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Periodicity generated by adding machines

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We show that a homeomorphism of the plane $\mathbb{R}^2$ with an invariant Cantor set $\mathbf{C}$, on which the homeomorphism acts as an adding machine, possesses periodic points arbitrarily close to $\mathbf{C}$. The existence of periodic points near an invariant Cantor set is related to a shape theory question whether a solenoid invariant in a flow defined on $\mathbb{R}^3$ must be contained in a larger movable invariant compactum.

1 Introduction

Let $\phi : X \to X$ be a homeomorphism ($\mathbb{Z}$-action) or a flow ($\mathbb{R}$-action) defined on a metric space X. A set $A \subset X$ invariant under ϕ is *Lyapunov stable* if for every neighborhood U of A there is a neighborhood V of A such that for every $p \in V$, the forward orbit of p is contained in U. J. Buescu and I. Stewart proved in [8] (see also [9]) that if $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism with an invariant Lyapunov stable Cantor set $\mathbf{C}$, and $h_{|\mathbf{C}}$ is an adding machine, then every neighborhood of $\mathbf{C}$ contains a periodic orbit of h. The theorem was also proved by H. Bell and K. Meyer in [2]. In addition, the authors construct in this paper a specific example of a Lyapunov stable adding machine in $\mathbb{R}^2$ invariant under a C^1 homeomorphism h of $\mathbb{R}^2$ and show that the theorem does not hold for a homeomorphism H on $\mathbb{R}^3$ and a Lyapunov stable adding machine invariant under H. We give a simple proof that without the assumption of Lyapunov stability a weaker version of the theorem holds: Every neighborhood of $\mathbf{C}$ contains a periodic point of h. The proof bears a similarity to the proof of the Cartwright-Littlewood Theorem given Morton Brown in [7]. The Cartwright-Littlewood Theorem asserts that if planar continuum Δ does not separate the plane $\mathbb{R}^2$ and is invariant under an orientation preserving homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$, then h has a fixed point $p \in \Delta$.

Much earlier E.S. Thomas considered in [18] one-dimensional solenoids invariant in a C^1 flow on a 3-manifold. A solenoid in this case is the inverse limit of circles with bonding maps being group homomorphisms. If almost all bonding maps are of degree one, then the solenoid is said to be trivial. Assuming that the flow on a non-trivial solenoid is minimal, the Poincaré first-return map on a local cross-section of the solenoid is an adding machine. The flow restricted to an invariant set is *minimal* on this set if every orbit is dense in the set. In case of a solenoid, this is equivalent to the fact that there are no fixed points in the solenoid, i.e., the flow is non-singular.

A compact invariant set is *isolated* if in some compact neighborhood it is the largest invariant set. The notion applies to both homeomorphisms and flows. Thomas uses isolating blocks, considered by C. Conley and R. W. Easton in [10] and previously by T. Ważewski in [20], in order to establish an Alexander-Spanier cohomology exact sequence involving the solenoid. He then shows that an invariant non-trivial solenoid in a nonsingular flow on a 3-manifold is not isolated. M. Kulczycki proved in [14] that under certain conditions, a planar adding machine is not isolated.

2 Adding machine

For a sequence of integers $(k_1, k_2, k_3, \ldots)$, each greater than one, denote by $\mathbf{C}(k_1, k_2, k_3, \ldots)$, or shortly by $\mathbf{C}$, the Cantor set $\Pi_{n=1}^{\infty} \mathbb{Z}/k_n\mathbb{Z}$.

Definition 1. *An adding machine is a homeomorphism $\alpha : \mathbf{C} \to \mathbf{C}$ such that if*

$$\alpha(i_1, i_2, i_3, \ldots) = (j_1, j_2, j_3, \ldots)$$

then

1. *if there is an* $m \geq 1$ *such that* $i_n = k_n - 1$ *for* $n < m$ *and* $i_m < k_m - 1$, *then* $j_n = 0$ *for* $n < m$, $j_m = i_m + 1$, *and* $j_n = i_n$ *for* $n > m$,

2. *otherwise* $j_m = 0$ *for all* m, *i.e., if* $i_m = k_m - 1$ *for* $m \geq 1$, *then* $j_m = 0$ *for* $m \geq 1$.

The map α is an *adding machine with base* $(k_1, k_2, k_3, \ldots)$ acting on $\mathbf{C}$. The Cantor set itself is ofter referred to as an adding machine; precisely, an adding machine is the pair $(\mathbf{C}, \alpha)$.

