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Rational homotopy type of free and pointed mapping spaces between spheres

1 Introduction

Denote by $\mathrm{map}(X,Y)$ (respec. $\mathrm{map}^*(X,Y)$) the space of free (respec. pointed) maps from X to Y. Whenever X is a finite CW-complex and Y is a nilpotent CW-complex of finite type over $\mathbb{Q}$, then [8] any path component of both $\mathrm{map}(X,Y)$ and $\mathrm{map}^*(X,Y)$ are nilpotent CW-complexes of finite type over $\mathbb{Q}$ and in particular, it can be rationalized in the classical sense. From the Sullivan approach to rational homotopy theory [9], and based in the fundamental work of Haefliger [7], there is a standard procedure [2, 3] to obtain Sullivan models of the path components $\mathrm{map}_f(X,Y)$ and $\mathrm{map}^*_f(X,Y)$ of $\mathrm{map}(X,Y)$ and $\mathrm{map}^*(X,Y)$ respectively, containing the map $f\colon X\to Y$. In this note, we show the advantage of this procedure and use it repeatedly to explicitly describe the rational homotopy type of free and pointed mapping spaces between spheres:

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Theorem 1.1. *(i) For m odd and any $n \geq 1$,*

$$map(S^n, S^m) \simeq_{\mathbb{Q}} \begin{cases} S^m \times K(\mathbb{Z}, m-n), & \text{if } m > n. \\ \bigcup_{\mathbb{N}} S^m, & \text{if } m = n. \\ S^m, & \text{if } m < n. \end{cases}$$

$$map^*(S^n, S^m) \simeq_{\mathbb{Q}} \begin{cases} K(\mathbb{Z}, m-n), & \text{if } m > n. \\ \bigcup_{\mathbb{N}} *, & \text{if } m = n. \\ *, & \text{if } m < n. \end{cases}$$

(ii) For m even and any $n \geq 1$,

$$map(S^n, S^m) \simeq_{\mathbb{Q}} \begin{cases} Y, & \text{if } m > n. \\ S^m \times K(\mathbb{Z}, 2m-n-1) \bigcup_{\mathbb{N}} S^{2m-1}, & \text{if } m = n. \\ S^m \times K(\mathbb{Z}, 2m-n-1), & \text{if } m < n < 2m-1. \\ \bigcup_{\mathbb{N}} S^m, & \text{if } m = 2n-1. \\ S^m, & \text{if } m = 2n-1. \end{cases}$$

$$map^*(S^n, S^m) \simeq_{\mathbb{Q}} \begin{cases} K(\mathbb{Z}, m-n) \times K(\mathbb{Z}, 2m-n-1), & \text{if } m > n. \\ \bigcup_{\mathbb{N}} K(\mathbb{Z}, 2m-n-1), & \text{if } m = n. \\ K(\mathbb{Z}, 2m-n-1), & \text{if } m < n < 2m-1. \\ \bigcup_{\mathbb{N}} *, & \text{if } m = 2n-1. \\ *, & \text{if } m < n. \end{cases}$$

Here, $\simeq_{\mathbb{Q}}$ means "rationally homotopy equivalent"; $\bigcup$ denotes the disjoint union; and Y is a rational space which sits in a fibration of the form

$$S^m_{\mathbb{Q}} \times K(\mathbb{Q}, m-n) \to Y \to K(\mathbb{Q}, 2m-n-1).$$

We should mention that the above result might be known, or easily deduced by specialists. However, to our knowledge, it has not been made explicit in the literature. Thus, this paper reviews in a particular a useful situation, the general procedure of obtaining the rational homotopy type of both free and pointed mappping spaces.

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2 Models of mapping spaces between spheres

In this section we prove the theorem above. We highly depend on known facts and techniques arising from rational homotopy theory. All of them can be found in the excellent reference [6] which is now standard on the subject. Here, we simply present a summary of some basic facts.

For any simply connected, or more generally, nilpotent CW-complex of finite type X, its rationalization $X_{\mathbb{Q}}$ is a rational space (i.e., its homotopy groups are rational vector spaces), together with a map $X \to X_{\mathbb{Q}}$ inducing isomorphisms in rational homotopy.

