Збірник праць Ін-ту математики НАН України 2013, том 6, N 6, 67–85

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Fixed Points on Trivial Surface Bundles

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The main purpose of this work is to study fixed points of fiber- preserving maps over S^1 on the trivial surface bundles $S^1 \times S_2$, where S_2 is the closed orientable surface of genus 2. We classify all such maps that can be deformed fiberwise to a fixed point free map.

Introduction

Given a fibration $E \to B$ and $f : E \to E$ a fiber-preserving map over B, the question if f can be deformed over B (by a fiberwise homotopy) to a fixed point free map has been considered for several years by many authors. Among others, see for example [Dol74], [FH81], [Gon87], [Pen97], [GPV04], [GPV09I] and [GPV09II]. More recently also the fiberwise coincidence case has been considered in [Kos11], [GK09], [GPV10], [SV12], [Vie12] and [GKLN], which certainly has intersection with the fixed point case.

In [FH81], Fadell, E. and Husseini, S. showed that the fiberwise fixed point problem can be stated in terms of obstructions (including higher ones) if the fibration satisfies certain hypothesis. This is the case if the base space, the total space and the fiber F are manifolds, and the dimension of F is greater or equal to 3. The project to study fixed point of fiberwise maps for surface bundles has been considered mainly in the case where the base is S^1 and it can be divided into several cases as follows.

If the fiber F is the projective real space RP^2 we never obtain a fixed point free fiberwise map, because RP^2 has the fixed point property. This case leads to a natural question about the minimal size of the fixed point set, namely, when is possible to have the fixed point set connected. Close related, if not equivalent, is the problem of classify maps which can be deformed to a map with exactly one fixed point in each fiber.

The case of fiber S^2, despite the fact that the approach of [FH81] can be used, using different techniques, it was studied in [Kos11], [GPV10] and [GKLN].

We note that if the fiber is a closed surface S distinct of S^2 and RP^2, the approach of [FH81] can not be used. A project to study surface bundles for closed surface distinct of S^2 and RP^2 has started looking the case where the base is S^1. The case where S is the torus has been solved by other methods in [GPV04] (see also [Kos11]). For S the Klein bottle the results were obtained in [GPV09II] (see also [SV12]) by similar methods as the case of the torus.

In the present work we start the case of a surface bundle over S^1

where the fiber is S_2 and S_2 is the closed orientable surface of genus 2. More precisely we study fiberwise maps of the trivial bundle $S^1 \times S_2$.

Let us consider the fibration $S^1 \times S_2 \to S^1$ and $h : S^1 \times S_2 \longrightarrow S^1 \times S_2$ a fiber-preserving map over S^1, where $h(x, y) = (x, f(x, y))$, $\forall (x, y) \in S^1 \times S_2$ and f is a map from $S^1 \times S_2$ into S_2.

The main result of this paper is:

Theorem 4.3 A fiberwise map h can be deformed over S^1 to a fixed point free map if and only if h is fiberwise homotopic to $id \times g$ where $g : S_2 \to S_2$ is a fixed point free map homotopic to f restricted to $1 \times S_2$.

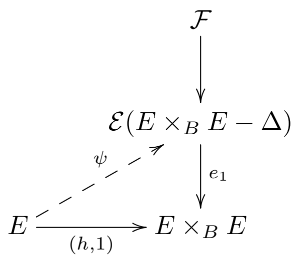
This paper is organized into 4 sections. In section 1 we review an approach to study fixed point of fiberwise maps and we adapt it for the case to be analyzed. In section 2 we make the main calculations where we compute the fundamental group of several spaces and homomorphisms to study a certain algebraic diagram. The main result of this section is Theorem 3.5. In section 3 we proof the main result of this work, which is Theorem 4.3. In section 4 we give a very brief view of the continuation of the study of the problems for the majority of the cases, which are still to be analyzed.

2 Preliminaries

Let $h : E \to E$ be a fiber-preserving map over B, i.e., $p \circ h = p$ where $p : E \to B$ is a fiber bundle with fiber a surface denoted by S. When is h deformable over B to a fixed point free map $h^{'}$ by a fiberwise homotopy over B ? We remark that in order to have a positive answer a necessary condition is that the map h restricted to a fiber is deformable to a fixed point free map.