Definition 2. *Let* α *be an adding machine with base* $(k_1, k_2, k_3, \ldots)$ *acting on* $\mathbf{C}$. *For a finite sequence of integers* $i_1, \ldots, i_n$, $0 \leq i_j < k_j$ *for* $j \leq n$, *define a cylinder of length* n *as the set*

$$C_{i_1,\ldots,i_n} = \{(x_1, x_2, \ldots) \,|\, x_1 = i_1, \ldots, x_n = i_n\}.$$

Note that the cylinder $C_{i_1,\ldots,i_n}$ is invariant under α^s , where s is a multiple of the product $k_1 \cdots k_n$.

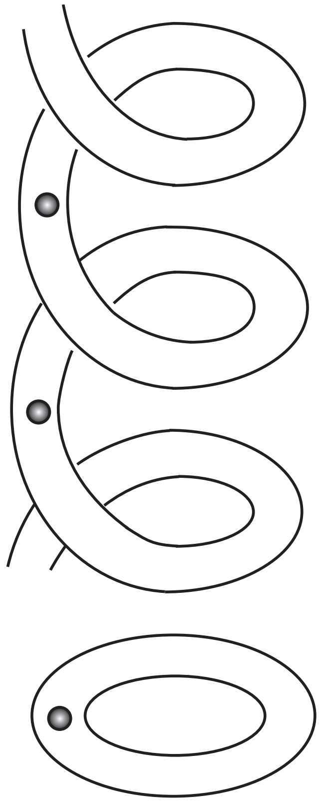
3 Periodic points near a planar adding machine

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and $\mathbf{C} = \mathbf{C}(k_1, k_2, k_3, \ldots) \subset \mathbb{R}^2$ an invariant Cantor set. Assume that $h_{|\mathbf{C}}$ is an adding machine with base $(k_1, k_2, k_3, \ldots)$. Let P be the set of periodic points of h in $\mathbb{R}^2$, including the fixed points although clearly the fixed points of h are away from C. Let $\mathrm{Cl}(P)$ be the closure of P. Each of the sets P and $\mathrm{Cl}(P)$ is invariant under h.

The theorem below shows that in every neighborhood of $\mathbf{C}$, there is a periodic point of h. Stability is not assumed.

Theorem. $\mathbf{C} \cap \mathrm{Cl}(P) \neq \emptyset$.

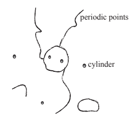
Proof. Suppose that $\mathbf{C} \cap \mathrm{Cl}(P) = \emptyset$. Let U be a component of $\mathbb{R}^2 \setminus \mathrm{Cl}(P)$ intersecting $\mathbf{C}$. Thus U contains a cylinder invariant under h^s, some power h^s of h. If U is simply connected, then by Brouwer's Theorem [6] we arrive at a contradiction that there is a fixed point of the orientation preserving homeomorphism $h^s \circ h^s$ in U, a periodic point of h outside P. The Brouwer Translation Theorem asserts that for a fixed point free,



orientation preserving homeomorphism of the plane no orbit of a point is bounded, hence there are no non-empty, compact, invariant sets.

In general, let $\widetilde{U}$ be the universal cover of U with $\pi : \widetilde{U} \to U$ the covering map. There is a cylinder $C_{i_1,\ldots,i_n}$ contained in an open, evenly covered disk $D \subset U$. Since $C_{i_1,\ldots,i_n}$ is invariant under $h^{k_1 \cdots k_n}$, so is U. Let $f = h^{k_1 \cdots k_n}_{|U}$. Since h has no periodic points in U, f as well as f^2, which is an orientation preserving homeomorphism of U, have no fixed points in U.

By composing a lift of f^2 with an appropriate deck transformation, we obtain an orientation preserving homeomorphism $\widetilde{f} : \widetilde{U} \to \widetilde{U}$ with an invariant compactum $\widetilde{C}$, a copy of $C_{i_1,\ldots,i_n}$ mapped homeomorphically by the projection π onto $C_{i_1,\ldots,i_n}$. Since $\widetilde{U}$ is homeomorphic to $\mathbb{R}^2$, by



Brouwer's theorem, $\widetilde{f}$ has a fixed point a. On the other hand since f^2 has no fixed points, no fiber $\pi^{-1}(p)$ is invariant under $\widetilde{f}$. Hence a cannot be a fixed point of $\widetilde{f}$.

