Збірник праць Ін-ту математики НАН України 2013, том 6, N 6, 209–223

Carlos Biasi , Thaís F. M. Monis

Weak local Nash equilibrium - part II

biasi@icmc.usp.br , tfmonis@rc.unesp.br

In the paper "Weak local Nash equilibrium" we define a concept of local equilibrium to non-cooperative games and we prove its existence applying the Lefschetz fixed point theorem. We was inspired by the original Nash's theorem and his proof.

1 Introduction

In the paper "Weak local Nash equilibrium" we define a concept of local equilibrium to non-cooperative games and we prove its existence applying the Lefschetz fixed point theorem. We was inspired by the original Nash's theorem and his proof.

The concept of Nash equilibrium says that an equilibrium for payoff functions

$$p_1, p_2, \ldots, p_n : S = S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$$

is a point $\tilde{s} = (\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n) \in S$ such that, for each $i \in \{1, 2, \ldots, n\}$,

$$p_i(\tilde{s}_1, \ldots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \ldots, \tilde{s}_n) \leq p_i(\tilde{s}), \text{ for all } s_i \in S_i.$$

Nash proved that:

Theorem 1.1 (Nash's Theorem). *Let $S_1, \ldots, S_n$ be compact convex subsets of an Euclidean space. Suppose that $p_1, \ldots, p_n : S = S_1 \times \cdots \times S_n \to \mathbb{R}$ are maps such that, for each $i = 1, \ldots, n$, $p_i(s_1, \ldots, s_n)$ is linear (afim) as a function of s_i. Then there exists at least one equilibrium to $p_1, \ldots, p_n$.*

The proof is the following: let $S_i \subset \mathbb{R}^{d_i}$, where d_i is the dimension of S_i. Thus, $S \subset \mathbb{R}^d$, where $d = d_1 + \cdots + d_n$. From the hypothesis, the payoff functions are of the type

$$p_i(s) = v_i(s) \cdot s_i + u_i(s)$$

where $v_i : S \to \mathbb{R}^{d_i}$ and $u_i : S \to \mathbb{R}$ are maps which don't depend on the coordinate s_i, $i = 1, \ldots, n$. Let $v : S \to \mathbb{R}^d$ be the vector field defined by $v(s) = (v_1(s), \ldots, v_n(s))$. Let $r : \mathbb{R}^n \to S$ be the natural retraction that assigns each point $p \in \mathbb{R}^n$ to the point $r(p) \in S$ which realizes the distance of p to S. Finally, let $f : S \to S$ be defined by $f(s) = r(s+v(s))$. Then, one can shown that $\tilde{s} \in S$ is a Nash equilibrium to $p_1, \ldots, p_n$ if and only if $\tilde{s}$ is a fixed point of f. Note that the existence of a fixed point to f is assured by Brouwer's fixed point theorem.

Based on the above proof, we investigated the existence of equilibrium in the context that the spaces of strategies are compact ENR's, not necessarily convex. This means that each space S_i is a subset of some euclidean space $\mathbb{R}^{d_i}$ and there is an open neighborhood V_i of S_i in $\mathbb{R}^{d_i}$ and a retraction $r_i : V_i \to S_i$. From this research, the following definitions arise.

Definition 1. *Let $(S_1, d_1), \ldots, (S_n, d_n)$ be metric spaces and $p_1, \ldots, p_n : S_1 \times \cdots \times S_n \to \mathbb{R}$ real functions. We say that $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n) \in S$ is a* **weak local equilibrium (abbrev., w.l.e.)** *for $p_1, \ldots, p_n$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$p_i(\tilde{s}_1, \ldots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \ldots, \tilde{s}_n) \leq p_i(\tilde{s}) + \varepsilon d_i(s_i, \tilde{s}_i),$$

for every $s_i \in B(\tilde{s}_i, \delta)$, $i = 1, 2, \ldots, n$, where $B(\tilde{s}_i, \delta)$ denotes the open ball with center in $\tilde{s}_i$ and radius $\delta > 0$ in (S_i, d_i).

