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Semi-free R^1 action and Bott map

1 Introduction

Let M^n be a compact closed manifold of dimension at least 3. We study the R^1-Bott functions on M^n. Separately investigated R^1-invariant Bott functions on M^{2n} with a semi-free circle action which has finitely many fixed points. The aim of this paper is to find exact values of minimal numbers of singular circles of some indices of R^1-invariant Bott functions on M^{2n}.

Closely related to R^1-Bott function on a manifold M^n is a more flexible object, the decomposition of round handle of M^n. In its turn, to study the round handles decomposition of M^n we use a diagram, i.e. a graph which carries the information about the handles.

2 R^1-Bott maps

Let M^n be a smooth manifold and $f : M^n \to \mathbf{R^1}$ smooth function or $f : M^n \to \mathbb{R}^1$ non-homotopy to zero a smooth map. Suppose that $x \in M^n$ one of its critical points of f. In neighborhood U of critical point x in both cases the map f can be viewed as a function with values in $\mathbf{R}$. Consider the Hessian $\Gamma_x(f) : T_x \times T_x \to \mathbf{R}$ at this point. Recall that the index of the Hessian is called the maximum dimension of T_x, where $\Gamma_x(f)$ is negative definite. The index of $\Gamma_x(f)$ is called the index of the critical point x, and the corank of $\Gamma_x(f)$ is called the corank of x. Suppose that the set of critical points of f forms a disjoint union of smooth submanifolds K^i_j whose their dimensions do not exceed $n-1$. A connected critical submanifold $K^{i_0}_{j_0}$ is called **non-degenerate** if the

Hessian is non-degenerate on subspaces orthogonal to $K^{i_0}_{j_0}$ (i.e. has corank equal to $n - i_0$) at each point $x \in K^{i_0}_{j_0}$.

Definition 2.1. *A mapping $f : M^n \to \mathbb{R}^1$ is called a Bott map if all of its critical points form nondegenerate critical submanifolds which do not intersect the boundary of M^n.*

Consider the following important example of Bott map:

Definition 2.2. *A mapping $f : M^n \to \mathbb{R}^1$ is called an R^1-Bott map if all of its critical points form nondegenerate critical circles.*

Note that an R^1-Bott map do not exist on any smooth manifold (see Theorem 2.3).

Theorem 2.1. *Let M^n be a smooth closed manifold and suppose that on M^n there is R^1- Bott map $f : M^n \to \mathbb{R}^1$. Denote by $\gamma \subset M^n$ its critical circle and let $f(\gamma) = a$. Then there is interval $(a - \varepsilon, a + \varepsilon) \subset \mathbb{R}^1$ and a system of coordinates in a neighborhood of γ of one of the following types:*

1) *Trivial $\nu : S^1 \times D^{n-1}(\varepsilon) \to M^n$; where $D^{n-1}(\varepsilon)$, a disc of radius ε, $\nu(R^1 \times 0) = \gamma$, and $f(\nu(\theta, x)) = a - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_{n-1}^2$, for $(\theta, x) \in S^1 \times D^{n-1}(\varepsilon)$.*

2) *Twisted $\tau : ([0,1] \times D^{n-1}(\varepsilon)/ \sim) \to M^n$, where τ is a smooth embedding such that $(\tau([0,1]) \times 0/ \sim) = \gamma$ and $f(\tau(t,x)) = a - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_{n-1}^2$, for $(t,x) \in (\tau : [0,1] \times D^{n-1}(\varepsilon)/ \sim)$. Here $([0,1] \times D^{n-1}(\varepsilon)/ \sim)$ is diffeomorphic to $S^1 \times D^{n-1}(\varepsilon)$ by identifying $0 \times D^{n-1}(\varepsilon)$ and*
   $1 \times D^{n-1}(\varepsilon)$ by the mapping:
   $(0, x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_{n-1}) \leftrightarrow (1, -x_1, ..., x_\lambda, -x_{\lambda+1}, ..., x_{n-1})$.

The number λ is called the index of the critical circle γ.

Let M^n be a smooth manifold, and $f : M^n \to R^1$ an R^1-Bott map. Each nice R^1-Bott map defines a filtration on manifold $M^n : \mathbb{M}_0(f) \subset \mathbb{M}_1(f) \subset ... \subset \mathbb{M}_{n-1}(f) \subset M^n$. The existence of a nice R^1-Bott map from manifold M^n into the circle is equivalent to existance of a R^1-round handle decomposition on the manifold M^n. We recall some necessary definitions.