On the other hand, to any space X there corresponds, in a contravariant way, its *minimal Sullivan model* which is a particular *Sullivan algebra* $(\Lambda V, d)$, unique up to isomorphism, which algebraically models the rational homotopy type of the space X, or equivalently, the homotopy type of its rationalization $X_{\mathbb{Q}}$. By ΛV we mean the free commutative algebra generated by the graded vector space V, i.e., $\Lambda V = TV/I$ where TV denotes the tensor algebra over V and I is the ideal generated by $v \otimes w - (-1)^{|w||v|} w \otimes v$, $\forall v, w \in V$. The differential d satisfies a certain minimality condition which, in the simply connectid case it translates to: for any element of $v \in V$, dv is a polynomial in ΛV with no linear term.

This correspondence yields an equivalence between the homotopy categories of 1-connected rational spaces of finite type and that of 1-connected rational commutative differential graded algebras of finite type. Indeed, this equivalence is the restriction to the appropriate subcategories of the classical adjoint functors [1]

$$\mathrm{SimplSets} \underset{\langle\,\rangle}{\overset{A_{PL}}{\rightleftarrows}} \mathrm{CDGA}$$

between the homotopy categories of commutative differential graded algebras and simplicial sets.

One can precise, through these functors, the notion of models of non connected spaces. As in [3], a *model* of a general space X, not necessarily connected, is a $\mathbb{Z}$-graded free CDGA $(\Lambda W, d)$ such that its simplicial realization $\langle(\Lambda W, d)\rangle$ has the same homotopy tye of the Milnor simplicial approximation of $X_{\mathbb{Q}}$, $S_*(X_{\mathbb{Q}})$.

We now introduce the Haefliger model [7] of the free and pointed mapping spaces $\mathrm{map}(X, Y)$, $\mathrm{map}^*(X, Y)$, via the functorial description of Brown-Szczarba [2].

Let B be a finite dimensional CDGA (commutative differential graded algebra) model of the finite CW-complex X and let $A = (\Lambda V, d)$ be a Sullivan model of the nilpotent CW-complex of finite type Y.

Denote by $B^\sharp = Hom(B, \mathbb{Q})$ the differential graded coalgebra, dual of B and therefore negatively graded, and consider the $\mathbb{Z}$-graded CDGA $\Lambda(A \otimes B^\sharp)$ with the natural differential induced by the one on A and by the dual δ of the differential on B. Now, consider the differential ideal $I \subset \Lambda(A \otimes B^\sharp)$ generated by $1 - 1 \otimes 1^\sharp$ and by the elements of the form

$$v_1 v_2 \otimes \beta - \sum_j (-1)^{|v_2||\beta'_j|}(v_1 \otimes \beta'_j)(v_2 \otimes \beta''_j),$$

with $v_1, v_2 \in V$, $\beta \in B$ and $\Delta\beta = \sum_j \beta'_j \otimes \beta''_j$. Then, the composition

$$\rho \colon \Lambda(V \otimes B^\sharp) \hookrightarrow \Lambda(A \otimes B^\sharp) \twoheadrightarrow \Lambda(A \otimes B^\sharp)/I$$

is an isomorphism of graded algebras [2, Thm.1.2]. Thus, we may consider on $\Lambda(V \otimes B^\sharp)$ the differential $\widetilde{d}$ for which the above becomes an isomorphisms of CDGA's. To explicitly determine $\widetilde{d}$ on the generator $v \otimes \beta \in V \otimes B^\sharp$, first compute $dv \otimes \beta + (-1)^{|v|} v \otimes \delta\beta$ and then use the relations which generate the ideal I to express $dv \otimes \beta$ as an element of $\Lambda(V \otimes B^\sharp)$.

Then, it turns out [2, Thm.1.3] that $\big(\Lambda(V \otimes B^\sharp), \widetilde{d}\big)$ is a model of $\mathrm{map}(X, Y)$ Moreover, if $B^\sharp_+$ denotes the subspace of $B^\sharp$ of strictly negative elements, $\big(\Lambda(V \otimes B^\sharp_+), \widetilde{d}\big)$ is a model of $\mathrm{map}^*(X, Y)$.

