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INTEGRAL THEOREMS FOR MONOGENIC FUNCTIONS IN COMMUTATIVE ALGEBRAS

Let $\mathbb{A}_n^m$ be an arbitrary n-dimensional commutative associative algebra over the field of complex numbers with m idempotents. Let $e_1 = 1, e_2, \ldots, e_k$ with $2 \leq k \leq 2n$ be elements of $\mathbb{A}_n^m$ which are linearly independent over the field of real numbers. We consider monogenic (i. e. continuous and differentiable in the sense of Gateaux) functions of the variable $\sum_{j=1}^{k} x_j \, e_j$, where $x_1, x_2, \ldots, x_k$ are real, and prove curvilinear analogues of the Cauchy integral theorem, the Morera theorem and the Cauchy integral formula in k-dimensional ($2 \leq k \leq 2n$) real subspace of the algebra $\mathbb{A}_n^m$. The present results are generalizations of the corresponding results obtained in [1] *for the case $k = 3$.*

1. Introduction. The Cauchy integral theorem and Cauchy integral formula for the holomorphic function of the complex variable are fundamental results of the classical complex analysis. Analogues of these results are also important tools in commutative algebras of dimension more than 2.

In the E. R. Lorch's paper [2], for functions differentiable in the sense of Lorch in an arbitrary convex domain of commutative associative Banach algebra, some properties similar to properties of holomorphic functions of complex variable (in particular, the curvilinear integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem) are established. E. K. Blum [3] withdrew the convexity of domain in the mentioned results from [2].

Let us note that *a priori* the differentiability of a function in the sense of Gateaux is a restriction weaker than the differentiability of this function in the sense of Lorch. Therefore, we consider a *monogenic* functions defined as a continuous and differentiable in the sense of Gateaux. Also we assume that a monogenic function is given in a domain of three-dimensional subspace of an arbitrary commutative associative algebra with unit over the field of complex numbers. In this situation the results established in the papers [2, 3] is not applicable for a mentioned monogenic function, because it deals with an integration along a curve on which the function is not given, generally speaking.

In the papers [4–6] for monogenic function the curvilinear analogues of the Cauchy integral theorem, the Cauchy integral formula and the Morera theorem are obtained in special finite-dimensional commutative associative algebras. The results of the papers [4, 5, 6] are generalized in the paper [1] to an arbitrary commutative associative algebra. At the same time, in [1] monogenic functions are given in a domain of a real three-dimensional subspace of the algebra.

In this paper, assuming that monogenic functions are given in a domain of a real k-dimensional subspace of the algebra, we generalize results of the papers [1].

Let us note that some analogues of the curvilinear Cauchy's integral theorem and the Cauchy's integral formula for other classes of functions in special commutative algebras are established in the papers [7–10].

2. The algebra $\mathbb{A}_n^m$. Let $\mathbb{N}$ be the set of natural numbers. We fix numbers $m,n\in\mathbb{N}$ such that $m\leq n$. Let $\mathbb{A}_n^m$ be an arbitrary commutative associative algebra with unit over the field of complex number $\mathbb{C}$. E. Cartan [11, p. 33] proved that in the algebra $\mathbb{A}_n^m$ there exist a basis $\{I_k\}_{k=1}^n$ which satisfies the following multiplication rules:

1. $\forall\ r,s\in[1,m]\cap\mathbb{N}:\qquad I_rI_s=\begin{cases}0 & \text{if}\quad r\neq s,\\ I_r & \text{if}\quad r=s;\end{cases}$

2. $\forall\ r,s\in[m+1,n]\cap\mathbb{N}:\qquad I_rI_s=\sum\limits_{k=\max\{r,s\}+1}^{n}\Upsilon_{r,k}^{s}I_k\,;$

3. $\forall\ s\in[m+1,n]\cap\mathbb{N}\ \ \exists!\ u_s\in[1,m]\cap\mathbb{N}\ \ \forall\ r\in[1,m]\cap\mathbb{N}:$

$$I_rI_s=\begin{cases}0\ \text{ if }\ r\neq u_s\,,\\ I_s\ \text{ if }\ r=u_s\,.\end{cases}\tag{1}$$

Furthermore, the structure constants $\Upsilon_{r,k}^{s}\in\mathbb{C}$ satisfy the associativity conditions:

(A 1). $(I_rI_s)I_p=I_r(I_sI_p)\quad\forall\ r,s,p\in[m+1,n]\cap\mathbb{N};$
(A 2). $(I_uI_s)I_p=I_u(I_sI_p)\quad\forall\ u\in[1,m]\cap\mathbb{N}\ \ \forall\ s,p\in[m+1,n]\cap\mathbb{N}.$

Obviously, the first m basis vectors $\{I_u\}_{u=1}^m$ are idempotents and, therefore, generate the semi-simple subalgebra. The vectors $\{I_r\}_{r=m+1}^n$ generate a nilpotent subalgebra of the algebra $\mathbb{A}_n^m$. The unit of $\mathbb{A}_n^m$ is the element of form $1=\sum_{u=1}^m I_u$. Therefore, we can state that the algebra $\mathbb{A}_n^m$ is a semi-direct sum of the m-dimensional semi-simple subalgebra S and $(n-m)$-dimensional nilpotent subalgebra N, i. e.

$$\mathbb{A}_n^m = S\oplus_s N. \tag{2}$$

Let us note that nilpotent algebras are fully described for the dimensions $1,2,3$ in the paper [12], and some four-dimensional nilpotent algebras can be found in the papers [13], [14].

The algebra $\mathbb{A}_n^m$ contains m maximal ideals

$$\mathcal{I}_u := \left\{ \sum_{k=1,\, k\neq u}^{n} \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}, \quad u=1,2,\ldots,m,$$

the intersection of which is the radical

$$\mathcal{R} := \left\{ \sum_{k=m+1}^{n} \lambda_k I_k : \lambda_k \in \mathbb{C} \right\}.$$

We define m linear functionals $f_u : \mathbb{A}_n^m \to \mathbb{C}$ by putting

$$f_u(I_u)=1, \quad f_u(\omega)=0 \quad \forall\, \omega\in\mathcal{I}_u\,, \quad u=1,2,\ldots,m.$$

Since the kernel of every functional f_u are the corresponding maximal ideal $\mathcal{I}_u$, these functionals are continuous and multiplicative (see [15, p. 147]) also.

3. Monogenic functions in E_k. Consider vectors $e_1=1,e_2,\ldots,e_k$ in $\mathbb{A}_n^m$, where $2\leq k\leq 2n$. Let these vectors be linearly independent over the field of real numbers $\mathbb{R}$ (see [6]). It means that the equality

$$\sum_{j=1}^{k} \alpha_j e_j = 0, \qquad \alpha_j\in\mathbb{R},$$

holds if and only if $\alpha_j=0$ for all $j=1,2,\ldots,k$.

Let the vectors $e_1 = 1, e_2, \ldots, e_k$ have the following decompositions with respect to the basis $\{I_r\}_{r=1}^{n}$:

$$e_1 = \sum_{r=1}^{m} I_r\,, \quad e_j = \sum_{r=1}^{n} a_{jr}\, I_r\,, \quad a_{jr} \in \mathbb{C}, \quad j = 2, 3, \ldots, k. \tag{3}$$

Let $\zeta := \sum\limits_{j=1}^{k} x_j\, e_j$, where $x_j \in \mathbb{R}$. It is obvious that

$$\xi_u := f_u(\zeta) = x_1 + \sum_{j=2}^{k} x_j\, a_{ju}, \quad u = 1, 2, \ldots, m.$$

Let $E_k := \{\zeta = \sum\limits_{j=1}^{k} x_j e_j \,:\, x_j \in \mathbb{R}\}$ be the linear span of vectors $e_1 = 1, e_2, \ldots, e_k$ over the field $\mathbb{R}$.

Everywhere below, we make the following essential assumption: $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Obviously, it holds if and only if for every fixed $u = 1, 2, \ldots, m$ at least one of the numbers $a_{2u}, a_{3u}, \ldots, a_{ku}$ belongs to $\mathbb{C} \setminus \mathbb{R}$.

With a set $Q_{\mathbb{R}} \subset \mathbb{R}^k$ we associate the set $Q := \{\zeta = \sum\limits_{j=1}^{k} x_j e_j \,:\, (x_1, \ldots, x_k) \in Q_{\mathbb{R}}\}$ in E_k. Note that topological properties of a set Q in E_k are understood as corresponding topological properties of the set $Q_{\mathbb{R}}$ in $\mathbb{R}^k$. For example, the homotopy of a curve $\gamma \subset E_k$ to the zero means the homotopy of $\gamma_{\mathbb{R}} \subset \mathbb{R}^k$ to the zero; the rectifiability of a curve $\gamma \subset E_k$ is understood as the rectifiability of the curve $\gamma_{\mathbb{R}} \subset \mathbb{R}^k$, etc.

