Збірник праць Ін-ту математики НАН України 2013, том 6, N 6, 190–208

Sebastian Ruszkowski

(Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, Toruń, Poland)

Time scale version of the Ważewski retract method

sebrus@mat.umk.pl

In the paper there are discussed approaches to the Ważewski retract method on time scales. In particular there is presented planar case without a restrictive assumption that the whole boundary of a set of constraints, where we look for solutions, is a set of egress points. One example illustrating the main theorem is presented.

Introduction

In 1947 Tadeusz Ważewski (see [1]) gave a simple but excellent topological principle, now called the Ważewski retract method, which has been used by many authors to prove the existence of solutions of a given differential equation which remain in a prescribed set of constraints. In particular, the method helps to find bounded solutions in several differential problems. It generalizes the direct method of Lyapunov and is based on examining so-called 'egress' and 'strict egress' points on a boundary of the set of constraints. It is worth noting that the set does not need to be an attractor or repellor. It is sufficient to check that the set of egress points, which is usually assumed to be equal to the set of strict egress points, is not a retract or, more generally, strong deformation retract of the whole set. This topological principle became a base and a motivation for a construction of a very well known and useful topological invariant, the Conley index (see, e.g., [2] for a comparison of these two topological tools).

© Sebastian Ruszkowski, 2013

The Ważewski retract method was generalized and adopted to: differential inclusions (see, e.g., [2] or [3] and references therein), difference equations (e.g. [4, 5]) or, recently, dynamic equations on time scales ([6, 7, 8]). This last area of research has been intensively developed since 90's as a unification and generalization of the theory of difference equations and differential equations, and has found applications in many mathematical models in biology and physics, where discrete and continuous dynamics have to be studied simultaneously. Moreover, various impulsive differential problems can be transformed to dynamic equations on time scales.

While several results on dynamic equations on time scales are just simple transformations of continuous or discrete analogs, the ones concerning qualitative theory are not. The results on the Ważewski topological principle for dynamic equations on time scales are still not satisfactory. In fact, the only cases explored enough are the ones where the set of constraints is negatively invariant (see [6, 7]).

When we drop the above simplification, we meet several essential problems. The main of them is to construct a retraction, which has to be a continuous map, from an initial section Ω_{t_0} of the tube of constraints onto the t_0-section E_{t_0} of the set of egress points. We need a deep geometrical study to overcome this problems. The Shöenflies theorem, a convexity and strict convexity play an important role in proofs of non-whole boundary case (see [8]).

The paper is organized as follows. In section 2 we recall some information on the calculus on time scales that will be useful in the sequel. Section 3 shows topological ideas contained in [6] and [7]. Section 4 presents results from [8], where the positive or negative invariance as well as a repulsivity of the set is not assumed anymore. One transparent example is given to illustrate the results.

2 Preliminaries

2.1 Basics of a calculus on time scales

The interested reader can consult [9, 10] to get a complete introduction or to find proofs of statements of this section.

A time scale is any closed subset of the set $\mathbb{R}$ of real numbers and we denote it by $\mathbb{T}$.

Basic functions describing $\mathbb{T}$ are jump operators $\sigma, \rho\colon \mathbb{T} \to \mathbb{T}$ i $\mu\colon \mathbb{T} \to \mathbb{R}$, defined as follows:

- $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (forward jump operator)
- $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ (backward jump operator)
- $\mu(t) = \sigma(t) - t$ (graininess function)

where we assume: $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Proposition 2.1 (Induction Principle)**.** *Let $t_0 \in \mathbb{T}$ and assume that $\{S(t)\colon t \in [t_0, \infty) \cap \mathbb{T}\}$ is a family of statements satisfying:*

- *The statemnt $S(t_0)$ is true.*
- *If $t \in [t_0, \infty) \cap \mathbb{T}$ is right-scattered and $S(t)$ is true for all $s \in [t_0, t) \cap \mathbb{T}$, then $S(\sigma(t))$ is also true.*
- *If $t \in [t_0, \infty) \cap \mathbb{T}$ is right-dense and $S(t)$ is true, then there is a neighborhood U of t such that $S(s)$ is true for all $s \in U \cap (t, \infty) \cap \mathbb{T}$.*
- *If $t \in (t_0, \infty) \cap \mathbb{T}$ is left-dense and $S(s)$ is true for all $s \in [t_0, t) \cap \mathbb{T}$, then $S(t)$ is true.*

Then $S(t)$ is true for all $t \in [t_0, \infty) \cap \mathbb{T}$.