For the model of the components of $\mathrm{map}(X, Y)$ and/or $\mathrm{map}^*(X, Y)$ we follow the approach and notation of [3, 4]:

For any free CDGA $(\Lambda W, d)$, in which W is $\mathbb{Z}$-graded, and any algebra morphism $u \colon \Lambda W \longrightarrow \mathbb{Q}$ consider the differential ideal K_u generated by $A_1 \cup A_2 \cup A_3$, being

$$A_1 = W^{<0},\ A_2 = dW^0,\ A_3 = \{\alpha - u(\alpha) : \alpha \in W^0\}.$$

$(\Lambda W, d)/K_u$ is again a free CDGA of the form $(\Lambda(\overline{W}^1 \oplus W^{\geq 2}), d_u)$ in which $\overline{W}^1$ is a complement in W^1 of $d(W^0)$ modulo identifications via A_1 and A_3, see [3, §4] for details. Note that, $\overline{W}^1$ depends also on u. Moreover, if $(\Lambda W, d)$ is a model of a non-connected space X and u corresponds to a 0-simplex of X, as remarked in [2, 4.3], $(\Lambda(\overline{W}^1 \oplus W^{\geq 2}), d_u)$ is a Sullivan model of the path component of X containing the fixed 0-simplex.

Next, consider $\big(\Lambda(V\otimes B^\sharp), d\big)$ the model of $\mathrm{map}(X,Y)$ which we have just recalled and let $\varphi\colon (\Lambda V,d)\to B$ be a model of a given map $f\colon X\to Y$. The morphism φ clearly induces a natural augmentation which shall be denoted also by $\varphi\colon \big(\Lambda(V\otimes B^\sharp), \widetilde{d}\big)\to\mathbb{Q}$. Applying the process above to this particular case yields the Sullivan algebra

$$\big(\Lambda\big(\overline{V\otimes B^\sharp}^1\otimes(V\otimes B^\sharp)^{\geq 2}\big),\widetilde{d}_\varphi\big)$$

which constitutes a Sullivan model of $\mathrm{map}_f(X,Y)$. In the same way,

$$\big(\Lambda\big(\overline{V\otimes B^\sharp_+}^1\otimes(V\otimes B^\sharp_+)^{\geq 2}\big),\widetilde{d}_\varphi\big)$$

is a Sullivan model of $\mathrm{map}^*_f(X,Y)$.

To prove our Theorem we will apply all of the above to the particular case of choosing $X=S^m$ and $Y=S^n$ to be spheres, $m,n\geq 1$. For it, recall that, if m is an odd integer, the minimal model of S^m is the exterior algebra on a generator of degree m with zero differential $(\Lambda x_m,0)$. On the other hand, if m is even, the minimal model of S^m is $(\Lambda x_m, y_{2m-1},d)$, $dx_m=0$, $dy_{2m-1}=x_m^2$. From now on, subscripts will always denote degree.

On the other hand, for any n, a coalgebra model of S^n is $B=\langle 1,\alpha_n\rangle$, in which α_n is a primitive cycle of degree $-n$, i.e., $\Delta\alpha_n=\alpha_n\otimes 1+1\otimes\alpha_n$.

We will now distinguish different cases:

Case 1: m odd.

A model of $\mathrm{map}(S^n,S^m)$ is therefore,

$$(\Lambda(x_m\otimes 1, x_m\otimes\alpha_n),0).$$

To avoid excessive notation we set $x_m\otimes 1=a_m$ and $x_m\otimes\alpha_n=b_{m-n}$ and rewrite the above as:

$$(\Lambda(a_m,b_{m-n}),0).$$

On the other hand, taking into account that the evaluation fibration

$$\mathrm{map}^*(S^n,S^m)\to\mathrm{map}(S^n,S^m)\to S^m$$

is modelled by

$$(\Lambda a_m,0)\to(\Lambda(a_m,b_{m-n}),0)\to(\Lambda b_{m-n},0),$$

a model for $\mathrm{map}^*(S^n,S^m)$ is simply $(\Lambda b_{m-n},0)$.