Let Ω be a domain in E_k and

$$\Omega_{\mathbb{R}} := \Big\{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k \,:\, \zeta = \sum_{j=1}^{k} x_j\, e_j \in \Omega\Big\}.$$

We say that a continuous function $\Phi : \Omega \to \mathbb{A}_n^m$ is *monogenic* in Ω if Φ is differentiable in the sense of Gateaux in every point of Ω, i. e. if for every $\zeta \in \Omega$ there exists an element $\Phi'(\zeta) \in \mathbb{A}_n^m$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall\, h \in E_k. \tag{4}$$

$\Phi'(\zeta)$ is the *Gateaux derivative* of the function Φ at the point ζ.

Consider the decomposition of a function $\Phi : \Omega \to \mathbb{A}_n^m$ with respect to the basis $\{I_r\}_{r=1}^n$:

$$\Phi(\zeta) = \sum_{r=1}^{n} U_r(x_1, x_2, \ldots, x_k)\, I_r\,. \tag{5}$$

In the case where the functions $U_r : \Omega_{\mathbb{R}} \to \mathbb{C}$ are $\mathbb{R}$-differentiable in $\Omega_{\mathbb{R}}$, i. e. for every $(x_1, x_2, \ldots, x_k) \in \Omega_{\mathbb{R}}$,

$$U_r\,(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_k + \Delta x_k) - U_r(x_1, x_2, \ldots, x_k) =$$

$$= \sum_{j=1}^{k} \frac{\partial U_r}{\partial x_j}\, \Delta x_j + o\left(\sqrt{\sum_{j=1}^{k} (\Delta x_j)^2}\right), \qquad \sum_{j=1}^{k} (\Delta x_j)^2 \to 0\,,$$

the function Φ is monogenic in the domain Ω if and only if the following Cauchy – Riemann conditions are satisfied in Ω:

$$\frac{\partial \Phi}{\partial x_j} = \frac{\partial \Phi}{\partial x_1}\, e_j \qquad \text{for all} \quad j = 2, 3, \ldots, k. \tag{6}$$

An expansion of the resolvent is of the form (see [16]):

$$(te_1 - \zeta)^{-1} = \sum_{u=1}^{m} \frac{1}{t - \xi_u}\, I_u + \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \frac{Q_{r,s}}{(t - \xi_{u_s})^r}\, I_s \tag{7}$$

$$\forall\, t \in \mathbb{C} :\ t \neq \xi_u, \quad u = 1, 2, \ldots, m,$$

where the coefficients $Q_{r,s}$ are determined by the following recurrence relations:

$$Q_{2,s} = T_s\,, \quad Q_{r,s} = \sum_{q=r+m-2}^{s-1} Q_{r-1,q}\, B_{q,\,s}\,, \quad r = 3, 4, \ldots, s - m + 1, \tag{8}$$

with

$$T_s := \sum_{j=2}^{k} x_j a_{js}\,, \quad B_{q,s} := \sum_{p=m+1}^{s-1} T_p \Upsilon_{q,s}^{p}\,, \ \ p = m+2, m+3, \ldots, n, \tag{9}$$

and the natural numbers u_s are defined in the rule 3 of the multiplication table of algebra $\mathbb{A}_n^m$.

In the paper [17] an expansion of the resolvent is obtained for the case $k=3$.

Consider the sets $M_u := \{\zeta \in E_k : f_u(\zeta) = 0\}$ for $u = 1, 2, \ldots, m$.

By $D_u \subset \mathbb{C}$ we denote the image of Ω under the mapping f_u, $u = 1, 2, \ldots, m$. We say that a domain $\Omega \subset E_k$ is *convex with respect to the set of directions* M_u if Ω contains the segment $\{\zeta_1 + \alpha(\zeta_2 - \zeta_1) : \alpha \in [0,1]\}$ for all $\zeta_1, \zeta_2 \in \Omega$ such that $\zeta_2 - \zeta_1 \in M_u$.

In the next theorem we give a constructive description of all monogenic functions given in domains of E_k and taking values in $\mathbb{A}_n^m$ by means of holomorphic functions of the complex variable.