Now we review an approach which was used in [GPV04] and [GPV09II]. Assuming the necessary condition, h is deformable over B to a fixed point free map $h^{'}$ by a fiberwise homotopy over B if and only if there exists a lifting ψ such that the following diagram is commutative, up to homotopy:

(2.1)



Here $E \times_B E$ is the pullback of p by p, Δ is the diagonal in $E \times_B E$ and the inclusion $E \times_B E - \Delta \hookrightarrow E \times_B E$ is changed by the fibration $e_1 : \mathcal{E}(E \times_B E - \Delta) \to E \times_B E$ with fiber $\mathcal{F}$, where $\pi_i(\mathcal{F}) \simeq \pi_{i+1}(E \times_B E, E \times_B E - \Delta)$. Also $\mathcal{E}(E \times_B E - \Delta)$ is the pullback of the fibration $e_0 : (E \times_B E)^{[0,1]} \to E \times_B E$ by the inclusion $E \times_B E - \Delta \to E \times_B E$. The fibration $e_0 : (E \times_B E)^{[0,1]} \to E \times_B E$ is the evaluation at 0 and $e_1 : \mathcal{E}(E \times_B E - \Delta) \to E \times_B E$ is the evaluation at 1.

Let us observe that if E, B and S are closed manifolds then $\pi_{i+1}(E \times_B E, E \times_B E - \Delta) \simeq \pi_{i+1}(S, S - y_0)$ (see [FH81]).

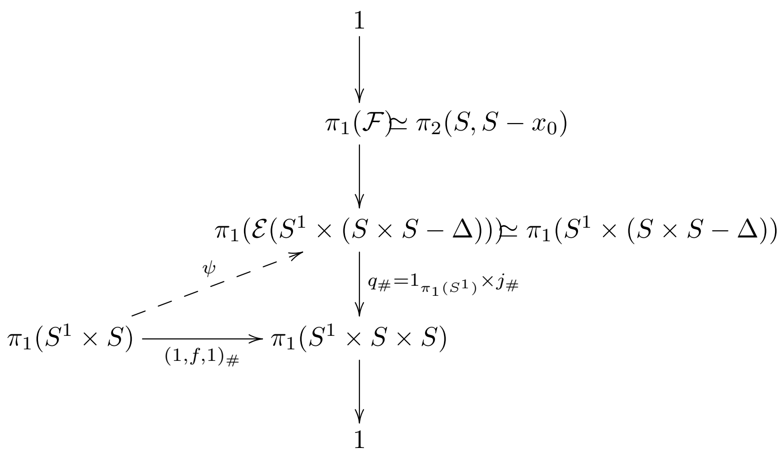
When $E = B \times S$ is the trivial bundle and $h : B \times S \to B \times S$ is a fiber-preserving map over B, the map h can be write in the form $h(x, y) = (x, f(x, y))$ for some $f : B \times S \to S$. Then the diagram 2.1 can be modified and becomes equivalent to the following diagram:

$$\begin{array}{ccc} & & \mathcal{F} \\ & & \downarrow \\ & & \mathcal{E}(B \times (S \times S - \Delta)) \\ & \overset{\psi}{\nearrow} & \downarrow {\scriptstyle e_1} \\ B \times S & \xrightarrow[(1,f,1)]{} & B \times S \times S \end{array} \tag{2.2}$$

3 Trivial $S-$bundles over S^1 with $\chi(S) < 0$

Let S be a surface with $\chi(S) < 0$ and let us consider the fibration $S^1 \times S \to S^1$ and $h : S^1 \times S \longrightarrow S^1 \times S$ a fiber-preserving map over S^1, where $h(x, y) = (x, f(x, y)), \forall (x, y) \in S^1 \times S$ and f is a map from $S^1 \times S$ into S. We also consider x_0 and y_0 base points of S^1 and S, respectively, and $f : (S^1 \times S, (x_0, y_0)) \longrightarrow (S, f(x_0, y_0))$, with $f(x_0, y_0) \neq y_0$. From the map f we obtain the maps $g = f\mid_{\{x_0\}\times S}$ and $l = f\mid_{S^1\times\{y_0\}}$. Recall that we are assuming the necessary condition: *the map g is deformable to a fixed point free map.*

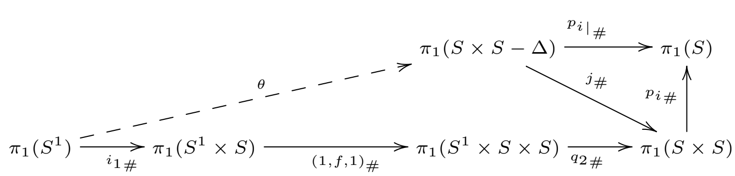
Using the approach developed in [GPV04] and [GPV09II] we will study in our case the existence of an algebraic lifting ψ to the diagram



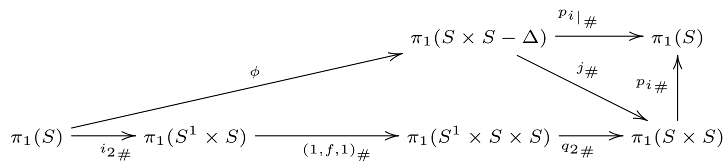
(3.1)

where $\pi_1(\mathcal{F}) \simeq \pi_1(S \times S - \triangle)$ is the pure braid group of S on 2-strings.