Definition 2. *We say that a subset X of $\mathbb{R}^m$ has the* **property of convenient retraction (abbrev., p.c.r.)** *if there exists a retraction $r : V \to X$, where V is an open neighborhood of X in $\mathbb{R}^m$, satisfying: given $x_0 \in V$ and $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\langle x_0 - r(x_0), x - r(x_0) \rangle \leq \varepsilon \|x - r(x_0)\|,$$

for all $x \in X$ with $\|x - r(x_0)\| < \delta$, where $\langle\ ,\ \rangle$ is the usual inner product in $\mathbb{R}^m$ and $\|\cdot\|$ is the norm induced by it. In this case, we say that $r : V \to X$ is a convenient retraction.

Example 1. *Every closed convex subset K of $\mathbb{R}^m$ has the p.c.r.. In fact, there is a natural retraction $r : \mathbb{R}^m \to K$ such that to each $x \in \mathbb{R}^m$ assigns the point $r(x) \in K$ which realizes the distance of x to K. This retraction satisfies $\langle x_0 - r(x_0), x - r(x_0)\rangle \leq 0$ for every $x_0 \in \mathbb{R}^m$ and $x \in K$.*

Example 2 ([3], Proposition 4.3)**.** *Every submanifold M of $\mathbb{R}^n$, of class C^2, with or without boundary, has the p.c.r..*

Let X be a closed subset of the Euclidean space $\mathbb{R}^n$ and let V be an open neighborhood of X in $\mathbb{R}^n$. A map $r : V \to X$ is called a proximative retraction (or metric projection) if

$$\|r(y) - y\| = \text{dist}(y, X), \text{ for every } y \in V,$$

where

$$\text{dist}(y, X) = \inf\{\|x - y\| \mid x \in X\}$$

is the distance of y to X.

Evidently, every proximative retraction is a retraction map but not conversely.

A compact subset $K \subset \mathbb{R}^n$ is called a proximative neighborhood retract (written $K \in$ PANR) if there exists an open neighborhood V of K in $\mathbb{R}^n$ and a proximative retraction $r : V \to K$.

We have the following statement:

Example 3 ([2])**.** *Let K be a compact subset of $\mathbb{R}^n$. If $K \in$ PANR then K is an ENR with the p.c.r..*

In the previous paper, we was able to prove the following result.

Theorem 1.2 ([2])**.** *Let $p_1, \ldots, p_n : S_1 \times \ldots \times S_n \to \mathbb{R}$ be maps, where each $S_i \subset \mathbb{R}^{m_i}$ is a compact ENR with the p.c.r.. Also, suppose $p_i(s_1, \ldots, s_n)$ continuously differentiable in a neighborhood of s_i when the other variables are kept fixed, $i = 1, 2, \ldots, n$. If $\chi(S_i) \neq 0$ for $i = 1, 2, \ldots, n$ then $p_1, p_2, \ldots, p_n$ have at least one w.l.e..*

Our goal in this paper is to prove a more general version of Theorem 1.2 changing the hypothesis of the continuously differentiable on the payoffs by a weaker hypothesis.

2 Preliminaires

In this section, we define a concept of an upper semi differentiable(u.s.d.) function.

The open ball in $\mathbb{R}^n$ with center in x_0 and radius $r > 0$ will be denoted by $B(x_0, r)$.

Definition 3. *Let $f : A \to \mathbb{R}$ be a function, where A is an open nonempty subset of $\mathbb{R}^n$. Given $x_0 \in A$, we say that f is upper semi differrentiable(u.s.d.) at x_0 if there exists at least one point $v \in \mathbb{R}^n$ together with a function $r : B(0, \varepsilon) \to \mathbb{R}$ such that $\lim\limits_{h \to 0} \dfrac{r(h)}{\|h\|} = 0$ and*

$$f(x_0 + h) \leq f(x_0) + v \cdot h + r(h)$$

for every h such that $x_0 + h \in A$.

We denote by $DSf(x_0)$ the set of such vectors v.