Theorem 1 [16]. *Let a domain $\Omega \subset E_k$ be convex with respect to the set of directions M_u and $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Then every monogenic function $\Phi : \Omega \to \mathbb{A}_n^m$ can be expressed in the form*

$$\Phi(\zeta) = \sum_{u=1}^{m} I_u \, \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1}\, dt +$$

$$+ \sum_{s=m+1}^{n} I_s \, \frac{1}{2\pi i} \int\limits_{\Gamma_{u_s}} G_s(t)(te_1 - \zeta)^{-1}\, dt, \tag{10}$$

where F_u and G_s are certain holomorphic functions in the domains D_u and D_{u_s}, respectively, and Γ_q is a closed Jordan rectifiable curve in D_q which surrounds the point ξ_q and contains no points ξ_ℓ, $\ell, q = 1, 2, \ldots, m$, $\ell \neq q$.

Note that in the paper [17] the expression of the form (10) is proved for the case $k = 3$.

4. Cauchy integral theorem for a curvilinear integral. Let γ be a Jordan rectifiable curve in E_k. For a continuous function $\Psi : \gamma \to \mathbb{A}_n^m$ of the form

$$\Psi(\zeta) = \sum_{r=1}^{n} U_r(x_1, x_2, \ldots, x_k)\, I_r + i \sum_{r=1}^{n} V_r(x_1, x_2, \ldots, x_k)\, I_r\,, \tag{11}$$

where $(x_1, x_2, \ldots, x_k) \in \gamma_{\mathbb{R}}$ and $U_r : \gamma_{\mathbb{R}} \to \mathbb{R}$, $V_r : \gamma_{\mathbb{R}} \to \mathbb{R}$, we define an integral along a Jordan rectifiable curve γ by the equality:

$$\int_{\gamma} \Psi(\zeta) d\zeta := \sum_{j=1}^{k} e_j \sum_{r=1}^{n} I_r \int_{\gamma_{\mathbb{R}}} U_r(x_1, x_2, \ldots, x_k) dx_j +$$

$$+ i \sum_{j=1}^{k} e_j \sum_{r=1}^{n} I_r \int_{\gamma_{\mathbb{R}}} V_r(x_1, x_2, \ldots, x_k) dx_j \,,$$

where $d\zeta := e_1 dx_1 + e_2 dx_2 + \ldots + e_k dx_k$.

Let us define a surface integral also. Let Σ be a piece-smooth hypersurface in E_k. For a continuous function $\Psi : \Sigma \to \mathbb{A}_n^m$ of the form (11), where $(x_1, x_2, \ldots, x_k) \in \Sigma_{\mathbb{R}}$ and $U_r : \Sigma_{\mathbb{R}} \to \mathbb{R}$, $V_r : \Sigma_{\mathbb{R}} \to \mathbb{R}$, we define a surface integral on Σ with the differential form $dx_p \wedge dx_q$, by the equality

$$\int_{\Sigma} \Psi(\zeta) dx_p \wedge dx_q := \sum_{r=1}^{n} I_r \int_{\Sigma_{\mathbb{R}}} U_r(x_1, x_2, \ldots, x_k) dx_p \wedge dx_q +$$

$$+ i \sum_{r=1}^{n} I_r \int_{\Sigma_{\mathbb{R}}} V_r(x_1, x_2, \ldots, x_k) dx_p \wedge dx_q .$$

If a function $\Phi : \Omega \to \mathbb{A}_n^m$ is continuous together with partial derivatives of the first order in a domain Ω, and Σ is a piece-smooth hypersurface in Ω, and the edge γ of surface Σ is a rectifiable Jordan curve, then the following analogue of the Stokes formula is true:

$$\int_{\gamma} \Psi(\zeta) d\zeta = \int_{\Sigma} \left(\frac{\partial \Psi}{\partial x_1} e_2 - \frac{\partial \Psi}{\partial x_2} e_1 \right) dx_1 \wedge dx_2 +$$

$$+ \left(\frac{\partial \Psi}{\partial x_2} e_3 - \frac{\partial \Psi}{\partial x_3} e_2 \right) dx_2 \wedge dx_3 + \ldots + \left(\frac{\partial \Psi}{\partial x_k} e_1 - \frac{\partial \Psi}{\partial x_1} e_k \right) dx_k \wedge dx_1 . \quad (12)$$

Now, the next theorem is a result of the formula (12) and the equalities (6).