We now obtain Sullivan models for the path components and identify the homotopy type of their realizations.

Case 1.1: free maps.

$\mathbf{m > n}$:

In this case $(\Lambda(a_m, b_{m-n}), 0)$ is already a Sullivan model as b_{m-n} has positive degree. Hence the only component of $\mathrm{map}(S^n, S^n)$ has the rational homotopy type of the product $S^m \times K(\mathbb{Z}, m-n)$ of S^m with the Eilenberg-MacLane space of type $(\mathbb{Z}, m-n)$.

$\mathbf{m = n}$:

In this case b_{m-n} has degree 0 and there are a countable number of non homotopic morphisms $\varphi_\lambda \colon (\Lambda(a_m, b_{m-n}), 0) \to \mathbb{Q}$, one for each $\lambda \in \mathbb{Q}$, sending b_{m-n} to 1. Then, the procedure above give rise to a countable number of components, just like in the integral case, each of which with Sullivan model $(\Lambda a_m, 0)$ whose realization is just $S^m_{\mathbb{Q}}$.

Observe, as in [5, Ex. 3], that in this case, since $\mathrm{map}(S^m, S^n)$ has infinitely many components, its rational homology in degree zero is infinite dimensional. Thus, its rational cohomology, also in degree zero, has uncountable dimension. This sharply contrasts with the rational cohomology of its model $(\Lambda(x_m \otimes 1, x_m \otimes \alpha_n), 0)$, which in degree zero has countable dimension. This illustrates why, in the non-connected case, a model of a space does not preserve, in general, rational homotopy invariants.

$\mathbf{m < n}$:

In this case b_{m-n} has negative degree and therefore, it vanishes when considering models of components. Therefore, there is only one component with Sullivan model $(\Lambda a_m, 0)$ whose realization is again $S^m_{\mathbb{Q}}$.

Case 1.2: pointed maps.

$\mathbf{m > n}$:

As in this case b_{m-n} is of positive degree there is only one component with Sullivan model $(\Lambda b_{m-n}, 0)$ whose realization is $K(\mathbb{Q}, m-n)$.

$\mathbf{m = n}$:

As in the free case, there are a countable number of non homotopic morphisms $\varphi_\lambda \colon (\Lambda b_{m-n}, 0) \to \mathbb{Q}$, one for each $\lambda \in \mathbb{Q}$, sending b_{m-n} to λ. Thus, when replacing b_{m-n} by λ we obtain $\mathbb{Q}$ as a model for the corresponding component and therefore, each component is rationally trivial.

m < n:

In this case b_{m-n} has negative degree so there is only one component which is rationally trivial.

Case 2: m even.

In this case, a model of $\mathrm{map}(S^n, S^m)$ is again computed via the methods above:

$$(\Lambda(x_m \otimes 1, y_{2m-1} \otimes 1, x_m \otimes \alpha_n, y_{2m-1} \otimes \alpha_n), d)$$

To avoid excessive notation, as before, we set $x_m \otimes 1 = a_m$, $y_{2m-1} \otimes 1 = c_{2m-1}$, $x_m \otimes \alpha_n = b_{m-n}$, $y_{2m-1} \otimes \alpha_n = z_{2m-n-1}$ and rewrite this model as:

$$(\Lambda(a_m, c_{2m-1}, b_{m-n}, z_{2m-n-1}), d),$$

in which the differential is given by

$$da_m = db_{m-n} = 0, \quad dc_{2m-1} = a_m^2, \quad dz_{2m-n-1} = 2a_m b_{m-n}.$$

Concerning pointed maps and taking into account that the evaluation fibration

$$\mathrm{map}^*(S^n, S^m) \to \mathrm{map}(S^n, S^m) \to S^m$$

is modelled by

$$(\Lambda(a_m, c_{2m-1}), d) \to (\Lambda(a_m, c_{2m-1}, b_{m-n}, z_{2m-n-1}), d), \to (\Lambda(b_{m-n}, z_{2m-n-1}), 0)$$

a model for $\mathrm{map}^*(S^n, S^m)$ is simply $(\Lambda(b_{m-n}, z_{2m-n-1}), 0)$.