Example 4. *If $f : A \to \mathbb{R}$ is differentiable at x_0 then f is u.s.d.. Moreover, $DSf(x_0) = \{f'(x_0)\}$. In fact, suppose $v \in \mathbb{R}^n$ and $r : B(0, \varepsilon) \to \mathbb{R}$ such that $\lim\limits_{h \to 0} \dfrac{r(h)}{h} = 0$ and $f(x_0 + h) \leq f(x_0) + v \cdot h + r(h)$ for every h. Thus, for $0 < t < \varepsilon$,*

$$\frac{f(x_0 + te_i) - f(x_0)}{t} \leq v \cdot e_i + \frac{r(te_i)}{t}.$$

It follows that

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{t \to 0^+} \frac{f(x_0 + te_i) - f(x_0)}{t} \leq v \cdot e_i.$$

On the other hand, for $-\varepsilon < t < 0$,

$$\frac{f(x_0 + te_i) - f(x_0)}{t} \geq v \cdot e_i + \frac{r(te_i)}{t}.$$

It follows that

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{t \to 0^-} \frac{f(x_0 + te_i) - f(x_0)}{t} \geq v \cdot e_i.$$

Therefore, $\dfrac{\partial f}{\partial x_i}(x_0) = v \cdot e_i$.

Thus, $v = f'(x_0) = \left(\dfrac{\partial f}{\partial x_1}(x_0), \ldots, \dfrac{\partial f}{\partial x_n}(x_0) \right)$.

The next result shows that the set $DSf(x_0)$ is convex.

Theorem 2.1. *If f is u.s.d. at x_0 then $DSf(x_0)$ is a convex subset of $\mathbb{R}^n$.*

Proof. Let $v_1, v_2 \in DSf(x_0)$ be arbitraires and let $r_1, r_2 : B(0,\varepsilon) \to \mathbb{R}$ be such that

$$\begin{array}{rcl} f(x_0+h) & \leq & f(x_0) + v_1 \cdot h + r_1(h) \\ f(x_0+h) & \leq & f(x_0) + v_2 \cdot h + r_2(h) \end{array}$$

with $\lim_{h\to 0} \frac{r_1(h)}{\|h\|} = \lim_{h\to 0} \frac{r_2(h)}{\|h\|} = 0$.

Let $v = \alpha v_1 + (1-\alpha)v_2$, with $\alpha \in (0,1)$. We have

$$\begin{array}{rcl} f(x_0+h) & = & \alpha f(x_0+h) + (1-\alpha) f(x_0+h) \\ & \leq & \alpha f(x_0) + \alpha v_1 \cdot h + \alpha r_1(h) + (1-\alpha) f(x_0) \\ & & +(1-\alpha) v_2 \cdot h + (1-\alpha) r_2(h) \\ & = & f(x_0) + v \cdot h + \alpha r_1(h) + (1-\alpha) r_2(h). \end{array}$$

Since

$$\lim_{h\to 0} \frac{\alpha r_1(h) + (1-\alpha) r_2(h)}{\|h\|} = \alpha \lim_{h\to 0} \frac{r_1(h)}{\|h\|} + (1-\alpha) \lim_{h\to 0} \frac{r_2(h)}{\|h\|} = 0,$$

it follows that $v \in DSf(x_0)$.

Therefore, $DSf(x_0)$ is convex. □

In the next theorems, we give conditions to $DSf(x_0)$ be compact.

Theorem 2.2. *Let $f : J \to \mathbb{R}$ be a function, where $J \subset \mathbb{R}$ is open interval, and let $x_0 \in J$. Suppose the existence of the right and left-hand limits*

$$c = \lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h}$$

and

$$d = \lim_{h\to 0^-} \frac{f(x_0+h) - f(x_0)}{h}$$

Then, f is u.s.d. if and only if $c \leq d$. Moreover, $DSf(x_0) = [c,d]$.

Proof. Suppose f u.s.d. at x_0 and let $v \in DSf(x_0)$. If $0 < h < \varepsilon$, we have

$$\frac{f(x_0+h)-f(x_0)}{h} \leq v + \frac{r(h)}{h},$$

following that $c = \lim\limits_{h\to 0^+} \dfrac{f(x_0+h)-f(x_0)}{h} \leq v$.

Analogously, if $-\varepsilon < h < 0$, we have

$$\frac{f(x_0+h)-f(x_0)}{h} \geq v + \frac{r(h)}{h},$$

following that $d = \lim\limits_{h\to 0^-} \dfrac{f(x_0+h)-f(x_0)}{h} \geq v$.

Therefore, $c \leq d$.

On the other hand, suppose $c \leq d$. Note that, above, we show that $DSf(x_0) \subset [c,d]$. Now, to conclude that $DSf(x_0) = [c,d]$, since $DSf(x_0)$ is convex, it is sufficient to show that $c, d \in DSf(x_0)$.