Theorem 2. *Suppose that $\Phi : \Omega \to \mathbb{A}_n^m$ is a monogenic function in a domain Ω, and Σ is a piece-smooth surface in Ω, and the edge γ of surface Σ is a rectifiable Jordan curve. Then*

$$\int_{\gamma} \Phi(\zeta)d\zeta = 0. \tag{13}$$

In the case where a domain Ω is convex, the equality (13) can be proved for an arbitrary closed Jordan rectifiable curve γ_ζ by the usual way (see, e. g., [18]).

In the case where Ω is an arbitrary domain, similarly to the proof of Theorem 3.2 [3], one can prove the following statement.

Theorem 3. *Let $\Phi : \Omega \to \mathbb{A}_n^m$ be a monogenic function in a domain Ω. Then for every closed Jordan rectifiable curve γ homotopic to a point in Ω, the equality* (13) *is true.*

5. The Morera theorem. To prove an analogue of Morera theorem in the algebra $\mathbb{A}_n^m$, we introduce auxiliary notions and prove some auxiliary statements.

Let us consider the algebra $\mathbb{A}_n^m(\mathbb{R})$ with the basis $\{I_k, iI_k\}_{k=1}^n$ over the field $\mathbb{R}$ which is isomorphic to the algebra $\mathbb{A}_n^m$ over the field $\mathbb{C}$. In the algebra $\mathbb{A}_n^m(\mathbb{R})$ there exist another basis $\{e_r\}_{r=1}^{2n}$, where the vectors $e_1, e_2, \ldots, e_k$ are the same as in the Section 3.

For every element $a := \sum\limits_{r=1}^{2n} a_r e_r,\ a_r \in \mathbb{R}$, we define the Euclidian norm

$$\|a\| := \sqrt{\sum_{r=1}^{2n} a_r^2}\,.$$

Accordingly, $\|\zeta\| = \sqrt{\sum\limits_{j=1}^{k} x_j^2}$ and $\|e_j\| = 1$ for all $j = 1, 2, \ldots, k$.

Using the equivalence of norms in any finite-dimensional space, for every element $b := \sum\limits_{r=1}^{n} (b_{1r} + ib_{2r})I_r$ with $b_{1r}, b_{2r} \in \mathbb{R}$, we have the following inequalities:

$$|b_{1r} + ib_{2r}| \leq \sqrt{\sum_{r=1}^{2n} \left(b_{1r}^2 + b_{2r}^2\right)} \leq c\|b\|, \tag{14}$$

where c is a positive constant independent of b.

Lemma 1. *If γ is a closed Jordan rectifiable curve in E_k and a function $\Psi : \gamma \to \mathbb{A}_n^m$ is continuous, then*

$$\left\| \int\limits_{\gamma} \Psi(\zeta)\, d\zeta \right\| \leq c \int\limits_{\gamma} \|\Psi(\zeta)\|\, \|d\zeta\|, \tag{15}$$

where c is a positive absolute constant.

Proof. Using the representation of function Ψ in the form (11) for $(x_1, x_2, \ldots, x_k) \in \gamma$, we obtain

$$\left\| \int\limits_{\gamma} \Psi(\zeta) d\zeta \right\| \leq \sum_{r=1}^{n} \|e_1 I_r\| \int\limits_{\gamma_{\mathbb{R}}} \big|U_r(x_1, x_2, \ldots, x_k) + iV_r(x_1, x_2, \ldots, x_k)\big|\, dx_1 +$$

$$\ldots + \sum_{r=1}^{n} \|e_k I_r\| \int\limits_{\gamma_{\mathbb{R}}} \big|U_r(x_1, x_2, \ldots, x_k) + i\, V_r(x_1, x_2, \ldots, x_k)\big|\, dx_k.$$

Now, taking into account the inequality (14) for $b = \Psi(\zeta)$ and the inequalities $\|e_j I_r\| \leq c_j,\ j = 1, 2, \ldots, k$, where c_j are positive absolute constants, we obtain the relation (15). The lemma is proved.

We understand a triangle $\triangle$ as a plane figure bounded by three line segments connecting three its vertices. Denote by $\partial\triangle$ the boundary of triangle $\triangle$ in relative topology of its plane. We assume that the triangle $\triangle$ includes the boundary $\partial\triangle$ also.

Using Lemma 1, for functions taking values in the algebra $\mathbb{A}_n^m$, the following Morera theorem can be established in the usual way.