Definition 2.3. *We define an n-dimensional round handle R_λ of index λ by $M_\lambda = M^1 \times D^\lambda \times D^{n-\lambda-1}$, where D^i is a disc of dimension i.*
Define twisted n-dimensional round handle TM_λ of index λ $(0 < \lambda < n-1)$ by $TM_\lambda = [0,1] \times D^\lambda \times D^{n-\lambda-1}/\sim$, where identification is given by the map: $(0, x_1, ..., x_\lambda, x_{\lambda+1}, ..., x_{n-1}) \leftrightarrow (1, -x_1, ..., x_\lambda, -x_{\lambda+1}, ..., x_{n-1})$.

Definition 2.4. *We say that the manifold M^n_λ is obtained from a smooth manifold M^n by attaching a round handle of index λ if*
$M^n_\lambda = M^n \bigcup_\varphi S^1 \times D^\lambda \times D^{n-\lambda-1}$, where $\varphi : R^1 \times \partial D^\lambda \times D^{n-\lambda-1} \longrightarrow \partial N^n$ is a smooth embedding.

Manifold M^n_λ is obtained from a smooth manifold M^n by gluing a twisted round handles of index λ, if $M^n_\lambda = N^n \bigcup_\varphi [0,1] \times D^\lambda \times D^{n-\lambda-1}/\sim$, where $\varphi : ([0,1] \times \partial D^\lambda \times D^{n-\lambda-1}/\sim) \to M^n$ is a smooth embedding.

Definition 2.5. *The M^1- round handle decomposition on the closed manifold M^n is called a filtration*

$$M^{n-1} \times [0, \varepsilon] \bigcup M^n_0(R) \subset M^n_1(R) \subset ... \subset M^n_{n-1}(R) = M^n,$$

where M^{n-1} is a closed submanifold of M^n, the manifold $M^n_i(R)$ obtained from the manifold $M^n_{i-1}(R)$ by gluing round and twisted round handles of index i .

In what follows we recall the relationship between S^1 and the decomposition by round handles ([11]).

Theorem 2.2. *Let M^n be a smooth closed manifold. The following two conditions are equivalent:*

1) *On the manifold M^n there is a nice R^1-Bott map with the critical circles $\gamma_1, ..., \gamma_k$ of index $\lambda_1, ..., \lambda_k$ with trivial coordinate systems and critical circles $\tilde{\gamma}_1, ..., \tilde{\gamma}_l$ of indices $\mu_1, ..., \mu_l$ with twisted coordinate systems.*

2) *Manifold M^n admits a decomposition by round handles consisting of round handles $R_{\lambda_1}, ..., R_{\lambda_k}$ of index $\lambda_1, ..., \lambda_k$ and of twisted round handles*
$TR_{\mu_1}, ..., TR_{\mu_l}$ of indices $\mu_1, ..., \mu_l$ so that the critical circle γ_i corresponds to a round handle R_{λ_i} $(1 \le i \le k)$, and the critical circle $\tilde{\gamma}_j$ corresponds to a twisted round handle TR_{μ_j} $(1 \le j \le l)$.

Thus each nice R^1-Bott map from manifold M^n into the $\mathbb{R}^1$ generates a round handle decomposition of M^n and vice versa.

We are interested in conditions when an R^1-Bott map on M^n has the property that all of its critical circles have trivial coordinate system. We recall the necessary facts from an [4].

Lemma 2.1. *Let M^n be a smooth closed manifold, $f : M^n \to \mathbb{R}^1$ an R^1-Bott map, and c its critical value. Suppose $\varepsilon > 0$, and that on the interval $[c-\varepsilon, c+\varepsilon]$ there are no other critical values. Assume that on the surface level $f^{-1}(c)$ there are critical circles $\gamma_1, ..., \gamma_k$ of indices $\lambda_1, ..., \lambda_k$ with trivial coordinate systems and there are critical circles $\tilde{\gamma}_1, ..., \tilde{\gamma}_l$ of indices $\mu_1, ..., \mu_l$ with twisted coordinate systems, then the homology groups $H_*(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$ is generated exactly by the handles which correspond to the critical circles $\gamma_1, ..., \gamma_k, \tilde{\gamma}_1, ..., \tilde{\gamma}_l$. Each circle γ_i generates two subgroups that are isomorphic to $\mathbf{Z}$, a direct product of the homology group $H_{\lambda_i}(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$, and the other in the homology group $H_{\lambda_{i+1}}(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$. Each circle $\tilde{\gamma}_j$ generates a subgroup $\mathbf{Z}_2$ which is direct product in a group $H_{\mu_j}(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z})$.*