Define $r(h) = \begin{cases} f(x_0+h)-f(x_0)-ch & \text{se } h > 0 \\ 0 & \text{se } h = 0 \\ f(x_0+h)-f(x_0)-dh & \text{se } h < 0 \end{cases}$

Then $\lim\limits_{h\to 0} \dfrac{r(h)}{h} = 0$. Moreover, for $h > 0$, we have

$$f(x_0+h) = f(x_0) + ch + f(x_0+h) - f(x_0) - ch$$

and, for $h < 0$, we have

$$f(x_0+h) = f(x_0) + ch + f(x_0+h) - f(x_0) - ch \leq f(x_0) + ch + \\ + f(x_0+h) - f(x_0) - dh$$

Therefore, $c \in DSf(x_0)$.

Analogously, for $h > 0$, we have

$$f(x_0+h) = f(x_0) + dh + f(x_0+h) - f(x_0) - dh \leq f(x_0) + dh + \\ + f(x_0+h) - f(x_0) - ch$$

and for $h < 0$,

$$f(x_0 + h) = f(x_0) + dh + f(x_0 + h) - f(x_0) - dh$$

Therefore, $d \in DSf(x_0)$. $\square$

Example 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by* $f(x) = \begin{cases} x, & \text{if } x < 0 \\ -x, & \text{if } x \geq 0 \end{cases}$. *The function f is u.s.d. at 0. In fact, we have*

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = -1 < 1 = \lim_{h\to 0^-} \frac{f(h) - f(0)}{h}$$

Then, by Theorem 2.2, f is u.s.d. at 0 and $DSf(0) = [-1, 1]$.

Notation: Let $f : A \to \mathbb{R}$ be a map, where A is an open subset of $\mathbb{R}^n$. Let $x_0 \in A$. We denote the right-hand partial derivatives and the left-hand partial derivatives, respectively, by

$$\frac{\partial f^+}{\partial x_i}(x_0) = \lim_{t\to 0^+} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

and

$$\frac{\partial f^-}{\partial x_i}(x_0) = \lim_{t\to 0^-} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

$i = 1, \ldots, n$

Theorem 2.3. *Let $f : A \to \mathbb{R}$ be a map, $A \subset \mathbb{R}^n$ open. Suppose well defined the right-hand and the left-hand partial derivatives of f at every $x_0 \in A$. Also, suppose the functions*

$$\frac{\partial f^+}{\partial x_i}, \frac{\partial f^-}{\partial x_i} : A \to \mathbb{R}$$

continuous and that

$$\frac{\partial f^+}{\partial x_i}(x_0) \leq \frac{\partial f^-}{\partial x_i}(x_0), \ \forall \ x_0 \in A,$$

$i = 1, \ldots, n$. Then, f is u.s.d. and

$$DSf(x_0) = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n],$$

where $a_i = \dfrac{\partial f^+}{\partial x_i}(x_0)$, $b_i = \dfrac{\partial f^-}{\partial x_i}(x_0)$, $i = 1, \ldots, n$. *Thus,* $DSf : A \multimap \mathbb{R}^n$ *is an u.s.c. multivalued map with convex compact values.*

Proof. Given $x_0 \in A$, let $a_i = \dfrac{\partial f^+}{\partial x_i}(x_0)$, $b_i = \dfrac{\partial f^-}{\partial x_i}(x_0)$, $i = 1, \ldots, n$. The technique used to prove that

$$DSf(x_0) \subset [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

is the same used in Theorem 2.2: let $v = (v_1, \ldots, v_n) \in DSf(x_0)$ arbitrary. Thus,

$$f(x_0 + h) \le f(x_0) + v \cdot h + r(h),$$

with $\lim_{h \to 0} \dfrac{r(h)}{\|h\|} = 0$. In particular, if $h = te_i$ then

$$f(x_0 + te_i) \le f(x_0) + tv \cdot e_i + r(te_i),$$

with $\lim_{h \to 0} \dfrac{r(te_i)}{t} = 0$. It follows that, for every $t > 0$,

$$\frac{f(x_0 + te_i) - f(x_0)}{t} \le v_i + \frac{r(te_i)}{t}.$$

Therefore

$$a_i = \frac{\partial f^+}{\partial x_i}(x_0) \le v_i.$$

Also, for every $t < 0$, we have

$$\frac{f(x_0 + te_i) - f(x_0)}{t} \ge v_i + \frac{r(te_i)}{t}.$$

Therefore,

$$b_i = \frac{\partial f^-}{\partial x_i}(x_0) \ge v_i.$$

Hence, $v \in [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

Since $DSf(x_0)$ is convex, in order to prove the equality

$$DSf(x_0) = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

it is sufficient to show that each vertex of that parallelepiped is contained in $DSf(x_0)$.