The existence of the lifting mentioned above is equivalent to find liftings θ and ϕ described in diagrams 3.2 and 3.3 below where θ and ϕ satisfy certain conditions. Since we are assuming the necessary condition then the lifting ϕ exists. So, we have the following two diagrams, where $i_{1\#}, i_{2\#}$ and $j_\#$ are induced homomorphisms on fundamental groups by the injective maps $i_1 : S^1 \to S^1 \times S$, $i_2 : S \to S^1 \times S$ and $j : S \times S - \Delta \to S \times S$, respectively, and $q_{2\#}$ and $p_{i\#}$ are induced homomorphisms by the projection maps $q_2 : S^1 \times S \times S \to S \times S$ and $p_i : S \times S \to S$, respectively.



(3.2)



(3.3)

We remark that in these diagrams we are omitting base points.

The following theorem provides some conditions that the liftings θ and ϕ must satisfy which are equivalent to a positive solution of the fixed point problem for the trivial bundle.

Theorem 3.1. *There exists ψ on the diagram 3.1 if and only if there exist θ and ϕ in the diagrams 3.2 and 3.3, respectively, such that Imθ commutes with Imϕ.*

Proof. Let us suppose that there exists a lifting ψ in the diagram (2.1). Define $\phi = q_{2|_\#} \circ \psi \circ i_{2\#}$ and $\theta = q_{2|_\#} \circ \psi \circ i_{1\#}$, where $i_1 : S^1 \to S^1 \times S$ and $i_2 : S \to S^1 \times S$ denote the inclusion maps and $q_{2|} : S^1 \times (S \times S - \Delta) \to (S \times S - \Delta)$ denotes the projection on the second factor. Therefore θ and ϕ are lifting for the diagrams 3.2 and 3.3, respectively, because $q_\# \circ \psi = (1, f, 1)_\#$ and $q_{2\#} \circ q_\# = j_\# \circ q_{2|_\#}$.

Now, for all $x \in \text{Im}\theta$ and for all $y \in \text{Im}\phi$ we have

$$\begin{aligned}
xy &= q_{2|_\#} \circ \psi \circ i_{1\#}([b]) q_{2|_\#} \circ \psi \circ i_{2\#}([s]) \\
&= q_{2|_\#} \circ \psi(i_{1\#}([b]) i_{2\#}([s])) \\
&= q_{2|_\#} \circ \psi(([b], 1)(1, [s])) \\
&= q_{2|_\#} \circ \psi((1, [s])([b], 1)) \\
&= q_{2|_\#} \circ \psi(i_{2\#}([s]) i_{1\#}([b])) \\
&= q_{2|_\#} \circ \psi \circ i_{2\#}([s]) q_{2|_\#} \circ \psi \circ i_{1\#}([b]) \\
&= yx
\end{aligned}$$

Conversely, suppose that θ and ϕ exist and we define ψ by $\psi([b], [s]) = ([b], \theta([b])\ \phi([s]))$ where $b : S^1 \to S^1$ and $s : S^1 \to S$ denote loops based at x_0 and at y_0, respectively. Since Imθ commutes with Imϕ, we have that ψ is a homomorphism and denoting by $s_0 : S^1 \to S$ and $b_0 : S^1 \to S^1$ the constant maps at y_0 and at x_0, respectively, it follows that $(q_\# \circ \psi)([b],[s]) = (1, f, 1)_\#([b],[s]), \forall([b],[s]) \in \pi_1(S^1 \times S)$. □

From now on we specialize for the case where the fiber S is the surface S_2. So we consider the trivial bundle $S^1 \times S_2$.

If ϕ is a lifting of the diagram 3.3 to discuss the existence of the lifting θ we will denote by 1 a generator of $\pi_1(S^1) \equiv \mathbb{Z}$ and by $\theta(1) = \omega \in \pi_1(S_2 \times S_2 - \Delta)$. A presentation of $\pi_1(S_2 \times S_2 - \Delta)$ is given in [FH82]. We will use the following notation: let $a_i = \rho_{1,i} \in \pi_1(S_2 \times S_2 - \Delta), i = 1, 2, 3, 4$ and by $b_i = \rho_{2,i} \in \pi_1(S_2 \times S_2 - \Delta)$.

So $\pi_1(S_2 \times S_2 - \Delta)$ has the following presentation:

(I) $[a_1, a_2^{-1}][a_3, a_4^{-1}] =: B_1 = B_2^{-1} := [b_1, b_2^{-1}][b_3, b_4^{-1}]$ (which defines the elements B_1 and B_2^{-1}).