Definition 2.2. Δ-derivative of a function $f : \mathbb{T} \to \mathbb{X}$ in a point t, where $\mathbb{X}$ is a linear normed space, is the point $f^{\Delta}(t) \in \mathbb{X}$ (if it exists) such that:

$$\forall_{\varepsilon>0}\ \exists_{\delta>0}\ \forall_{s\in B(t,\delta)\cap\mathbb{T}}\ \ \|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\| \le \varepsilon|\sigma(t) - s|$$

Proposition 2.3. *If a function f is continuous in t and:*

- $t = \sigma(t)$*, then:* $f^{\Delta}(t) = \lim_{s\to t} \frac{f(t)-f(s)}{t-s}$
- $t \neq \sigma(t)$*, then:* $f^{\Delta}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)}$
- *in general, for $t \in \mathbb{T}^{\kappa}$ we have*

$$f^{\Delta}(t) = \lim_{s\to t^{\mathbb{T}}} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}$$

where $\mathbb{T}^{\kappa}$ is the set $\mathbb{T}$ without the point $\max \mathbb{T}$ *if this point exists and is isolated.*

2.2 Δ-differential equations

Definition 2.4. By a local solution of a system of equations:

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \tag{1}$$

we will mean a continuous function $x\colon \mathbb{T} \cap (a, b) \to \mathbb{X}$ such that $a < \rho(t_0)$, $b > \sigma(t_0)$, $x(t_0) = x_0$ and for all $t \in \mathbb{T}^{\kappa} \cap (a, b)$ equation $x^{\Delta}(t) = f(t, x(t))$ is fulfilled.

Definition 2.5. A solution x_2 is an extension of a solution x_1, if x_1 and x_2 are local solutions of the same system of equations, $\mathrm{Dom}(x_2) \supsetneq \mathrm{Dom}(x_1)$ and $x_2|_{\mathrm{Dom}(x_1)} = x_1$.

If we cannot extend a local solution, then we call it a global solution.

Proposition 2.6. *If for all $t_0 \in \mathbb{T}^{\kappa}$ and $x_0 \in \mathbb{X}$ there exists a unique local solution of system (1), then for all $t_0 \in \mathbb{T}^{\kappa}$ and $x_0 \in \mathbb{X}$ there exists a unique global solution of the same equation $x : \mathbb{T} \cap (a, b) \to \mathbb{X}$, where $\mu(a) = 0$ or $a = -\infty$, and $b - \rho(b) = 0$ or $b = \infty$.*

In analogy to standard local processes on $\mathbb{R}$ we can define a local Δ-process. We have then a formal definition:

Definition 2.7. A continuous function $\Pi\colon M \to \mathbb{X}$ (where $M \subset \mathbb{X} \times \mathbb{T}^2$) is a local Δ-process if:

P1 $\forall_{x \in \mathbb{X}, t \in \mathbb{T}} \exists_{\alpha < t < \beta}\ (\mu(\alpha) = 0 \vee \alpha = -\infty) \wedge (\beta - \rho(\beta) = 0 \vee \beta = \infty) \wedge$
$\wedge \{s \in \mathbb{T}\ ;\ (x, t, s) \in M\} = (\alpha, \beta) \cap \mathbb{T}$,

P2 $\forall_{x \in \mathbb{X}, t \in \mathbb{T}}\ \Pi(x, t, t) = x$,

P3 $\forall_{(x,t,s),(x,t,r) \in M}\ (\Pi(x, t, s), s, r) \in M \wedge \Pi(\Pi(x, t, s), s, r) = \Pi(x, t, r)$.

