], \end{aligned} \tag{20}$$

which is a function of the mean vortex $\boldsymbol{\Omega}(r,\theta,z;\tau)$.

To find $\boldsymbol{\Omega}(r,\theta,z;\tau)$, the time-averaged (4/3)-approximation

$$\left\langle \tfrac{1}{2}\epsilon^{2/3}\partial_\tau \boldsymbol{\Omega} + \dot{\boldsymbol{\omega}}_{4/3} = \operatorname{rot}\left[\boldsymbol{v}_{1/3} \times \boldsymbol{\omega}_{3/3} + (\nabla\phi_{2/3} + \boldsymbol{W}) \times \boldsymbol{\Omega}\right]\right\rangle_t \tag{21}$$

is needed, which leads, accounting for (18), the *necessary solvability* condition

$$\begin{aligned}&\tfrac{1}{2}\epsilon^{2/3}\partial_\tau \boldsymbol{\Omega} = \operatorname{rot}\left[\boldsymbol{W} \times \boldsymbol{\Omega}\right] \\ &+ \boxed{\tfrac{1}{2}(ab - \bar{a}\bar{b})}\operatorname{rot}\,\left[\boldsymbol{v}_b \times \operatorname{rot}(\boldsymbol{v}_a \times \boldsymbol{\Omega}) - \boldsymbol{v}_a \times \operatorname{rot}(\boldsymbol{v}_b \times \boldsymbol{\Omega})\right] \ \text{ in } Q_0,\end{aligned} \tag{22}$$

where $\operatorname{div}\boldsymbol{\Omega} = \operatorname{div}\operatorname{rot}\boldsymbol{W} \equiv 0$ and $\boldsymbol{v}_b, \boldsymbol{v}_a$ are defined by (15).

**Remark 2.1.** *Getting a unique $\boldsymbol{W}$ from $\boldsymbol{\Omega}$ by using the Biot-Savart law requires an appropriate boundary condition for $\boldsymbol{W}$. If exist, the restored $\boldsymbol{W}$ converts (22) to a nonlinear integral-and-differential equation with respect to $\boldsymbol{\Omega}$. Alternatively, substituting $\boldsymbol{\Omega} = \operatorname{rot}\boldsymbol{W}$ into (22) and addiing the aforementioned boundary conditions transform (22) to a boundary value problem.*

**Remark 2.2.** *The mean vortex $\boldsymbol{\Omega}$ is coupled with sloshing, if and only if, $ab - \bar{a}\bar{b} \not\equiv 0$, namely, only for swirl-type wave regimes (17).*

**Remark 2.3.** *Within the framework of the adopted hydrodynamic model, one can suggest the zero boundary condition*

$$\boldsymbol{W}\cdot\boldsymbol{n} = 0 \ \textit{on} \ \ S_0 + \Sigma_0,\ k \geq 2. \tag{23}$$

*However, this condition would lead, generally speaking, to the trivial solution* $\boldsymbol{W} = \boldsymbol{\Omega} = \mathbf{0}$*. To have a non-trivial one, an inhomogeneous boundary condition is needed instead of (23). Appropriate inhomogeneous condition follows from the Stokes steady-streaming at the wetted tank surface and beneath the free surface. Because the steady-streaming is of the second order* $O(\epsilon^{2/3})$ *with respect to the lowest-order velocity field (14d), its usage for the second-order variable* $\boldsymbol{W}$ *is consistent with the Moiseev-type asymptotic technique and one can postulate*

$$\boldsymbol{W} = \tfrac{1}{2}(ab - \bar{a}\bar{b})\boldsymbol{W}_0 \quad \text{on walls/bottom/free surface walls.} \tag{24}$$

*Dedicated studies on the analytical form (24) are needed.*

**Remark 2.4.** *When introducing*

$$\boldsymbol{\Omega}(r, \theta, z; \tau) = \Omega_r \,\hat{\boldsymbol{r}} + \Omega_\theta \,\hat{\boldsymbol{\theta}} + \Omega_z \,\hat{\boldsymbol{z}} \tag{25}$$

*the rot-term at the* $\frac{1}{2}(ab - \bar{a}\bar{b})$ *multiplier of (22) reads as*

$$- \left[ Z^2(z) f(r) - \frac{J_1^2(r)}{r^2 \sinh^2 h} \right] \partial_\theta \,\boldsymbol{\Omega} \\ - \hat{\boldsymbol{\theta}} \left( 2\,\Omega_r \left[ 3g(r) Z^2(z) + \frac{g_1(r)}{\sinh^2 h} \right] - r f(r) \left[ Z^2(z) \right]_z' \Omega_z \right) \tag{26}$$

*where*

$$g(r) = \frac{(r J_1'(r) - J_1(r))^2}{r^4}, \quad f(r) = g(r) + \frac{J_1^2(r)}{r^2}, \\ g_1(r) = \frac{J_1(r)(r J_1'(r) - J_1(r))}{r^2}. \tag{27}$$

*The graphs of* $f(r) > 0$, $g(r) > 0$ *and* $g_1(r)$ *by (27) are demonstrated in figure 2. One can prove that*

$$f(r) = \frac{1}{4} \exp \left( -6 \int_0^r \frac{g(r)}{r f(r)} dr \right). \tag{28}$$

**The secular (solvability) equations** for the four dominant amplitude parameters $a(\tau), \bar{a}(\tau), \bar{b}(\tau), b(\tau)$ were derived in [3] for the potential flow model. The procedure can be generalised to the studied case.