To elucidate, we will write the proof to the case $n = 2$ and for the vertex (a_1, a_2). The general case is analogous.

Let $x_0 = (x_1, x_2)$ and $h = (h_1, h_2)$. We need to show that

$$f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2) - h_1 a_1 - h_2 a_2 \le r(h)$$

with $\lim_{h \to 0} \dfrac{r(h)}{\|h\|} = 0$.

Consider the functions $g(y) = f(x_1 + h_1, y)$ and $l(x) = f(x, x_2)$. Note that

$$\begin{aligned}
\frac{\partial g^+}{\partial y}(x_2) &= \frac{\partial f^+}{\partial x_2}(x_1 + h_1, x_2) \\
\frac{\partial g^-}{\partial y}(x_2) &= \frac{\partial f^-}{\partial x_2}(x_1 + h_1, x_2) \\
\frac{\partial l^+}{\partial x}(x_1) &= \frac{\partial f^+}{\partial x_1}(x_1, x_2) \\
\frac{\partial l^-}{\partial x}(x_1) &= \frac{\partial f^-}{\partial x_1}(x_1, x_2)
\end{aligned}$$

>From Theorem 2.2, g and l are u.s.d.. Moreover,

$$DSg(x_2) = \left[\frac{\partial f^+}{\partial x_2}(x_1 + h_1, x_2), \frac{\partial f^-}{\partial x_2}(x_1 + h_1, x_2)\right]$$

and

$$DSl(x_1) = \left[\frac{\partial f^+}{\partial x_1}(x_1, x_2), \frac{\partial f^-}{\partial x_1}(x_1, x_2)\right].$$

Thus,

$$\begin{aligned}
g(x_2 + h_2) - g(x_2) - h_2 \frac{\partial f^+}{\partial x_2}(x_1 + h_1, x_2) &\le r_1(h_2) \\
l(x_1 + h_1) - l(x_1) - h_1 \frac{\partial f^+}{\partial x_1}(x_1, x_2) &\le r_2(h_1)
\end{aligned}$$

with $\lim_{x \to 0} \dfrac{r_2(x)}{x} = \lim_{y \to 0} \dfrac{r_1(y)}{y} = 0$.

We have that

$$
\begin{aligned}
& f(x_1+h_1,x_2+h_2)-f(x_1,x_2)-h_1a_1-h_2a_2 = \\
& g(x_2+h_2)-g(x_2)-h_2\frac{\partial f^+}{\partial x_2}(x_1+h_1,x_2)+l(x_1+h_1)-l(x_1)- \\
& -h_1\frac{\partial f^+}{\partial x_1}(x_1,x_2)+\ h_2\left[\frac{\partial f^+}{\partial x_2}(x_1+h_1,x_2)-\frac{\partial f^+}{\partial x_2}(x_1,x_2)\right] \\
& \leq\ r(h)
\end{aligned}
$$

where $r(h) = r_1(h_2)+r_2(h_1)+h_2\left[\frac{\partial f^+}{\partial x_2}(x_1+h_1,x_2)-\frac{\partial f^+}{\partial x_2}(x_1,x_2)\right]$.
Now, it is easy to see that $\lim_{h\to 0}\frac{r(h)}{\|h\|}$. □

3 The main theorem

In this section, we will stablish a generalization of the Theorem 1.2. It is the following:

Theorem 3.1. *Let $p_1,\dots,p_n : S_1\times\dots\times S_n \to \mathbb{R}$ be maps, where each $S_i\subset\mathbb{R}^{m_i}$ is a compact ENR with the p.c.r.. Also, suppose that $p_i(s_1,\dots,s_i,\dots,s_n)$ as a function of $s_i=(s_1^1,\dots,s_1^{m_i})$ satisfies:*