Corolary 2.1. *Let M^n be a smooth closed manifold, $f : M^n \to \mathbb{R}^1$ an S^1-Bott map, and $c_1, ..., c_k$ its critical values. Suppose $\varepsilon_i > 0 (1 \le i \le k)$ such that the interval $[c_i-\varepsilon_i, c_i+\varepsilon_i]$ has no other critical values. Then on a level surface $f^{-1}(c_i)$ there are only critical circles with trivial coordinate systems if and only if the nonzero homology groups $H_*(f^{-1}[c_i-\varepsilon_i, c_i+\varepsilon_i], f^{-1}(c_i-\varepsilon_i), \mathbf{Z})$ are free Abelian groups.*

Thus we have a homological criterion when R^1-Bott map do not have critical circle with twisted coordinate systems.

In the next section, we give another class of R^1-Bott map which do not possess the critical circle with twisted coordinate systems.

Definition 2.6. *Let M^n be a smooth closed manifold. The number $\chi_i(M^n) = \mu(H_i(M^n, \mathbf{Z})) - \mu(H_{i-1}(M^n, \mathbf{Z})) + \ldots + (-1)^{i+1}\mu(H_0(M^n, \mathbf{Z}))$ is called the i-th Euler characteristic of M^n, where $\mu(H)$ is a minimal number of generators H.*

Definition 2.7. *A dimension λ of closed manifold M^n is called singular if $H_\lambda(M^n, \mathbf{Z})$ is a nonzero finite group distinct from $\mathbf{Z}_2 \oplus ... \oplus \mathbf{Z}_2$ and $\chi_{\lambda-1}(M^n) = \chi_{\lambda+1}(M^n) = 0$.*

Definition 2.8. *Let M^n be a smooth closed manifold. A round handle decomposition is called quasiminimal, if one of the following holds:*

1) *the number of round handles of index i equals to $\rho(\chi_i(M^n)) + \varepsilon_i$, where $\varepsilon_i = 0$,if dimension $i+1$ is nonsingular and $\varepsilon_i = 1$,if dimension $i+1$ is singular,*

2) *the number of round handles of index i equals to $\rho(\chi_i(M^n))$, if dimension $i+1$ is singular, then there is only one handle of index $i+2$.*

In both cases, the number of round handles of index $i+1$ equals to $\rho(\chi_{i+1}(M^n))$. A round handle decomposition is called minimal, if number of round handles of index i equals to $\rho(\chi_i(M^n))$ for all i.

Using the decomposition of manifold on handles and the diagram technique, we can easily prove the following fact [4].

Proposition 2.1. *Let M^n be a smooth closed simply connected manifold $(n > 5)$. Then M^n admits a* **quasiminimal** *decomposition into round handles. If manifold M^n have not singular dimensions, then M^n admits a* **minimal** *decomposition into round handles.*

Definition 2.9. *Let the manifold M^n admits R^1-Bott function, then* **R^1-Morse number** $M_i^{R^1}(M^n)$ **of index** *i is the minimum number of singular circles of index i taken over all R^1-Bott functions on M^n.*

Lemma 2.2. *Let on a closed manifold M^n exist a smoth function $f : M^n \to \mathbb{R}$ such that each connected component of the singular set Σ_f of f is either a nondegenerate critical point $p_i (i = 1, ..., k)$ or a non-degenerate critical circle $S^1_j (j = 1, ..., l)$. Then the Euler characteristic of the manifold M^n is equal to $\chi(M^n) = \sum_{i=1}^{k} (-1)^{index(p_i)}$.*

Proof. It is known that for any Morse function on the manifold M^n $g : M^n \to \mathbb{R}$ with critical points $p_i (i = 1, ..., q)$ there is the formula $\chi(M^n) = \sum_{i=1}^{q} (-1)^{index(p_i)}$. By small perturbation of the function f any non-degenerate critical circle S^1_j of index λ can be replaced by non-degenerate critical points of idexes λ and $\lambda + 1$ [1]. Therefore the contribution in the formula of Euler characteristic this critical points will not give and we obtain the desired formula. $\square$

3 Manifolds with free R^1-action

Let on smooth manifold M^n there is smooth free circle action. Then of course the set M^n/S^1 is a manifold and natural projection $p : M^n \to$

M^n/S^1 is fibre bundle. Any smooth R^1-invariant map $f : M^n \to \mathbb{R}^1$ from the manifold M^n into the circle $\mathbb{R}^1$ is called an R^1**-invariant round Bott map** if each connected component of the singular set Σ_f is non-degenerate critical circle.