(II) $b_l a_j b_l^{-1} = a_j$ where $1 \le j, l \le 4$, and $j < l$(resp. $j < l - 1$) if l is odd (resp. l is even).

(III) $b_k a_k b_k^{-1} = a_k[a_k^{-1}, B_1]$ and $b_k^{-1} a_k b_k = a_k[B_1^{-1}, a_k]$ for all $1 \le k \le 4$.

(IV) $b_k a_{k+1} b_k^{-1} = B_1 a_{k+1}[a_k^{-1}, B_1]$ and $b_k^{-1} a_{k+1} b_k = B_1^{-1}[B_1, a_k] a_{k+1}[B_1^{-1}, a_k]$, for all k odd, $1 \le k \le 4$.

(V) $b_{k+1} a_k b_{k+1}^{-1} = a_k B_1^{-1}$, and $b_{k+1}^{-1} a_k b_{k+1} = a_k B_1[B_1^{-1}, a_{k+1}]$, for all k odd, $1 \le k \le 4$.

(VI) $b_l a_j b_l^{-1} = [B_1, a_l^{-1}] a_j [a_l^{-1}, B_1]$ and $b_l^{-1} a_j b_l = [a_l, B_1^{-1}] a_j [B_1^{-1}, a_l]$ for all $1 \le l < j \le 4$ and $(j, l) \ne (2t, 2t - 1)$ for all $t \in \{1, 2\}$.

We also observe that from the fibration $p_2 | : S_2 \times S_2 - \Delta \longrightarrow S_2$ we get the following exact sequence:

$$1 \longrightarrow \pi_1(S_2 - y_0) \longrightarrow \pi_1(S_2 \times S_2 - \Delta) \xrightarrow{p_2|_\#} \pi_1(S_2) \longrightarrow 1 . \tag{3.4}$$

The group $\pi_1(S_2 - y_0)$ is free and from the sequence above it is identified with the subgroup of $\pi_1(S_2 \times S_2 - \Delta)$ freely generated by a_1, a_2, a_3, a_4. Also the image of the set of elements b_1, b_2, b_3, b_4 projects to a set of generators of $\pi_1(S_2)$ giving a presentation of $\pi_1(S_2) =< \bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4 \mid [\bar{b}_1, \bar{b}_2^{-1}][\bar{b}_3, \bar{b}_4^{-1}] >$. More details see [FH82].

Given a group G the central series of G is defined recursively by

$$G_1 = G, G_{n+1} = [G, G_n], n = 1, 2, \ldots.$$

For any group G we have that G_m is a normal subgroup of G_n for all $n \leq m$. In case G is free group of finite rank r then it is well known that G_n/G_{n+1} is a free abelian of rank

$$N_n = \frac{1}{n} \sum_{d|n} \mu(d) r^{\frac{n}{d}}$$

(see [[MKS76],Theorem 5.11, p.330]). Here $\mu(d)$ denotes the *Moebius Function* defined for all positive integers by $\mu(1) = 1$, $\mu(p) = -1$ if p is a prime number, $\mu(p^k) = 0$ for $k > 1$, and $\mu(b \cdot c) = \mu(b) \cdot \mu(c)$ if b and c are coprime integers.

For any group G denote the commutator $[[a, b], c]$ by (a, b, c). If a, b, c are elements of a group G and k, m, n are positive integers such that $a \in G_k, b \in G_m, c \in G_n$ then $(a, b, c) \cdot (b, c, a) \cdot (c, a, b) \equiv 1 \mod G_{k+m+n+1}$ (see [[MKS76], Theorem 5.3, p.293]).

For the next Lemma, let $G = G_1 = \pi_1(S_2 - y_0)$ which is a free group, and it is identified with a subgroup of $\pi_1(S_2 \times S_2 - \Delta)$ using the short exact sequence 3.4.

Lemma 3.2. *If $v \in G_2 = [G_1, G_1]$, then $[b_j, v] \in G_3$, for $j = 1, 2, 3, 4$.*

Proof. We will prove the statement for b_3. The other cases are similar. If $v \in G_2 = [G_1, G_1]$, then v is a finite product of $[a_i, a_j]$ and of its inverses. If $v_1, v_2 \in G_2$ then $[b_3, v_1 v_2] = b_3 v_1 v_2 b_3^{-1} v_2^{-1} v_1^{-1} = b_3 v_1 b_3^{-1} v_1^{-1} v_1 b_3 v_2 b_3^{-1} v_2^{-1} v_1^{-1} = [b_3, v_1] v_1 [b_3, v_2] v_1^{-1}$. We know that the conjugation by v_1 preserves the central series and then if we prove that $[b_3,[a_i,a_j]] = 1$ mod G_3 the result follows.