Therefore the assumption that $\mathbf{C} \cap \mathrm{Cl}(P) = \emptyset$ is not valid. There are periodic points of h arbitrarily close to the Cantor set C. □

Remark. *The above theorem does not address the periods of the periodic points that are close to the Cantor set equipped with the adding machine with base* $(k_1, k_2, k_3, \ldots)$*. The almost periodicity of the adding machine yields natural relations of these periods to the products of numbers* $k_1, k_2, \ldots$ *multiplied by the number* 2 *in case of orientation reversing homeomorphisms.*

4 Shape theory

The notion of movability is one of the most important concepts of shape theory. A compact subset F of the Hilbert cube Q is *movable* [4] if for every neighborhood U of F there exists a neighborhood V of F such that for every neighborhood W of F there is a deformation of V into W within U. This property does not depend on the embedding of F in Q and the Hilbert cube can be replaced in the definition by any metric ANR. For the basic notions of the theory of shape the reader is referred to [3] and K. Borsuk's monograph [5]. The notion of movability seems closely related to notion of Lyapunov stability and thus it is of importance in dynamics.

Non-trivial solenoids were the first and most obvious examples of non-movable compacta. On the other hand, the Denjoy continua [11], which by construction are in a natural manner embedded in the surface of a torus, are movable. A description of a C^1 Denjoy set (conitnuum) is easily accessible in [15] or [16]. Denjoy continua are completely classified in [1] and [12]. Let D be a Denjoy continuum embedded in the surface of a torus $S^1 \times S^1$. Let $\pi : S^1 \times S^1 \to S^1 \times S^1$ be a covering projection with finite fibers. The set $\pi^{-1}(D)$ is *Denjoy-like*. The complement of a

Denjoy continuum in $S^1 \times S^1$ is connected, whereas the complement of a Denjoy-like continuum in $S^1 \times S^1$ may have several components.

Let $\phi : \mathbb{R} \times M \to M$ be a non-singular flow on a 3-manifold M. Let Σ be a solenoid in M approximated in terms of the Hausdorff distance by a sequence of pairwise disjoint simple closed curves $\{C_n\}_{n=1}^{\infty}$ disjoint from the solenoid. It is easy to show that the compactum $X = \Sigma \cup \bigcup_{n=1}^{\infty} C_n$ is movable.

Question 1. *If a solenoid Σ is invariant under ϕ, is Σ contained in a larger movable compact set invariant under ϕ?*

The next question is a slight variation of Question 1.

Question 2. *If a solenoid Σ is invariant under ϕ and U is a neighborhood of Σ, is Σ contained in a larger movable compact set invariant under ϕ and contained in U?*

Question 3. *Could the larger movable invariant set in Questions 2 always consist of Σ and a sequence of invariant approximating circles?*

In [17], P. Šindelářová constructed a flow on $\mathbb{R}^3$ with an invariant non-movable one-dimensional continuum Ω. The continuum is not a solenoid, but maps continuously onto a non-trivial solenoid and therefore by [19] or [13] it is not movable. In Šindelářová's flow, Ω is approximated by invariant Denjoy-like continua $\{D_n\}_{n=1}^{\infty}$ and the union $\Omega \cup \bigcup_{n=1}^{\infty} D_n$ is movable.

Question 4. *Is every compact invariant set in flow on a 3-manifold contained in a movable invariant set?*

Question 5. *If a compactum Y is invariant under a flow ϕ on a 3-manifold and U is a neighborhood of Y, is Σ contained in a movable compact set invariant under ϕ and contained in U?*

Question 6. *Would Questions 4 and 5 pose a different challenge if one assumed that the flow ϕ on the non-movable invariant set were minimal?*

Let D be a Cantor set in $\mathbb{R}^2$ invariant under an orientation preserving homeomorphism $g : \mathbb{R}^2 \to \mathbb{R}^2$. By Brouwer's theorem, g has a fixed point $p \in \mathbb{R}^2$. (It is easy to construct an example such that $g_{|D}$ is a Denjoy homeomorphism and g has no periodic points other than one fixed point.) This suggest the following:

Question 7. *Let Z be a compact invariant set in a flow on $\mathbb{R}^3$ such that there exists a sequence of invariant Denjoy-like continua $\{D_n\}_{n=1}^{\infty}$ so that the union $Z \cup \bigcup_{n=1}^{\infty} D_n$ is movable. Does there exist an invariant simple closed curve? Do there exist invariant simple closed curves arbitrarily close to Z?*

Finally let's recall the main problem:

Question 8. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and let $\mathbf{C}$ be a Cantor set invariant under h. If $h_{|\mathbf{C}}$ is an adding machine, does there exist a periodic orbit in every neighborhood of $\mathbf{C}$?*

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