It is clear that if f be a R^1-invariant round Bott map from the manifold M^n then it projection $\pi_*(f) : M^n/S^1 \to \mathbb{R}^1$, is a Morse map. And conversaly, if $g : M^n/S^1 \to \mathbb{R}^1$ be a Morse map from the manifold M^n/S^1 then $\pi_*^{-1}(g) = g \circ \pi : M^n \to \mathbb{S}^1$ is R^1-invariant round Bott map from the manifold M^n. The critical point of the index λ of the map g correspond to critical circle of the index λ of the map $\pi_*^{-1}(g)$.

Definition 3.1. *Let on smooth manifold M^n there are smooth free circle action $\theta : M^n \times S^1 \to M^n$ and R^1-invariant round Bott map $f : M^n \to \mathbb{S}^1$. For the triple (M^n, θ, f)* R^1**-equivariant round Morse-Bott number of index** *i, $\mathfrak{M}_i^{eqS^1}(M^n, \theta, f)$ is the minimum number of singular circles of index i taken over all homotopic to f R^1-invariant round Bott map from M^n into $\mathbb{R}^1$.*

Definition 3.2. *Let on smooth manifold M^n there is Morse maps $f : M^n \to \mathbb{R}^1$. For the couple (M^n, f)* **Morse-Novikov number of index** *i, $\mathfrak{M}_i(M^n, f)$ is the minimum number of critical points of index i taken over all homotopic to f Morse maps from M^n into $\mathbb{R}^1$.*

It is clear that there is following fact.

Corolary 3.1. *Let on smooth manifold M^n there is smooth free circle action $\theta : M^n \times R^1 \to M^n$ and let $p : M^n \to M^n/R^1$ is natural projection. Suppose that $f : M^n/R^1 \to \mathbb{R}^1$ be a Morse map. Then $\mathfrak{M}_i^{eqR^1}(M^n, \theta, f \cdot p) = \mathfrak{M}_i(M^n/S^1, f)$.*

Definition 3.3. *Let on smooth manifold M^n there is smooth free circle action $\theta : M^n \times R^1 \to M^n$. Then this circle action is* **minimal** *if there exist R^1-invariant round Bott map $f : M^n \to \mathbb{R}^1$ such that $\mathfrak{M}_i^{eqR^1}(M^n, \theta, f) = \mathfrak{M}_i^{S^1}(M^n, f)$ for all i.*

Suppose that on smooth compact manifold $M^n (n > 6)$ there is smooth free circle action $\theta : M^n \times R^1 \to M^n$ and let $p : M^n \to M^n/R^1$ is natural projection. Suppose that $\pi_1(M^n) \approx \pi_1(M^n/R^1) \approx \mathbf{Z}$. Then from from results of Novikov [2] it follows that

$$\mathfrak{M}_i(M^n/R^1, f) = \mu(H_i(M^n/R^1, Z)) + \mu(TorsH_{i-1}(M^n/R^1, Z))$$

for any non-homotopy to zero Morse map $f : M^n/R^1 \to \mathbb{R}^1$. Therefore corollary 3.1 implies that $\mathfrak{M}_i^{eqR^1}(M^n, \theta, f \cdot p) = \mathfrak{M}_i^{R^1}(M^n, f)$

Theorem 3.1. *Let on smooth compact manifold $M^n (n > 6)$ there is smooth free circle action. Suppose that $\pi_1(M^n) \approx \pi_1(M^n/S^1) \approx \mathbf{Z}$. Then this circle action is minimal if and only if*

$$\mu(H_i(M^n/S^1, Z) + \mu(TorsH_{i-1}(M^n/S^1, Z) = \rho(\chi_i(M^n))$$

for all i.