Now in G_1/G_3, we have that

$[b_3,[a_i,a_j]] = b_3[a_i,a_j]b_3^{-1}[a_i,a_j]^{-1} = b_3a_ia_ja_i^{-1}a_j^{-1}\ b_3^{-1}[a_i,a_j]^{-1}$.

In the case where $i \neq 4$ and $j \neq 4$ and recalling that in G_1/G_3, b_3 commutes with a_i and a_j (i.e, the action is trivial) we have the desired result.

If $i = 4$ and $j \neq 4$ we also have that b_3 commutes with a_j because $j \neq 4$ and the action in a_4 results in B_1a_4. Therefore in G_1/G_3 the action of b_3 in $[a_4,a_j]$ is $b_3a_4a_ja_4^{-1}a_j^{-1}b_3^{-1} = B_1a_4a_ja_4^{-1}B_1^{-1}a_j^{-1}$ and so

$b_3[a_4,a_j]b_3^{-1}[a_4,a_j]^{-1} = B_1a_4a_ja_4^{-1}B_1^{-1}a_4a_j^{-1}a_4^{-1} = [B_1\ ,\ a_4a_ja_4^{-1}]$
$= [a_4a_ja_4^{-1},B_1]^{-1} \in G_3$.

The case where $i \neq 4$ and $j = 4$ is analogue. □

Let $C(\theta(1))$ be the centralizer of $\theta(1)$ in $\pi_1(S_2 \times S_2 - \Delta)$.

Proposition 3.3. Let θ be a lifting such that $(p_2\ |_\#)(C(\theta(1)) = \pi_1(S_2,y_0)$. Then $\theta(1) \in G$ and there exist u_1,u_2,u_3,u_4 elements of G such that $u_jb_j \in C(\theta(1))$, $j = 1,2,3,4$. If $\theta(1) = xv$, with $v \in G_2, x \in G$, then we have that $[u_j^{-1},x^{-1}][x^{-1},b_j] = 0$ in G_2/G_3.

Proof. From the hypothesis $(p_2\ |_\#)(C(\theta(1))) = \pi_1(S_2,y_0)$ follows that $p_{2\#}(\theta(1))$ is in the centralizer of $\pi_1(S)$. This implies that this element is trivial and then $\theta(1) \in G$. Also, given $\bar{b}_j$ from the hypothesis follows that there exists x_j which is in the centralizer of $\theta(1)$ which projects to $\bar{b}_j$. Therefore $x_j = u_jb_j$ for some $u_j \in G$.

Since $u_jb_j \in C(\theta(1))$, $j = 1,2,3,4$ we have that

$$\begin{aligned} xvu_jb_j &= u_jb_jxv \\ b_jxb_j^{-1}[b_j,v] &= u_j^{-1}xu_j[u_j^{-1},v] \\ u_j^{-1}x^{-1}u_jb_jxb_j^{-1} &= [u_j^{-1},v][v,b_j] \\ [u_j^{-1},x^{-1}][x^{-1},b_j] &= [u_j^{-1},v][v,b_j] \end{aligned}$$

where $v \in G_2$, $u_j \in G_1$. Then in G_2/G_3 it follows from the Lemma 3.2 that

$$[u_j^{-1}, x^{-1}][x^{-1}, b_j] = 0. \tag{3.5}$$

□

The group G_2/G_3 is a **Z**- free module and let us consider the basis

$$\{[a_1, a_2], [a_1, a_3], [a_1, a_4], [a_2, a_3], [a_2, a_4], [a_3, a_4]\}$$

which we refer as the canonical basis of G_2/G_3.

Lemma 3.4. *In G_2/G_3 we have:*

a) $[a_i a_j, x] = [a_j a_i, x]$, *where a_i, a_j are generators of G and $x \in G$.*

b) *If $B = [a_1, a_2^{-1}][a_3, a_4^{-1}] \in G$, then $B = -[a_1, a_2] - [a_3, a_4]$ and its coordinate in relation to the canonical basis is given by* $(-1, 0, 0, 0, 0, -1)$.

c) *The element $[a_i^{x_i}, a_j^{x_j}]$ is given in the following form:*

$$[a_i^{x_i}, a_j^{x_j}] = \begin{cases} 0 \;\; se \; i = j \\ x_i x_j [a_i, a_j] \;\; se \; i < j \\ -x_i x_j [a_j, a_i] \;\; se \; i > j \end{cases}$$

Proof. Since $[a_i a_j, x] = [a_i, [a_j, x]][a_j, x][a_i, x]$ and $[a_j a_i, x] = [a_j, [a_i, x]][a_i, x][a_j, x]$, the result of item a) follows by observing that in G_2/G_3 we have that $[a_i, [a_j, x]] = 0 = [a_j, [a_i, x]]$ which is commutative. The items b) and c) are easy. □