- *The map $x_i \longmapsto p(s_{-i},x_i)$ can be continuously defined on a neighborhood V_i of S_i. The symbol (s_{-i},x_i) denotes the point $(s_1,\dots,s_{i-1},x_i,s_{i+1},\dots,s_n)$.*
- *$p_i(s_{-i},\_\ ):V_i\to\mathbb{R}$ has continuous lateral partial derivatives*

$$\frac{\partial {p_i}^+}{\partial x_i^j}(s_{-i},\_\ ),\frac{\partial {p_i}^-}{\partial x_i^j}(s_{-i},\_\ ):V_i\to\mathbb{R}$$

  $j=1,\dots,m_i$ and

- $$\frac{\partial {p_i}^+}{\partial x_i^j}(s_{-i},x_i)\leq\frac{\partial {p_i}^-}{\partial x_i^j}(s_{-i},x_i),\ \forall\ x_i\in V_i$$

With these assumptions, if $\chi(S_i)\neq 0$ for $i=1,2,\dots,n$ then $p_1,p_2,\dots,p_n$ have at least one w.l.e..

The proof of Theorem 3.1 is an application of a fixed point theorem of multivalued maps.

3.1 The Lefschetz Fixed Point Theorem for Admissible Multivalued Mappings

The spaces considered here are metric. Also, we are considering the Čech homology functor with compact carriers and with coefficients in $\mathbb{Q}$.

A proper map $f : X \to Y$ is a map such that, for all $K \subset X$ compact, $f^{-1}(K)$ is compact.

A compact space X is called acyclic if $H_0(X) = \mathbb{Q}$ and $H_q(X) = 0$ for $q > 0$.

A map $p : (X, X_0) \to (Y, Y_0)$ is called a Vietoris map if $p : X \to Y$ is proper, $p^{-1}(Y_0) = X_0$ and $p^{-1}(y)$ is acyclic, for every $y \in Y$. Symbol: $p : (X, X_0) \Rightarrow (Y, Y_0)$.

Theorem 3.2 (Vietoris Mapping Theorem)**.** *If $p : (X, X_0) \Rightarrow (Y, Y_0)$ is a Vietoris map then $p_* : H_*(X, X_0) \to H_*(Y, Y_0)$ is an isomorphism.*

Let X and Y be two spaces and assume that for each point $x \in X$ a nonempty closed subset $\varphi(x)$ of Y is given; in this case, we say that φ is a multivalued map from X into Y and we write $\varphi : X \multimap Y$.

A multivalued map $\varphi : X \multimap Y$ is called upper semicontinuous (u.s.c.) if for every open subset U of Y the set $\varphi^{-1}(U) = \{x \in X \mid \varphi(x) \subset U\}$ is an open subset of X.

An u.s.c. multivalued map $\varphi : X \multimap Y$ is called acyclic if for every $x \in X$ the set $\varphi(x)$ is an acyclic subset of Y.

An u.s.c. multivalued map $\varphi : X \multimap Y$ is called admissible if there exists a space Γ and mappings $p : \Gamma \Rightarrow X$, $q : \Gamma \to Y$ such that:

- p is a Vietoris map,
- $q(p^{-1}(x)) \subset \varphi(x)$, for every $x \in X$.

(p, q) is called a selected pair of φ (written $(p, q) \subset \varphi$).

Let $\varphi : X \multimap Y$ be an admissible multivalued map. The set $\{\varphi\}_*$ of linear induced mappings is defined by

$$\{\varphi\}_* = \{q_* p_*^{-1} : H_*(X) \to H_*(Y) \mid (p, q) \subset \varphi\}$$

Two admissible multivalued maps $\varphi, \psi : X \multimap Y$ are called homotopic (written $\varphi \sim \psi$) if there exists an admissible multivalued map $\chi : X \times [0,1]$ such that:

$$\chi(x,0) \subset \varphi(x) \quad \text{and} \quad \chi(x,1) \subset \psi(x) \quad \text{for every } x \in X$$

Theorem 3.3 ([5], Theorem (40.11)). *Let $\varphi : X \multimap Y$ be two admissible multivalued maps. Then $\varphi \sim \psi$ implies that there exists selected pairs $(p,q) \subset \varphi$ and $(\bar{p},\bar{q}) \subset \psi$ such that*

$$q_* p_*^{-1} = \bar{q}_* \bar{p}_*^{-1}$$

Let X be a compact ANR and let $\varphi : X \multimap X$ be an admissible multivalued map. Then, it is well defined the Lefschetz set $\Lambda(\varphi)$ of φ by putting

$$\Lambda(\varphi) = \{\Lambda(q_* p_*^{-1}) = \sum_i (-1)^i \mathrm{trace}_i(q_* p_*^{-1}) \mid (p,q) \subset \varphi\}$$