Proof. **Necessary.** Suppose that on M^n there is minimal smooth free circle action. If $n > 6$ from results of Novikov [2] it follows that Morse number in dimension i of the manifold M^n/R^1 is equal
$\mathfrak{M}_i(M^n/S^1) = \mu(H_i(M^n/R^1, Z)) + \mu(TorsH_{i-1}(M^n/R^1, Z))$. There is equality
$\mathfrak{M}_i(M^n/S^1) = \mathfrak{M}_i^{eqR^1}(M^n)$. Because of the condition of minimal free circle action there is equality $\mathfrak{M}_i(M^n/R^1) = \mathfrak{M}_i^{eqR^1}(M^n) = \mathfrak{M}_i^{R^1}(M^n) = \rho(\chi_i(M^n))$.

Sufficiently. Consider on manifold M^n/R^1 Morse function with the number of critical points of index i equal
$M_i(M^n/R^1) = \mu(H_i(M^n/R^1, Z)) + \mu(TorsH_{i-1}(M^n/R^1, Z))$. By the construction and condition of the theorem we have the equalities
$M_i(M^n/S^1) = M_i^{eqR^1}(M^n) = \rho(\chi_i(M^n))$. But $M_i^{R^1}(M^n) = \rho(\chi_i(M^n))$ and therefore free action of R^1 is minimal. □

4 Manifolds with semi-free R^1-action

Let M^{2n} be a closed smooth manifold with semi-free R^1-action which has only isolated fixed points. It is known that every isolated fixed point p of a semi-free R^1-action has the following important property: near such a point the action is equivalent to a certain linear $S^1 = SO(2)$-action on $\mathbb{R}^{2n}$. More precisely, for every isolated fixed point p there exist an open invariant neighborhood U of p and a diffeomorphism h from U to an open unit disk D in $\mathbb{C}^n$ centered at origin such that h is conjugate to the given S^1-action on U to the S^1-action on $\mathbb{C}^n$ with weight $(1, \ldots, 1)$. We will use both complex, $(z_1, \ldots, z_n)$, and real coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ with $z_j = x_j + \sqrt{-1}y_j$. The pair (U, h) will be called a **standard chart** at the point p. Let $f : M^{2n} \to \mathbb{R}^1$ be a smooth R^1-invariant map from the manifold M^{2n} into the circle $\mathbb{R}^1$. Denote by Σ_f

the set of singular points of the map f. It is clear that the set of isolated singular points $\Sigma_f(p_j) \subset \Sigma_f$ of f coincides with the set of fixed points M^{R^1}.

For a nondegenerate critical point p_j there exist a standard chart (U_j, h_j) such that on U_j the map f is given by the following formula:

$$f = f(p) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2.$$

Notice that the index of nondegenerate critical point p_j is always even.

Denote by $\Sigma_f(R^1)$ the set singular points of the function f that are disconnected union of circles. These circles will be called singular.

A circle $s \in \Sigma f_(R^1)$ is called nondegenerate if there is an R^1-invariant neighborhood U of s on which R^1 acts freely and such that the point $\pi(s)$ is nondegenerate for the function $\pi_*(f) : U/R^1 \to \mathbb{R}$, induced on U/R^1 by the natural map $\pi : U \to U/R^1$. An invariant version of Morse lemma says that there exist an R^1-invariant neighborhood U of the circle s and coordinates $(x_1, \ldots, x_{2n-1})$ on U/R^1 such that the function $\pi_*(f)$ has the following presentation:

$$\pi_*(f) = \pi_*(f(\pi(s))) - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_{2n-1}^2.$$

By definition λ is the **index** of singular circle s.

Definition 4.1. *A smooth S^1-invariant function $f : M^{2n} \to \mathbb{R}$ on a manifold M^{2n} with a semi-free circle action which has isolated fixed points is called : R^1_*-Bott function if each connected component of the singular set Σ_f is either a nondegenerate fixed point or a nondegenerate critical circle.*

Theorem 4.1. *Assume that M^{2n} is the closed manifold with a smooth semi-free circle action which has isolated fixed points $p_1, \ldots, p_k$. Let for any fixed point p_j consider standard chart (U_j, h_j) and function*

$$f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2$$

on U_j, where λ_j is an ***arbitrary integer*** *from $0, 1, \ldots, n$.*

Then there exist an R^1-invariant R^1_*-Bott function f on M^{2n} such that $f = f_j$ on U_j.