Theorem 4. *If a function $\Phi : \Omega \to \mathbb{A}_n^m$ is continuous in a domain*

Ω *and satisfies the equality*

$$\int\limits_{\partial\triangle} \Phi(\zeta)d\zeta = 0 \tag{16}$$

for every triangle $\triangle \subset \Omega$*, then the function* Φ *is monogenic in the domain* Ω*.*

6. Cauchy integral formula for a curvilinear integral. Let $\zeta_0 := \sum\limits_{j=1}^{k} x_j^{(0)} e_j$ be a point in a domain $\Omega \subset E_k$. Let us take any 2-dimensional plane containing the point ζ_0. In this plane let us take a circle $C(\zeta_0, R)$ of radius R with the center at the point ζ_0. Let R be such that $C(\zeta_0, R)$ is completely contained in Ω. By $f_u(C(\zeta_0, R))$ we denote the image of $C(\zeta_0, R)$ under the mapping f_u, $u = 1, 2, \ldots, m$.

We assume that the circle $C(\zeta_0, R)$ *embraces the set* $\{\zeta - \zeta_0 : \zeta \in \bigcup\limits_{u=1}^{m} M_u\}$. It means that the curve $f_u(C(\zeta_0, R))$ bounds some domain D'_u and $f_u(\zeta_0) \in D'_u$ for all $u = 1, 2, \ldots, m$.

We say that the curve $\gamma \subset \Omega$ *embraces once the set* $\{\zeta - \zeta_0 : \zeta \in \bigcup\limits_{u=1}^{m} M_u\}$, if there exists a circle $C(\zeta_0, R)$ which embraces the mentioned set and is homotopic to γ in the domain $\Omega \setminus \{\zeta - \zeta_0 : \zeta \in \bigcup\limits_{u=1}^{m} M_u\}$.

Let a circle $C(0, R)$ embrace the set $\bigcup\limits_{u=1}^{m} M_u$. Since the function ζ^{-1} is continuous on $C(0, R)$, there exists the integral

$$\int\limits_{C(0,R)} \zeta^{-1} d\zeta =: \lambda\,. \tag{17}$$

The next theorem is an analogue of Cauchy integral theorem for monogenic functions $\Phi : \Omega \to \mathbb{A}_n^m$ and is proved similarly to Theorem 4 in [1] .

Theorem 5. *Suppose that a domain* $\Omega \subset E_k$ *is convex with respect to the set of directions* M_u *and* $f_u(E_k) = \mathbb{C}$ *for all* $u = 1, 2, \ldots, m$*. Suppose* *also that $\Phi : \Omega \to \mathbb{A}_n^m$ is a monogenic function in Ω. Then for every point $\zeta_0 \in \Omega$ the following equality is true:*

$$\lambda\,\Phi(\zeta_0) = \int\limits_{\gamma} \Phi(\zeta)\,(\zeta - \zeta_0)^{-1}\,d\zeta, \tag{18}$$

where γ is an arbitrary closed Jordan rectifiable curve in Ω, that embraces once the set $\{\zeta - \zeta_0 : \zeta \in \bigcup\limits_{u=1}^{m} M_u\}$.

7. A constant λ. In certain special algebras (see [4–6]) the Cauchy integral formula (18) has the form

$$\Phi(\zeta_0) = \frac{1}{2\pi i}\int\limits_{\gamma_\zeta} \Phi(\zeta)\,(\zeta - \zeta_0)^{-1}\,d\zeta, \tag{19}$$

i. e.

$$\lambda = 2\pi i. \tag{20}$$

In this Section we indicate a set of algebras $\mathbb{A}_n^m$ for which (20) holds. First, let us consider some auxiliary statements.

As a consequence of the expansion (7), we obtain the following equality:

$$\zeta^{-1} = \sum_{r=1}^{n} \widetilde{A}_r\, I_r \tag{21}$$

with the coefficients $\widetilde{A}_r$ determined by the following relations:

$$\begin{gathered} \widetilde{A}_u = \frac{1}{\xi_u}\,, \quad u = 1, 2, \ldots, m, \\ \widetilde{A}_s = \sum_{r=2}^{s-m+1} \frac{\widetilde{Q}_{r,s}}{\xi_{u_s}^{r}}\,, \quad s = m+1, m+2, \ldots, n, \end{gathered} \tag{22}$$

where $\widetilde{Q}_{r,s}$ are determined by the following recurrence relations:

$$\widetilde{Q}_{2,s} = -T_s\,, \quad \widetilde{Q}_{r,s} = -\sum_{q=r+m-2}^{s-1} \widetilde{Q}_{r-1,q}\, B_{q,\,s}\,, \quad r = 3, 4, \ldots, s-m+1, \tag{23}$$

where T_s and $B_{q,s}$ are the same as in the equalities (9), and natural numbers u_s are defined in the rule 3 of the multiplication table of the algebra $\mathbb{A}_n^m$.