Definition 2.8. We say that an equation $x^{\Delta}(t) = f(t, x(t))$ generates a local Δ-process Π, if for all $x_0 \in \mathbb{X}$ and $t_0 \in \mathbb{T}$ a function $\Pi(x_0, t_0, \cdot)$ is a global and unique solution of (1) and Π is a local Δ-process.

Analogously as processes on $\mathbb{R}$, a Δ-process induce homeomorphisms along trajectories:

There is also possibility of understanding a solution as a function that fulfills the equation $x^{\Delta}(t) = f(t, x(t))$ $\mathbb{T}$-almost everywhere (that concept had been introduced in [11]). If we accept this definition the next part of this paper needs only nonsignificant changes.

Proposition 2.9. *If an equation $x^{\Delta}(t) = f(t, x(t))$ generates a local Δ-process Π, and if all solutions of the problem*

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)) \\ x(t_0) \in A \end{cases}$$

exist in time t_1, then $\Pi(\cdot, t_0, t_1)|_A$ is a homeomorphism between A and its image.

Proof. Π is continuous so $\Pi(\cdot, t_0, t_1)$ and $\Pi(\cdot, t_1, t_0)$ are continuous on theirs domains which implies what was to prove. □

We will need the preservation of orientation by Π. Below we show a simple theorem which gives an example of a class of functions implying that property.

Definition 2.10. A function $f : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is rd-continuous if it is continuous in all $t \in \mathbb{T}$ such that $\mu(t) = 0$, that is in so-called right dense points (this justifies "rd" in the name).

Proposition 2.11. *Let $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ be rd-continuous and Lipschitz continuous (with Lipschitz constant $L(t)$) with respect to the second variable. If for all $t \in \mathbb{T}$ inequality $L(t)\mu(t) < 1$ is fulfilled, then an equation $x^{\Delta}(t) = f(t, x(t))$ generates a local Δ-process Π and for all $t \in \mathbb{T}$ we have that function $\Pi(\cdot, t, \sigma(t))$ preserves an orientation of $\mathbb{R}^n$.*

Proof. An equation $x^{\Delta}(t) = f(t, x(t))$ has a global and unique solution (see [9, p.322, 324]) with a continuous dependence on the initial conditions, so this equation generates a local Δ-process Π. Moreover:

$$\Pi(x, t, \sigma(t)) - \Pi(0, t, \sigma(t)) = x + \mu(t) f(t, x) - (0 + \mu(t) f(t, 0)) =$$

$$= x + \mu(t)(f(t, x) - f(t, 0))$$

so, by $L(t)\mu(t) < 1$, for $x \neq 0$ we have:

$$\langle \Pi(x, t, \sigma(t)) - \Pi(0, t, \sigma(t)), x \rangle > 0$$

which means that vectors $\Pi(x, t, \sigma(t)) - \Pi(0, t, \sigma(t))$ and x are in the same halfspace, so $\Pi(\cdot, t, \sigma(t))$ preserves an orientation of $\mathbb{R}^n$. □

3 Ważewski method for the whole boundary egress set

There are shown two approaches to basic Ważewski Theorem in this chapter, which means: the case of set Ω, for which all trajectories starting from the boundary immediately leaves that set. In this section we assume, that $\mathbb{X} = \mathbb{R}^n$.

3.1 Approach 1.

If local Δ-process Π generated by Δ-equation is not well defined in (x, t_0, t_1), it means, that solution starting in (x, t_0) reaches to boundary of $\mathbb{R}^n$ - infinity (one point compactification of $\mathbb{R}^n$). This observation leads to convenient notation for points (x, t_0, t_1) outside of the domain of local Δ-process Π: $\Pi(x, t_0, t_1) := \infty$.

We will use a function of positively closest point of change of interval charakter of $\mathbb{T}$.