Proof. Consider on U_j the function f_j. Let $\pi_*(f_j) : U_j/S^1 \to \mathbb{R}$, continuos function induced on U_j/R^1 by the natural map $\pi : U_j \to U_j/R^1$. It is clear that function $\pi_*(f_j)$ is smooth on manifold $(U_j \setminus$

$p_j)/R^1$. Denote by g smooth extension functions $\pi_*(f_j)$ on M^{2n}/R^1. By small deformation of the function g, that is fixed on U_j/R^1, we shall find function g_1 on M^{2n}/R^1 such that g_1 equal $\pi_*(f_j)$ on U_j/R^1 and g_1 have only non-degenerate critical points on $M^{2n} \setminus \bigcup(U_j/R^1)$. Then the function $f = g_1 \circ p$ satisfy conditions of the theorem. □

Theorem 4.2. *The number of fixed points of any smooth semi-free circle action on M^{2n} with isolated fixed points is always even and equal to the Euler characteristic of the manifold M^{2n}.*

$f_1 = f_1(p_1) + |z_1|^2 + \ldots + |z_n|^2$ on U_1 and $f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_n|^2$

on U_j $(2 \leq j \leq l)$ and extend such functions to S^1-invariant Bott function f on manifold $M^{2n} \setminus U_1 \bigcup U_2 \bigcup ... \bigcup U_l$. We suppose that U_j is diffeomorfic to open disk D^{2n} for any j. Consider manifold $V^{2n} = W^{2n} \setminus \bigcup U_j$. The boudary of manifod V^{2n} is disconnected union of spheres S^{2n-1}. By construction of manifold V^{2n} there is free cirle action. The boundary of the manifold V^{2n}/S^1 is disconnected union of complex projective spaces $\mathbb{CP}^{n-1}$. If the number of the boundary components of the manifold V^{2n}/S^1 is odd then we glue pairwise boundary components and obtain compact smoth manifold with with boundary $\mathbb{CP}^{n-1}$. From the well known fact that the manifold $\mathbb{CP}^{n-1}$ is non-cobordant to zero it follows that the number of fixed points of any smooth semi-free circle action on M^{2n} with isolated fixed points is even. The value of the Euler characteristic $\chi(M^{2n}) = 2k$ is follow from Lemma 3.4.□

Definition 4.2. *Let f be an R^1-invariant S^1_*-Bott function for smooth semi-free circle action with isolated fixed points $p_1, \ldots, p_{2k}$ on a closed manifold M^{2n}. Denote by λ_j the index of a critical point p_j of the function f. The* **state** *of the function f is the collection of numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, which we will be denoted by $St_f(\Lambda)$. It is clear that all numbers λ_j are even and $(0 \leq \lambda_j \leq 2n)$.*

Remark 4.1. *It follows from Theorem 4.2 that for every smooth semi-free circle action on a closed manifold M^{2n} with isolated fixed points $p_1, \ldots, p_{2k}$ and any collection even numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, such that $0 \leq \lambda_j \leq 2n$ there exists an R^1-invariant R^1_*-Bott functions f on M^{2n} with state $St_f(\Lambda)$.*

Definition 4.3. *Let M^{2n} be a closed smooth manifold with smooth semi-free circle action which has finitely many fixed points $p_1, \ldots, p_{2k}$. Fix any collection even numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, such that $0 \leq \lambda_j \leq 2n$.*

The R^1-Morse number $\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda))$ of index i is the minimum numbers of singular circles of index i taken over all R^1-invariant R^1_*-Bott functions f on M^{2n} with state $St_f(\Lambda)$.

There is an unsolved problem: for a manifold M^{2n} with a semi-free circle action which has finitely many fixed points **find exact values** of numbers $\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda))$.

5 About R^1-equivariant Morse numbers $\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda))$

Let M^{2n} be a compact closed manifold of dimension with semi-free circle action which has finite many fixed points $p_1, , ..., p_{2k}$. Denote by $\pi : M^{2n} \to M^{2n}/R^1$ canonical map. The set M^{2n}/R^1 is manifold with singular points $\pi(p_1), , ..., \pi(p_{2k})$. It is clear that neighborhood of any singular point is cone over $\mathbb{CP}^{n-1}$. If $f : M^{2n} \to \mathbb{R}$ be a smooth R^1-invariant R^1_*-Bott function on the manifold M^{2n}, then $\pi_*(f) : M^{2n}/R^1 \to \mathbb{R}$ is continuos function such that on smooth non-compact manifold $N^{2n-1} = M^{2n}/R^1 \setminus \bigcup_{j=1}^{2k} \pi(p_j)$ it is Morse function.