Theorem 3.5. *If there exists θ and $(p_2 \mid_{\#})(C(\theta(1)) = \pi_1(S_2, y_0)$ then $\theta(1) \in G_2 = [G_1, G_1]$.*

Proof. It follows from the exact sequence $1 \to [G_1, G_1] \to G_1 \to G_1/[G_1, G_1] \to 0$ that $\theta(1) \in G_1$ is of the form $\theta(1) = xv$ with

$x = a_1^a a_2^b a_3^c a_4^d$ and $v \in G_2 = [G_1, G_1]$. We are going to prove that the exponents $a = b = c = d = 0$.

It follows from Proposition 3.3 that in G_2/G_3 we have $[u_j^{-1}, x^{-1}][x^{-1}, b_j] = 0$ with $u_j \in G_1$.

In fact this is a system with j equations and four variables x:(a, b, c, d) and for each j four variables (e_j, f_j, g_j, h_j) corresponding to $u_j^{-1} = a_1^{e_j} a_2^{f_j} a_3^{g_j} a_4^{h_j}$.

Writing $[u^{-1}, x^{-1}]$ in the canonical basis of G_2/G_3 and observing that the exponents of $u^{-1} = a_1^e a_2^f a_3^g a_4^h$ must to appear with sub-index j (We are omitting such sub-index) we obtain:

$[u^{-1}, x^{-1}] = (af - be)[a_1, a_2] + (ag - ce)[a_1, a_3] + (ah - de)[a_1, a_4] + (bg - cf)[a_2, a_3] + (bh - df)[a_2, a_4] + (ch - dg)[a_3, a_4]$

Calculating $[x^{-1}, b_j] \in G_2$ with $x = a_1^a a_2^b a_3^c a_4^d$ we obtain:

$$\begin{aligned}
x^{-1}b_1xb_1^{-1} &= x^{-1}a_1^a(B_1a_2)^b a_3^c a_4^d \\
&= a_4^{-d}a_3^{-c}a_2^{-b}(B_1a_2)^b a_3^c a_4^d \\
&= a_4^{-d}a_3^{-c}a_2^{-b}(a_2^b a_2^{-b}B_1a_2^b \dots a_2^{-2}B_1a_2^2a_2^{-1}B_1a_2)a_3^c a_4^d \\
x^{-1}b_2xb_2^{-1} &= x^{-1}(a_1B_1^{-1})^a a_2^b a_3^c a_4^d \\
&= a_4^{-d}a_3^{-c}a_2^{-b}a_1^{-a}(a_1B_1^{-1})^a a_2^b a_3^c a_4^d \\
&= a_4^{-d}a_3^{-c}a_2^{-b}a_1^{-a}(a_1B_1^{-1}a_1^{-1}a_1^2B_1^{-1}a_1^{-2}\dots a_1^aB_1^{-1}a_1^{-a})\times \\
&\ \times a_1^a a_2^b a_3^c a_4^d \\
x^{-1}b_3xb_3^{-1} &= x^{-1}a_1^a a_2^b a_3^c(B_1a_4)^d \\
&= a_4^{-d}(B_1a_4)^d \\
&= a_4^{-d}(a_4^d a_4^{-d}B_1a_4^d \dots a_4^{-2}B_1a_4^2a_4^{-1}B_1a_4) \\
x^{-1}b_4xb_4^{-1} &= x^{-1}a_1^a a_2^b(a_3B_1^{-1})^c a_4^d \\
&= a_4^{-d}a_3^{-c}(a_3B_1^{-1})^c a_4^d \\
&= a_4^{-d}a_3^{-c}(a_3B_1^{-1}a_3^{-1}a_3^2B_1^{-1}a_3^{-2}\dots a_3^cB_1^{-1}a_3^{-c}a_3^c)a_4^d
\end{aligned}$$

Therefore in relation to the canonical basis of G_2/G_3 and by using Lemma 3.4 *b*) we obtain $(-b, 0, 0, 0, 0, -b), (a, 0, 0, 0, 0, a), (-d, 0, 0, 0, 0, -d), (c, 0, 0, 0, 0, c)$ as the coordinates of $[x^{-1}, b_j]$ respectively for $j = 1, 2, 3, 4$.