Then, on components:

Case 2.1: free maps.

m > n:

In this case both b_{m-n}, z_{2m-n-1} have positive degrees so the above is already a Sullivan model. Hence, there is only one component whose realization is a space Y which fits in a fibration of the form

$$S^m_{\mathbb{Q}} \times K(\mathbb{Q}, m-n) \to Y \to K(\mathbb{Q}, 2m-n-1).$$

m = n:

Now z_{2m-n-1} has positive degree but b_{m-n} has degree zero and there are a countable number of non homotopic morphisms

$$\varphi_\lambda\colon (\Lambda(a_m, c_{2m-1}, b_{m-n}, z_{2m-n-1}), d) \to \mathbb{Q},$$

one for each $\lambda \in \mathbb{Q}$, sending b_{m-n} to λ. This gives rise to a countable number of components. If $\lambda \neq 0$ then the corresponding component has Sullivan minimal model $(\Lambda c_{2m-1}, 0)$ whose realization is S^{2m-1}. On the other hand, if $\lambda = 0$, the corresponding component has Sullivan minimal model $(\Lambda(a_m, c_{2m-1}, z_{2m-n-1}), d)$, with $dz_{2m-n-1} = 0$ whose realization is of the rational homotopy type of $S^m \times K(\mathbb{Z}, 2m-n-1)$.

m < n < 2m − 1:

Now b_{m-n} has negative degree but z_{2m-n-1} has positive degree. Thus, there is only one component with model $(\Lambda(a_m, c_{2m-1}, z_{2m-n-1}), d)$, with $dz_{2m-n-1} = 0$ whose realization is again $S^m_{\mathbb{Q}} \times K(\mathbb{Q}, 2m-n-1)$.

n = 2m − 1:

Here, b_{m-n} has negative and z_{2m-n-1} has degree zero. Hence, we have a countable number of components arising from the CDGA morphisms $\varphi_\lambda\colon (\Lambda(a_m, c_{2m-1}, b_{m-n}, z_{2m-n-1}), d) \to \mathbb{Q}$, one for each $\lambda \in \mathbb{Q}$, sending z_{2m-n-1} to λ. Each of them produces via the procedure above the same Sullivan model $(\Lambda(a_m, c_{2m-1}), d)$ whose realization is $S^m_{\mathbb{Q}}$.

n > 2m − 1:

In this case both b_{m-n}, z_{2m-n-1} have negative degrees. Hence, there is only one component with model $(\Lambda(a_m, c_{2m-1}), d)$ whose realization is $S^m_{\mathbb{Q}}$.

Case 2.2: pointed maps.

m > n:

In this case, both b_{m-n}, z_{2m-n-1} have positive degrees and the model $(\Lambda(b_{m-n}, z_{2m-n-1}), 0)$ is already minimal. Thus, there is one component rationally equivalent to $K(\mathbb{Z}, m-n) \times K(\mathbb{Z}, 2m-n-1)$.

m = n:

Now, z_{2m-n-1} has positive degree but b_{m-n} has degree zero. Thus, as in precedent cases, it can be replaced by any rational number giving rise to a countable number of components each of which with model $(\Lambda z_{2m-n-1}, 0)$, whose realization is $K(\mathbb{Q}, 2m-n-1)$.

$\mathbf{m < n < 2m - 1:}$

Here, z_{2m-n-1} has positive degree but b_{m-n} is of negative degree. Hence, there is only one component with model $(\Lambda z_{2m-n-1}, 0)$ whose realization is again $K(\mathbb{Q}, 2m-n-1)$.

$\mathbf{n = 2m - 1:}$

In this case b_{m-n} has negative degree and z_{2m-n-1} is of degree zero. Hence, in the procedure of obtaining components, b_{m-n} vanishes while z_{2m-n-1} is replaced by any rational number giving rise to a countable number of rationally trivial components.

$\mathbf{n > 2m - 1:}$

Finally, both b_{m-n}, z_{2m-n-1} have negative degrees and there is only one component which is rationally trivial.

Summarizing all of the above finishes the proof of our theorem.

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