Definition 3.1. Essential forward jump operator is a function $\mathrm{ess}\sigma : \mathbb{T} \to \mathbb{T} \cup \sup \mathbb{T}$ with formula:

$$\mathrm{ess}\sigma(t) := \inf\{s \in \mathbb{T}\ ;\ s > t \wedge (\mu(s) > 0 \vee \mu(t) > 0)\}.$$

Let $\tilde{\Omega}$ be closed subset of $\mathbb{R} \times \mathbb{R}^n$, such that for each $r \in \mathbb{R}$ the set $\tilde{\Omega}_r := \{x \in \mathbb{R}^n\ ;\ (r, x) \in \tilde{\Omega}\}$ is nonempty and bounded, $\partial(\tilde{\Omega}_r)$ is not a retrakt of $\tilde{\Omega}_r$ and $\{r\} \times \partial(\tilde{\Omega}_r)$ is a retrakt of $\partial(\tilde{\Omega})$. We will use curtailment of $\tilde{\Omega}$ to the time scale $\mathbb{T}$:

$$\Omega := \bigcup_{t \in \mathbb{T}} \{t\} \times \tilde{\Omega}_t.$$

Theorem 3.2. *For above set Ω and equation $x^{\Delta}(t) = f(t, x(t))$, which generates local Δ-process Π, if for all $t \in \mathbb{T}$ and for all $s \in (t, \mathrm{ess}\sigma(t)]$ we have $\Pi(cl(\Omega^c)_t, t, s) \subset (\Omega_s)^c$() then for each $t_0 \in \mathbb{T}$ there exists point $x_0 \in \Omega_{t_0}$, such that the solution starting from (t_0, x_0) remain in Ω for every $t \in \mathbb{T}$ bigger than t_0.*

Similar theorem is proved in [7], but they are focused on simple time scales (with values of grainies function equal 0 or bigger than ϵ), and then using inverse systems and analitic means, they obtain general case.

This condition means that starting from the outside of set Ω there is no trajektory such that enters set Ω up to time of essential forward jump of starting time, which means that the whole boundary of Ω is a set of egress points in a specyfic sens.

Proof. We will prove by induction principle for time scales (Proposition 2.1), that $\Pi(\Omega_t, t, s) \subset \Omega_s$ for $s, t \in \mathbb{T}$ where $s \leqslant t$.

Obviously $\Pi(\Omega_t, t, t) = \Omega_t$.

We have $\mathrm{ess}\sigma(t) = \sigma(t)$ for right-scattered points, therefor: $\Pi(\mathrm{cl}(\Omega^c)_t, t, \sigma(t)) \in (\Omega_{\sigma(t)})^c$, so $\Pi(\Omega_{\sigma(t)}, \sigma(t), t) \subset \Omega_t$.

For points t in compact interval in time scale, that are not right boundery of that interval we have that $\mathrm{ess}\sigma(t)$ is right boundery of that interval. In particular we have $\mathrm{ess}\sigma(t) - t > \varepsilon > 0$, so for $s \in (t, t+\varepsilon)$ we have $\Pi(\Omega_t, t, s) \subset \Omega_s$.

For other right-dense points t we know that ther exists sequence $(t_n) \subset \mathbb{T}$ diminishing to t such that $\mu(t_n) > 0$ for all n, for which we have $\Pi(\Omega_{\sigma(t_n)}, \sigma(t_n), t_n) \subset \Omega_{t_n}$, so by continuity of Π we find $\varepsilon > 0$ such that for $s \in (t, t+\varepsilon) \cap \mathbb{T}$ we have $\Pi(\Omega_t, t, s) \subset \Omega_s$.

For left-dense points we obtain needed property also by continuity.

By induction we have $\Pi(\Omega_t, t, s) \subset \Omega_s$ for $s, t \in \mathbb{T}$ where $s \leqslant t$.