So, the system to be solved is

$$\left\{\begin{array}{lllllll} -be & +af & & & = & b, -a, d, -c \text{ respectively for } b_1, b_2, b_3, b_4 \\ -ce & & +ag & & = & 0 \\ -de & & & +ah & = & 0 \\ & -cf & +bg & & = & 0 \\ & -df & & +bh & = & 0 \\ & & -dg & +ch & = & b, -a, d, -c \text{ respectively for } b_1, b_2, b_3, b_4 \end{array}\right. \tag{3.6}$$

understanding that in the letters (e, f, g, h) must to appear sub-index j, but not in the letters (a, b, c, d).

a-) $d \neq 0$ in the system (3.6)

a1) If $b = 0$ we have that L_5 implies $f = 0$, making L_1 without solution.

a2) If $b \neq 0$, in the system $L_3 \to dL_1 - bL_3$ produces an incompatibility : new L_3 and L_5 .

b-) $b \neq 0$ in the system (3.6)

b1) If $d = 0$ we have that L_6 implies $c \neq 0$ and $h \neq 0$. Then L_3 implies $a = 0$ and from L_5 we conclude that $b = 0$, making L_1 without solution.

b2) If $d \neq 0$, in the system $L_4 \to bL_6 + dL_4$ produces an incompatibility: new L_4 and L_5.

c-) $c \neq 0$ in the system (3.6)

c1) If $a = 0$ then L_1 implies that $b \neq 0$ and $e \neq 0$ and from L_3 we obtain $d = 0$ and from L_5 we have that $h = 0$. Therefore L_6 is impossible.

c2) If $a \neq 0$ the system is impossible. In the system we make $L_4 \to cL_1 - aL_4$ and obtain an incompatibility: new L_4 and L_2.

d-) $a \neq 0$ in the system (3.6)

d1) If $c = 0$ the system is impossible. It follows from the fact that L_6 implies $d \neq 0$ and $g \neq 0$. Also L_4 implies $b = 0$ and L_5 implies $f = 0$ making L_1 impossible.

d2) If $c \neq 0$ the system is impossible, because in the system we make $L_3 \to aL_6 - cL_3$ which produces an incompatibility: new L_3 and L_2.

From the considerations above we conclude that $a = b = c = d = 0$ and therefore $x = 1$ and $\theta(1) \in G_2$. □

4 Main Result

Let $h : S^1 \times S_2 \to S^1 \times S_2$ given by $h(x, y) = (x, f(x, y))$. Let us consider $l : (S^1, x_0) \to (S_2, f(x_0, y_0))$ and $g : (S_2, y_0) \to (S_2, f(x_0, y_0))$ given by $l(x) = f(x, y_0)$ and $g(y) = f(x_0, y)$, respectively. Without loss of generality we are assuming that g is a fixed point free map.

To prove our main result we need the following

Lemma 4.1. *Let $t : S_2 \to S_2$ be a continuous map and $t_\# : \pi_1(S_2) \to \pi_1(S_2)$ the induced homomorphism of the map t. Suppose that $t_\#(b_i) = \alpha^{n_i}$, where $b_i, i = 1, 2, 3, 4$ is a generator of $\pi_1(S_2)$. If the map t can be deformed to a fixed point free map then $\sum_1^4 n_i|\alpha|_i = 1$ where $|\alpha|_i$ denotes the sum of the exponents of b_i in the word α.*

Proof. Let $\iota : S^1 \to S_2$ be a map which represents the element $\alpha \in \pi_1(S_2)$. We can define $t' : S_2 \to S^1$ such that $\iota \circ t' = t$. By the commutativity property for fixed point we know that the Nielsen number of t is the same as the Nielsen number of $t' \circ \iota$, which is a self map of the circle. So if t is deformable to a fixed point free map then we have that the Nielsen number of $t' \circ \iota$ is trivial which is equivalent to say that the

Lefschetz number of $t' \circ \iota$ is 0, which is the same to say $\sum_1^4 n_i|\alpha|_i = 1$. So the result follows. $\square$

The main result will follows from the Proposition below.

Proposition 4.2. The fiberwise map h is deformable to a fixed point free map over S^1 if and only if $l_\#(1) = e$, where $l_\# : \pi_1(S^1; x_0) \to \pi_1(S_2; f(x_0, y_0))$.

Proof. Let h be a fiberwise map where $h(x,y) = (x, f(x,y))$. To prove that h can be deformed fiberwise to a fixed point free map it is suffice to show that f is homotopic to the map $f^{'}(x,y) = f(x_0, y)$.

Because $S^1 \times S_2$ and S_2 are $K(\pi, 1)$ the two maps are homotopic if the induced homomorphisms on the fundamental group are equal. Because $\pi_1(S^1 \times S_2) = \pi_1(S^1) \times \pi_1(S_2)$ to show that the two homomorphisms are the same it suffices to show that these homomorphisms coincide when restricted to each of the two subgroups $\pi_1(S^1), \pi_1(S_2)$. By hypothesis $l_\#(1) = e$ follows that they coincide on $\pi_1(S^1)$. By the definition of $f^{'}$ also follows that they coincide on $\pi_1(S_2)$, and this concludes the proof of one implication.