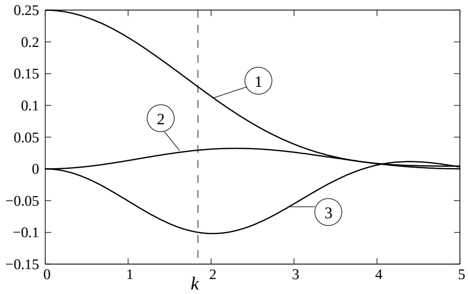


**Figure 2.** The graphs of $f(r)$ (marked by 1), $g(r)$ (by 2) and $g_1(r)$ (by 3).

Using the kinematic boundary condition (5c) gives

$$\partial_z \phi_{3/3} = \dot{\zeta}_{3/3} + \tfrac{1}{2}\epsilon^{2/3}\partial_\tau \zeta_{1/3} + \partial_r \zeta_{1/3} W_r + \tfrac{1}{r}\partial_\theta \zeta_{1/3} W_\theta - \partial_z W_z \zeta_{1/3} + \boxed{\text{n.p.}} \quad \text{on} \quad \Sigma_0, \quad (29)$$

where the framed quantity corresponds to the nonlinear potential flow components.

The Euler equation (7) leads to

$$\left(\dot{\boldsymbol{w}}_{3/3} - \nabla\phi_{1/3} \times \boldsymbol{\Omega} - \nabla\Psi\right) + \nabla\Big[p_{3/3} + \dot{\phi}_{3/3} + \tfrac{1}{2}\epsilon^{2/3}\partial_\tau \phi_{1/3} + \dot{\boldsymbol{v}}_O \cdot \boldsymbol{x} + \nabla\phi_{1/3} \cdot \nabla\phi_{2/3} + \nabla\phi_{1/3} \cdot \boldsymbol{W} + \Psi\Big] = \boldsymbol{0} \quad \text{in} \quad Q_0, \quad (30)$$

where $\boldsymbol{x} = (r, \theta, z)$ and an auxiliary function $\Psi$ is introduced to provide the both summands equal to zero. Applying the divergence to $\dot{\boldsymbol{w}}_{3/3} - \nabla\phi_{1/3} \times \boldsymbol{\Omega} - \nabla\Psi = 0$ and adding the zero boundary conditions $\boldsymbol{w}_{3/3} \cdot \boldsymbol{n} = \boldsymbol{0}$ on $\Sigma_0 + B_0 + V_0$ give

$$\Psi = [a \sin t - \bar{a} \cos t]\Psi_a(r, \theta, z; \tau) + [\bar{b} \sin t - b \cos t]\Psi_b(r, \theta, z; \tau), \quad (31)$$

where

$$\nabla^2 \Psi_{a,b} = \operatorname{div}[\boldsymbol{v}_{a,b} \times \boldsymbol{\Omega}] \text{ in } Q_0, \ \nabla\Psi_{a,b} \cdot \boldsymbol{n} = [\boldsymbol{v}_{a,b} \times \boldsymbol{\Omega}] \cdot \boldsymbol{n} \text{ on } V_0 + \Sigma_0 + B_0. \quad (32)$$

The second summand of (30) derives the (3/3) pressure approximation

$$p_{3/3} = -\dot{\phi}_{3/3} - \tfrac{1}{2}\epsilon^{2/3}\partial_\tau \phi_{1/3}$$

$$- \dot{\boldsymbol{v}}_O \cdot \boldsymbol{x} - \nabla\phi_{1/3} \cdot \nabla\phi_{2/3} - \nabla\phi_{1/3} \cdot \boldsymbol{W} - \Psi - C(t), \quad (33)$$

where $C(t)$ is an arbitrary time-depending function.

The dynamic boundary condition (8) in the (3/3)-approximation transforms to

$$p_{3/3} + \partial_z p_{0/3}\zeta_{3/3} + \partial_z p_{2/3}\zeta_{1/3} + \partial_z p_{1/3}\zeta_{2/3} + \tfrac{1}{2}\partial_z^2 p_{1/3}\zeta_{1/3}^2 + \left[p_{1/3} + \partial_z p_{0/3}\zeta_{1/3}\right] = C(t) \quad \text{on} \quad \Sigma_0, \quad (34)$$

where the square brackets expression is of the $O(\epsilon^{3/3})$ order due to the Moiseev detuning (as remarked ahead of (16)) but the rotational flow components are only due to $p_{3/3}$ by (33).