Theorem 3.4 (Lefschetz fixed point theorem for admissible multivalued mappings). *Let X be a compact ANR and $\varphi : X \multimap X$ be a compact admissible multivalued map. If $\Lambda(\varphi) \neq \{0\}$ then $Fix(\varphi) \neq \emptyset$.*

3.2 Proof of Theorem 3.1

In order to prove Theorem 3.1 we will define an admissible multivalued map $F : S \multimap S$ and we will prove that if $\tilde{s} \in F(\tilde{s})$ then $\tilde{s}$ is an w.l.e. for $p_1, \ldots, p_n$. The conclusion of the proof will follow from the Lefschetz fixed point theorem for admissible multivalued mappings. First, we need the following lemma.

Lemma 1. *Let X be a compact subset of $\mathbb{R}^m$ and let V be an open neighborhood of X in $\mathbb{R}^m$. Then, given a multivalued map $\varphi : X \multimap \mathbb{R}^m$ u.s.c. with compact values, there exists $t_1 > 0$ such that $x + tv \in V$ for all $x \in X$, $v \in \varphi(x)$ and $t \in [0, t_1]$.*

Proof. Let $\varphi : X \multimap \mathbb{R}^m$ be an u.s.c. multivalued map with compact values. If $\varphi(x) = \{0\}$ for every $x \in X$, there is nothing to prove. Suppose

$\varphi(x) \neq \{0\}$ for some $x \in X$. Since X is compact and φ is u.s.c. with compact values, the image $\varphi(X) = \bigcup_{x \in X} \varphi(x)$ is also compact. Then, the real number $u = \max_{v \in \varphi(X)} \{\|v\|\}$ is a finite positive number. For every $x \in X$, there is $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset V$. Since X is compact, we obtain a finite open subcover $\left\{B\left(x_i, \frac{\epsilon_{x_i}}{4}\right)\right\}_{i=1}^{l}$ with

$$X \subset \bigcup_{i=1}^{l} B\left(x_i, \frac{\epsilon_{x_i}}{4}\right) \subset \bigcup_{i=1}^{l} B(x_i, \epsilon_{x_i}) \subset V.$$

Let $\epsilon = \min_{1 \leq i \leq l} \left\{\frac{\epsilon_{x_i}}{4}\right\}$ and $t_1 = \frac{\epsilon}{u}$. Thus, $x + tv \in V$ for all $x \in X$, $v \in \varphi(x)$ and $t \in [0, t_1]$. In fact, given $x \in X$, we have $x \in B\left(x_i, \frac{\epsilon_{x_i}}{4}\right)$ for some x_i. If $v = 0$ the conclusion is obvious. If $v \neq 0$ then, given $t \in [0, t_1]$, we have

$$t \leq t_1 = \frac{\epsilon}{u} \leq \frac{\epsilon_{x_i}}{4u} \leq \frac{\epsilon_{x_i}}{4\|v\|}.$$

It follows that

$$\|x + tv - x_i\| \leq \|x - x_i\| + t\|v\| \leq \frac{\epsilon_{x_i}}{4} + \frac{\epsilon_{x_i}}{4\|v\|}\|v\| = \frac{\epsilon_{x_i}}{2} < \epsilon_{x_i}.$$

Therefore, $x + tv \in B(x_i, \epsilon_{x_i}) \subset V$.

Hence, for all $x \in X$, $v \in \varphi(x)$ and $t \in [0, t_1]$. □

Proof of Theorem 3.1. Since $S_1 \subset \mathbb{R}^{m_1}, \ldots, S_n \subset \mathbb{R}^{m_n}$ are compact ENR's with the p.c.r., the product $S = S_1 \times \cdots \times S_n \subset \mathbb{R}^m$ is also a space with the p.c.r, $m = m_1 + \cdots + m_n$. Thus, let $r : V \to S$ be a convenient retraction.