Reciprocally, let h be a map deformable to a fixed point free map over S^1. Then by Theorem 3.1 exist ϕ and θ such that the image of θ commutes with the image of ϕ. From the diagrams 3.2 and 3.3, $p_{1|_\#} \circ \theta = l_\#$ and $p_{1|_\#} \circ \phi = g_\#$. It is known that $g_\#(\pi_1(S_2))$ is a subgroup of $\pi_1(S_2)$ isomorphic to one of the following groups:

1. $\{e\}$.

2. a free group of rank 2 (see [LS89]and [Zie62]).

3. $\pi_1(S)$

4. $\mathbb{Z} = \langle \beta \rangle$

The first item does not occur, otherwise g is homotopic to the constant map so it can not be deformed to a fixed point free map.

In the second and third cases, from above $l_{\#}(1)$ commutes with all elements of $g_{\#}(\pi_1(S))$ but the centralizer of these two subgroups is trivial. Therefore $l_{\#}(1) = e$.

For the last case we have that $g_{\#}(\pi_1(S)) = \mathbb{Z} = \langle \beta \rangle = \langle \alpha^k \rangle$ where $\alpha \neq 0$, α has no roots and $\alpha^k = \beta$. Since $l_{\#}(1)$ commutes with the elements of $g_{\#}(\pi_1(S))$ then $l_{\#}(1) = \alpha^r$. If $r = 0$ the proof follows. So suppose that $r \neq 0$.

Writing $g_{\#}(b_i) = \alpha^{n_i}$ we have by the lemma 4.1 that if g is homotopic to a fixed point free map then $\sum_1^4 n_i|\alpha|_i \neq 0$.

We have that $p_{1|_{\#}} \circ \theta(1) = l_{\#}(1)$. We also have that $\text{im}\ \phi \subset C(\theta(1))$, and $p_{2|_{\#}}(\text{Im}(\phi))) = \pi_1(S)$. Therefore $p_{2|_{\#}}(C(\theta(1))) = \pi_1(S)$ and by theorem 3.5 follows that $\theta(1) \in G_2 = [G_1, G_1]$.

So, $\alpha^r = l_{\#}(1) \in \left[p_{1|_{\#}}(G_1), p_{1|_{\#}}(G_1)\right]$.

Therefore $\alpha \in \left[p_{1|_{\#}}(G_1), p_{1|_{\#}}(G_1)\right]$ and then $|\alpha|_i = 0$ and by using the above result we conclude that g is not homotopic to a fixed point free map, which contradicts the initial condition on g. So the result follows. □

In fact the proof above shows that if $\theta(1) = e$ then f does not depend of x, i.e. h is the unique fiberwise map homotopic to $id \times g$ where $g : S_2 \to S_2$ is a fixed point free map homotopic to f restricted to $1 \times S$. So we state the main result.

Theorem 4.3. *A fiberwise map h can be deformed over S^1 to a fixed point free map if and only if h is fiberwise homotopic to $id \times g$ where $g : S_2 \to S_2$ is a fixed point free map homotopic to f restricted to $1 \times S_2$.*

5 Other surface bundles

Here let us make few comments about the fixed point question studied in the previous sections in the case we have a more general surface bundle. Let $S \to E \to B$ be a surface bundle over a space B where S is a closed surface of negative Euler characteristic. We can consider three subfamilies of the family of these bundles, namely: I) let S be an arbitrary closed surface(orientable or nonorientable) of arbitrary genus $g > 1$, and $E = S^1 \times S$; II) let $E = B \times S$ be a bundle for B any connected CW complex; III) let E be a $S-$bundle over S^1.

The subcases I) and II) we expect that the answer of the problem should be similar to the answer of the case studied here where $S = S_2$.

The subcase III) is more subtle. First of all the formulation of the problem is already more elaborate. More precisely, let us consider the map $\phi : [E, E]_B \to [S, S]$ which associate to a homotopy class of a fibre preserve map $[f]$ the homotopy class of the restriction $f|_S : S \to S$. Then one would like to know first which homotopy class $[g] \in [S, S]$ which contains a fixed point free map are in the image of ϕ. Second, for a class $[g]$ in the image how many classes $[f] \in [E, E]$ we would like to compute the pre-image of $[g]$, i.e. $\phi^{-1}[g]$. For example in the case that we solved, we have that the $[g]$ is in the image for all maps g which are fixed point free and the pre-image contains exactly one element.

The study and full calculation of the questions above are in progress and should appear somewhere.

Acknowledgments

The authors were partly supported by the Projeto Temático - Topologia Algébrica, Geométrica e Diferencial - FAPESP (08/57607-6)

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