Finding $\phi_{1/3}$ from (29), substituting the result into (33) and using all together in (34) to get projections of (34) on $J_1(r)\cos\theta\cos t$, $J_1(r)\cos\theta\sin t$, $J_1(r)\sin\theta\cos t$, and $J_1(r)\sin\theta\sin t$ on $\Sigma_0$ derive the following four secular equations (the solvability condition)

$$\epsilon^{2/3}\bar{a}' + a\left(\Lambda + m_1(a^2 + \bar{a}^2 + \bar{b}^2) + m_3 b^2\right) + \bar{a}\left((m_1 - m_3)b\bar{b} + V_{a,\cos}[\boldsymbol{W}]\right) + bV_{b,\cos}[\boldsymbol{W}] = \epsilon_x, \quad (35a)$$

$$- \epsilon^{2/3}a' + \bar{a}\left(\Lambda + m - 1(a^2 + \bar{a}^2 + b^2) + m_3\bar{b}^2\right) + a\left((m_1 - m_3)b\bar{b} - V_{a,\cos}[\boldsymbol{W}]\right) - \bar{b}V_{b,\cos}[\boldsymbol{W}] = 0, \quad (35b)$$

$$- \epsilon^{2/3}\bar{b}' + b\left(\Lambda + m_1(b^2 + \bar{b}^2 + \bar{a}^2) + m_3 a^2\right) + \bar{b}\left((m_1 - m_3)a\bar{a} - V_{b,\sin}[\boldsymbol{W}]\right) - aV_{a,\sin}[\boldsymbol{W}] = \epsilon_y, \quad (35c)$$

$$\epsilon^{2/3}b' + \bar{b}\left(\Lambda + m_1(b^2 + \bar{b}^2 + a^2) + m_3\bar{a}^2\right) + b\left((m_1 - m_3)a\bar{a} + V_{b,\sin}[\boldsymbol{W}]\right) + \bar{a}V_{a,\sin}[\boldsymbol{W}] = 0, \quad (35d)$$

where $m_1$ and $m_3$ are the $h$-dependent coefficients, which are, within to the adopted normalisation, the same as in [3],

$$\epsilon_x = \eta_{1a}P,\ \epsilon_y = \eta_{2a}P; \quad P = \frac{1}{Z(0)||J_1||^2}\int_0^k r^2 J_1(r)dr, \quad ||J_1||^2 = \int_0^k rJ_1^2(r)dr, \quad (36)$$

and $V_{a,\cos}[\boldsymbol{W}], V_{b,\cos}[\boldsymbol{W}], V_{a,\sin}[\boldsymbol{W}]$ and $V_{b,\sin}[\boldsymbol{W}]$ and the linear operators acting on $\boldsymbol{W}$ as follows

$$V_{(*1),(*2)}[\boldsymbol{W}] = \frac{1}{\pi||J_1||^2}\int_0^k\int_{-\pi}^{\pi} r\,[\,(*2)\theta\,]\,J_1(r)V_{(*1)}[\boldsymbol{W}]\,d\theta dr, \qquad (37)$$

in which

$$V_a[\boldsymbol{W}] = \Bigg[2J_1'(r)\cos\theta\,W_r - \frac{2}{r}J_1(r)\sin\theta\,W_\theta + \frac{J_1(r)\cos\theta}{Z(0)}(W_z - \partial_z W_z) - \frac{\Psi_a[\boldsymbol{W}]}{Z(0)}\Bigg]_{z=0}, \qquad (38a)$$

$$V_b[\boldsymbol{W}] = \Bigg[2J_1'(r)\sin\theta\,W_r + \frac{2}{r}J_1(r)\cos\theta\,W_\theta + \frac{J_1(r)\sin\theta}{Z(0)}(W_z - \partial_z W_z) - \frac{\Psi_b[\boldsymbol{W}]}{Z(0)}\Bigg]_{z=0}. \qquad (38b)$$

*In summary*, the (Stokes) steady-streaming causes a mean rotational flow in the liquid bulk for the swirl-type sloshing. The swirl-type sloshing and the mean rotational flow are coupled and described by the nonlinear differential equations (35) with respect to the dominant wave amplitude parameters $a(\tau), \bar{a}(\tau), \bar{b}(\tau), b(\tau) \sim O(\epsilon^{1/3})$ and (22) governing $\boldsymbol{W}$.

## 3 Conclusion

By assuming known a steady-streaming at the wetted tank walls/bottom and beneath the free surface and using an inviscid incompressible liquid with rotational flows, we derive the governing equations *coupling* the wave amplitude parameters and the corresponding mean (rotational) liquid flow in bulk. The Moiseev-type asymptotic technique is employed for an upright circular cylindrical tank orbitally forced with the forcing frequency close to the lowest natural sloshing frequency. The coupling occurs, if and only if, a swirl-type sloshing is realised.

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