Let us fix $t_0 \in \mathbb{T}$. We can choose sequence $(t_n)_{n=0..\infty} \subset \mathbb{T}$ increasing to $\sup \mathbb{T}$ and define $(\Omega^n)_{n=1..\infty}$ by:

$$\Omega^n := \Pi(\Omega_{t_n}, t_n, t_0).$$

We know, that:

$$\Omega^{n+1} = \Pi(\Pi(\Omega_{t_{n+1}}, t_{n+1}, t_n), t_n, t_0) \subset \Omega^n,$$

therefore $(\Omega^n)_{n=1..\infty}$ is descending family of compact sets. Intersection of descending family of nonemty compact sets is nonempty and we choose x_0 in that intersection. We now know, that trajektory starting in (t_0, x_0) is in Ω up to any time t_n, and $t_n \to \sup \mathbb{T}$, so this is the searched trajektory.

□

3.2 Approach 2.

Let $b_i, c_i : \mathbb{T} \to \mathbb{R}$ (for $i = 1..n$) be Δ-differentiable functions where $b_i < c_i$. We define Ω using that functions:

$$\Omega := \{(t, x) \in \mathbb{T} \times \mathbb{R}^n \; ; \; b_i(t) \leqslant x_i \leqslant c_i(t) \text{ for all } i\}$$

and we will use notoation:

$$\partial_{\mathbb{T}}\Omega := \{(t,x) \in \mathbb{T}\times\mathbb{R}^n\ ;\ b_i(t) \leqslant x_i \leqslant c_i(t) \text{ for all } i$$
$$\text{wherein at least one inequality is equality}\}.$$

All points $p \in \partial_{\mathbb{T}}\Omega$ can be presented in one of the following ways:

$$p = (t, x_1, ..., x_{i-1}, b_i(t), x_{i+1}, ..., x_n) \in \Omega_b^i$$

or

$$p = (t, x_1, ..., x_{i-1}, c_i(t), x_{i+1}, ..., x_n) \in \Omega_c^i.$$

Theorem 3.3. *Let $b_i, c_i : \mathbb{T} \to \mathbb{R}$ be Δ-differentiable and $f : \mathbb{T}\times\mathbb{R}^n \to \mathbb{R}^n$ generates local Δ-process Π. If for all $(t,x) \in \Omega_b^i$ we have $f(t,x) < b_i^{\Delta}(t)$ and for all $(t,x) \in \Omega_c^i$ we have $f(t,x) > c_i^{\Delta}(t)$ then for each $t_0 \in \mathbb{T}$ there exists $x_0 \in \Omega_{t_0}$, such that solution starting in (t_0, x_0) remains in Ω for every $t \in \mathbb{T}$ bigger than t_0.*

Proof. (ad absurdum)

Let us notice, that for $(t,x) \in \Omega_b^i$, where $\mu(t) > 0$, we have $\Pi(x,t,\sigma(t)) = x + \int_t^{\sigma(t)} f(\tau,x)\Delta\tau < x + \int_t^{\sigma(t)} b_i^{\Delta}(\tau)\Delta\tau = b_i(\sigma(t))$, and similarly for $\mu(t) = 0$ we have $\Pi(x,t,t+\epsilon) = x + \int_t^{t+\epsilon} f(\tau,x)\Delta\tau < x + \int_t^{t+\epsilon} b_i^{\Delta}(\tau)\Delta\tau = b_i(t+\epsilon)$, which means that trajectories starting in Ω_b^i immiedietly leaves set Ω. By analogy, trajectories starting in Ω_c^i also immiedietly leaves set Ω.

Let us fix $t_0 \in \mathbb{T}$. We extend linearly Ω to continuous tube on $[t_0, \sup\mathbb{T}] \cap \mathbb{R}$:

$\Omega^* := \Omega \cup \{(t,x) \in ([t_0, \sup\mathbb{T}] \cap \mathbb{R} \setminus \mathbb{T}) \times \mathbb{R}^n\ ;$

$$b_i(t_a) + (b_i(t_b) - b_i(t_a))\frac{t-t_a}{t_b-t_a} \leqslant x_i \leqslant c_i(t_a) + (c_i(t_b) - c_i(t_a))\frac{t-t_a}{t_b-t_a}$$
$$\text{for all } i\}$$

where $t_a, t_b \in \mathbb{T}$ are such that $t_a < t < t_b$ and $(t_a, t_b) \cap \mathbb{T} = \emptyset$.