Taking into account the equality (21) and the relation

$$d\zeta=\sum_{j=1}^{k} dx_j\, e_j=\sum_{u=1}^{m}\Big(dx_1+\sum_{j=2}^{k} dx_j\, a_{ju}\Big)I_u+$$

$$+\sum_{r=m+1}^{n}\sum_{j=2}^{k} dx_j\, a_{js}\, I_r=\sum_{u=1}^{m} d\xi_u\, I_u+\sum_{r=m+1}^{n} dT_r\, I_r\,,$$

we have the following equality:

$$\zeta^{-1}d\zeta=\sum_{u=1}^{m}\widetilde{A}_u\, d\xi_u\, I_u+\sum_{r=m+1}^{n}\widetilde{A}_{u_r}\, dT_r\, I_r+$$

$$+\sum_{s=m+1}^{n}\widetilde{A}_s\, d\xi_{u_s}\, I_s+\sum_{s=m+1}^{n}\sum_{r=m+1}^{n}\widetilde{A}_s\, dT_r\, I_sI_r=:\sum_{r=1}^{n}\sigma_r\, I_r\,. \tag{24}$$

Now, taking into account the denotation (24) and the equalities (22), we calculate:

$$\int\limits_{C(0,R)}\sum_{u=1}^{m}\sigma_u\, I_u=\sum_{u=1}^{m} I_u\int\limits_{f_u(C(0,R))}\frac{d\xi_u}{\xi_u}=2\pi i\sum_{u=1}^{m} I_u=2\pi i.$$

Thus,

$$\lambda=2\pi i+\sum_{r=m+1}^{n} I_r\int\limits_{C(0,R)}\sigma_r\,. \tag{25}$$

Therefore, the equality (20) holds if and only if

$$\int\limits_{C(0,R)}\sigma_r=0\qquad\forall\, r=m+1,\ldots,n. \tag{26}$$

But, to satisfy the equality (26), the differential form σ_r must be the total differential of some function. Note that the property to be the total differential is invariant under admissible transformations of coordinates [19, p. 328, Theorem 2]. Thus, if to show that σ_r is the total differential of a function which depends on the variables $\frac{T_{m+1}}{\xi}, \ldots, \frac{T_k}{\xi}$, then it will mean that σ_r is a total differential of a function of $x_1, x_2, \ldots, x_k$.

7.1. In this subsection we indicate a set of algebras in which the vectors (3) are arbitrarily chosen and the equality (20) holds. In the next theorem we use the representation (2) of algebra $\mathbb{A}_n^m$.

Theorem 6. *The equality* (20) *holds if at least one of the following conditions are satisfied:*

1) $\mathbb{A}_n^m \equiv S$;

2) N *is a zero nilpotent subalgebra*;

3) $\dim_{\mathbb{C}} N \leq 3$, *where* $\dim_{\mathbb{C}} N$ *is the complex dimensionality of subalgebra* N;

4) $\dim_{\mathbb{C}} N = 4$ *and*

$$\Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+2}_{m+2,m+3} = \Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+2}_{m+2,m+4} = \Upsilon^{m+1}_{m+1,m+3}\Upsilon^{m+2}_{m+3,m+4} =$$
$$= \Upsilon^{m+3}_{m+3,m+4}\Upsilon^{m+1}_{m+1,m+3} = \Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+1}_{m+3,m+4} =$$
$$= \Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+2}_{m+3,m+4} = \Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+3}_{m+3,m+4} =$$
$$= \Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+2}_{m+3,m+4} = \Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+1}_{m+2,m+3}\Upsilon^{m+3}_{m+3,m+4} =$$
$$= \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+1}_{m+3,m+4} = \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+3}_{m+3,m+4} =$$
$$= \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+1}_{m+3,m+4} = \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+2}_{m+3,m+4} =$$
$$= \Upsilon^{m+2}_{m+2,m+3}\Upsilon^{m+1}_{m+1,m+2}\Upsilon^{m+3}_{m+3,m+4} = 0. \qquad (27)$$

The proof of Theorem 6 is analogous to the proofs of Theorems 5 — 8 in [1].