Let $\varphi : S \multimap \mathbb{R}^m$ be the multivalued map defined by

$$\varphi(s) = \varphi_1(s) \times \cdots \times \varphi_n(s)$$

where $\varphi_i(s) = DSp_i(s_{-i}, s_i)$.

>From Lemma 1, there exists $t_1 > 0$ such that $s + tv \in V$ for all $s \in S$, $t \in [0, t_1]$ and $v \in V(s)$.

Finally, we define $F : S \multimap S$ by

$$F(s) = \{r(s + t_1 v) \mid v \in \varphi(s)\}.$$

As defined, F is a compact admissible multivalued map. Moreover, F is homotopic to the identity map via homotopy $\psi : S \times [0, t_1] \to S$ given by $\psi(s,t) = \{r(s+tv) \mid v \in \varphi(s)\}$. Thus, by Theorema 3.3, there exists a selected pair $(p, q) \subset F$ such that

$$\Lambda(q_* p_*^{-1}) = \Lambda(id_S) = \chi(S) = \chi(S_1) \cdots \chi(S_n).$$

If $\chi(S_i) \neq 0$, $i = 1, \dots, n$, then $\Lambda(F) \neq \{0\}$. It follows, from Theorem 3.4, that F has a fixed point, ie, a point $\tilde{s} \in S$ such that $\tilde{s} \in F(\tilde{s})$. We affirm that a such fixed point $\tilde{s}$ is a w.l.e. for $p_1, \dots, p_n$. In fact, if $\tilde{s} \in F(\tilde{s})$ then $\tilde{s} = r(\tilde{s} + t_1 v)$ for some $v \in \varphi(\tilde{s})$. Since r is a convenient retraction, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - r(\tilde{s} + t_1 v)\| = \|x - \tilde{s}\| < \delta$$

implies that

$$\begin{aligned} \langle \tilde{s} + t_1 v - r(\tilde{s} + t_1 v), x - r(\tilde{s} + t_1 v) \rangle &= t_1 \langle v, x - \tilde{s} \rangle \\ &\leq \frac{t_1 \varepsilon}{2} \|x - \tilde{s}\|. \end{aligned}$$

Moreover, from the definition of φ, we can assume that if $\|\tilde{s} - s\| < \delta$ then

$$p_i(\tilde{s}_1, \dots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \dots, \tilde{s}_n) \leq p_i(\tilde{s}) + \langle v_i, s_i - \tilde{s}_i \rangle + \frac{\varepsilon}{2} \|s_i - \tilde{s}_i\|,$$

$1 \leq i \leq n$. It follows that, if $s \in S$ and $\|s - \tilde{s}\| < \delta$ then

$$p_i(\tilde{s}_1, \dots, \tilde{s}_{i-1}, s_i, \tilde{s}_{i+1}, \dots, \tilde{s}_n) \leq p_i(\tilde{s}) + \varepsilon \|s_i - \tilde{s}_i\|,$$

$1 \leq i \leq n$.

Hence, $\tilde{s}$ is a w.l.e. for $p_1, \dots, p_n$.

□

References

[1] Alós-Ferrer, C., Ania, A.B., *Local equilibria in economic games.* Econom. Lett., 70, no. 2, 165-173 (2001).

[2] Biasi, C., Monis, T.F.M., *Weak local Nash equilibrium.* Topological Methods in Nonlinear Analysis **41**, no. 2, 409-419 (2013).

[3] Biasi, C., Mendes Monis, T. F., *Some coincidence theorems and its applications to existence of local Nash equilibrium.* JP J. Fixed Point Theory Appl. **5**, no. 2, 81-102 (2010).

[4] Eilenberg, S., Montgomery, D., *Fixed point theorems for multi-valued transformations.* Amer. J. Math., 68, 214-222 (1946).

[5] Górniewicz, L., *Topological fixed point theory of multivalued mappings.* Second edition. Topological Fixed Point Theory and Its Applications, 4. Springer, Dordrecht, 2006.

[6] Milnor, J., *A nobel prize for John Nash.* Math. Intelligencer, 17, no. 3, 11-17 (1995).

Carlos Biasi
Instituto de Ciências Matemáticas e de Computação. Universidade de São Paulo.
e-mail: biasi@icmc.usp.br

Thaís Fernanda Mendes Monis
Instituto de Geociências e Ciências Exatas. Univ Estadual Paulista.
email: tfmonis@rc.unesp.br