Choose an invariant neighborhood U_i of the point p_j diffeomorphic to the open unit disc $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_{j=1}^{2k} U_j$. Consider compact manifold $V^{2n-1} = (M^{2n} \setminus U)/R^1$, its boundary is a disconnected union of complex projective spaces $\partial V^{2n-1} = \mathbb{CP}_1^{n-1} \cup \ldots \cup \mathbb{CP}_{2k}^{n-1}$. It is clear that manifold $V^{2n-1} \setminus \partial V^{2n-1}$ and manifold N^{2n-1} are diffeomorphic. We use a manifold V^{2n-1} for the study of R^1-invariant R^1_*-Bott functions on the manifold M^{2n} with states $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$. Let $\partial_0 V^{2n-1}$ be a part of boundary of V^{2n-1} consist from r component $\mathbb{C}P^{2n-2}$ $(2k-1 \geq r \geq 1)$, and $\partial_1 V^{2n-1} = \partial V^{2n-1} \setminus \partial_0 V^{2n-1}$. On the manifold with boundary V^{2n-1} constructed Morse function $f : V \to [0, 1]$, such that $f^{-1}(0) = \partial_0 V^{2n-1}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function f we constructed on the manifold M^{2n} R^1-equivariant R^1_*-Bott function F with the state $St(0, \ldots, 0, 2n, \ldots, 2n)$, such that restriction $\pi_*(F)$ on V coinside with f. Therefore Morse number of index i $M_i(V^{2n-1}, \partial_0 V^{2n-1})$ of manifold with boundary V^{2n-1} is equal $\mathcal{M}_i^{S^1}(M^{2n}, St(0, \ldots, 0, 2n, \ldots, 2n)$.

Theorem 5.1. *Let M^{2n} $(2n > 8)$ be a closed smooth manifold admits a smooth semi-free circle action with isolated fixed points $p_1, \ldots, p_{2k}$. Then* *for the manifold M^{2n} with the state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$*

$$\mathcal{M}_i^{R^1}(M^{2n}, St(\Lambda) = \mathbb{D}^i(V^{2n-1}, \partial_0 V^{2n-1}) + \widehat{S}^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) +$$

$$+\widehat{S}^{i+1}_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) + dim_{N(Z[\pi])}(H^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}))$$

for $3 \leq i \leq 2n-4$.

Proof. Choose an invariant neighborhood U_i of the point p_i diffeomorphic to the unit disc $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_i U_i$. Let f_i be a function on U_i equal

$$f_i = |z_1|^2 + \ldots + |z_n|^2, \text{ and } f_j \text{ on } U_j \text{ equal } f_j = 1 - |z_1|^2 - \ldots - |z_n|^2,$$

for $i = 1, ..., r$, $j = r+1, ..., 2k-r$. Consider the manifold $V^{2n} = (M^{2n} \setminus U)/R^1$. It is clear that its boundary is a disconnected union of complex projective spaces $\partial V^{2n} = \mathbb{C}P_1^{2n-2} \cup \ldots \cup \mathbb{C}P_{2k}^{2n-2}$.

Let $\partial_0 V^{2n}$ be a part of boundary of V^{2n} consist from r component $\mathbb{C}P^{2n-2}$, that corespondent U_i and $\partial_1 V^{2n}$ be a part of boundary consist from component $\mathbb{C}P^{2n-2}$, that corespondent U_j. On manifold $V^{2n} = (M^{2n} \setminus U)/R^1$ constructed Morse function $f : V \to [0,1]$, such that $f^{-1}(0) = \partial_0 V^{2n}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function f we constructed on manifold M^{2n} S^1-equivariant S^1_*-Bott function F with the state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$, such that restriction F on U_i coinside with f_i, restriction F on U_j coinside with f_j and restriction $\pi_*(F)$ on V coinside with f. Therefore Morse number of cobordism V equal $\mathcal{M}^\lambda_{R^1}(M^{2n}, St(\Lambda))$ In the paper [12] there is value of Morse number of a cobordism. $\square$

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