Further, we consider some examples of algebras which satisfy the relations (27).

Examples.

- The algebra with the basis $\{I_1 = 1, I_2, I_3, I_4, I_5\}$ and the multiplication rules

$$I_2^2 = I_3\,,\ I_2\,I_4 = I_5,$$

and other products of nilpotent elements I_2, I_3, I_4, I_5 are equal to zero (for nilpotent subalgebra see [14], Table 21, algebra $\mathcal{J}_{69}$ and [13], page 590, algebra $A_{1,4}$).

- The algebra with the basis $\{I_1 = 1, I_2, I_3, I_4, I_5\}$ and the multiplication rule
$$I_2^2 = I_3,$$
and other products of nilpotent elements I_2, I_3, I_4, I_5 are equal to zero (for nilpotent subalgebra see [13], page 590, algebra $A_{1,2} \oplus A_{0,1}^2$).

- The algebra with the basis $\{I_1 = 1, I_2, I_3, I_4, I_5\}$ and the multiplication rules
$$I_2^2 = I_3\,,\ I_4^2 = I_5,$$
and other products of nilpotent elements I_2, I_3, I_4, I_5 are equal to zero (for nilpotent subalgebra see [13], page 590, algebra $A_{1,2} \oplus A_{1,2}$).

- The algebra with the basis $\{I_1 = 1, I_2, I_3, I_4, I_5\}$ and the multiplication rules
$$I_2^2 = I_3\,,\ I_2\, I_3 = I_4,$$
and other products of nilpotent elements I_2, I_3, I_4, I_5 are equal to zero (for nilpotent subalgebra see [14], Table 21, algebra $\mathcal{J}_{71}$).

An example of algebra, which does not satisfy the relations (27), is considered in the paper [1]. Moreover, in [1] the vectors e_1, e_2, e_3 of form (3) are selected such that the equality (20) is not true.

7.2 In this subsection we indicate sufficient conditions on a choose of the vectors (3) for which the equality (20) is true. We use the representation (2) of algebra $\mathbb{A}_n^m$.

Theorem 7. *If $C(0,R) \subset E_k \subset S$, then the equality* (20) *holds.*

Proof. Note that the condition $E_k \subset S$ means that $a_{jr} = 0$ for all $j = 2, 3, \ldots, k$ and $r = m+1, \ldots, n$ in the decomposition (3).

If $\zeta \in S$, then $T_s = 0$ for $s = m+1, \ldots, n$ (see the denotation (9)). It follows from the relation (24) that

$$\begin{gathered}\sigma_{m+1} = \frac{dT_{m+1}}{\xi_{u_{m+1}}} + \widetilde{A}_{m+1}\, d\xi_{u_{m+1}}\,, \\ \sigma_r = \frac{dT_r}{\xi_{u_r}} + \widetilde{A}_r\, d\xi_{u_r} + \sum_{q,s=m+1}^{r-1} \widetilde{A}_q\, dT_s \Upsilon_{q,r}^s\,, \quad r = m+2, \ldots, n.\end{gathered} \tag{28}$$

Now, it follows from (23), (22) that $\widetilde{A}_s = 0$, and then $\sigma_r = 0$ for $r = m+1, \ldots, n$ due to the equalities (28). Finally, the equality (20) is a consequence of the equality $\sigma_r = 0$ and the relation (25). The theorem is proved.

Theorem 7 generalizes Theorem 6 [20] and Theorem 9 [1].

Now, let us consider some results for the case where $E_k \not\subset S$. First, if $\dim_{\mathbb{C}} N \leq 3$, then by Theorem 6 the equality (20) holds for $C(0,R) \subset E_k$ and any E_k. Furthermore, using Theorem 6, we prove the next theorem similarly to Theorem 10 in [1].

Theorem 8. *Let* $\dim_{\mathbb{C}} N = 4$. *Then the equality* (20) *holds if the following two conditions are satisfied:*

1) $a_{j,m+1} = 0$ *for all* $j = 2, 3, \ldots, k$;

2) *for every* $j = 2, 3, \ldots, k$ *at least one of the equalities* $a_{j,m+2} = 0$ *or* $a_{j,m+3} = 0$ *holds.*

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