Naturally, for above t_a and t_b we know that set $\Omega^*_{[t_a,t_b]}$ is convex.

Note that $r : \partial\Omega^* \to \{t_0\} \times \partial\Omega_{t_0}$

$$r(t,x) := (t_0, (b_i(t_0) + \frac{c_i(t_0) - b_i(t_0)}{c_i(t) - b_i(t)}(x_i - b_i(t)))_{i=1..n})$$

This is Theorem from [6], with assumptions given in the the language of Δ-processes.

is a retraction.

Now we will find continuous function from the set Ω_{t_0} to the boundary of Ω^*.

Negation of the thesis means that for all $x \in \Omega_{t_0}$ we have finite time of exit
$t_e(x) := \sup\{t \in \mathbb{T} \; ; \; \forall_{s\in\mathbb{T}\cap[t_0,t]}(t, \Pi(x, t_0, t)) \in \Omega\} < \sup \mathbb{T}$.

If $\mu(t_e(x)) = 0$, then $(t_e(x), \Pi(x, t_0, t_e(x)))$ is in boundary of Ω^*.

If $\mu(t_e(x)) \neq 0$, then $(t_e(x), \Pi(x, t_0, t_e(x))) \in \Omega$ and $(\sigma(t_e(x)), \Pi(, t_0, \sigma(t_e(x)))) \notin \Omega$ and by convexity of $\Omega^*_{[t_e(x),\sigma(t_e(x))]}$ we obtain unique intersection of $\Omega^*_{[t_e(x),\sigma(t_e(x))]}$ with interval connecting points $(t_e(x), \Pi(x, t_0, t_e(x)))$ and $(\sigma(t_e(x)), \Pi(x, t_0, \sigma(t_e(x))))$. Denote this point by $(t_e^*(x), x_e^*)$.

Therefore, we can define function $p : \Omega_{t_0} \to \partial\Omega^*$:

$$p(x) := \begin{cases} (t_e(x), \Pi(x, t_0, t_e(x))), & \text{gdy } \mu(t_e(x)) = 0 \\ (t_e^*(x), x_e^*), & \text{gdy } \mu(t_e(x)) > 0. \end{cases}$$

By continuities of Π and tube Ω^* we have continuous dependence $(t_e^*(x), x_e^*)$ and $(t_e(x), \Pi(x, t_0, t_e(x)))$ in respect to x. It is enough to show continuous dependence beetwen $(t_e^*(x), x_e^*)$ and $(t_e(x), \Pi(x, t_0, t_e(x)))$.

For point x_0 such that $t_e(x_0)$ is left-dense and right-scatered and $(t_e(x_0), \Pi(x_0, t_0, t_e(x_0))) \in \Omega$ we have $p(x) \to p(x_0)$ for $x \to x_0$ with the time of exit $t_e(x) < t_e(x_0)$, and $p(x) \to p(x_0)$ for $x \to x_0$ with the time of exit $t_e(x) = t_e(x_0)$. By continuity of Π we can choose small enough neighborhood of x_0, which do not contains another points, so p is continuous in x_0.

By analogy, for left-scattered and rigth-dense points $t_e(x_0)$ we obtain continuity of p in such points. Therefore p is continuous.

Note that function $R : \{t_0\} \times \Omega_{t_0} \to \{t_0\} \times \partial\Omega_{t_0}$

$$R(t_0, x) := r(p(x))$$

is a composition of continuous functions, so it is a retraction, which is in contradiction with the construction of set Ω. $\square$

4 Ważewski method for non-whole boundary egress set

In this section there are presented results from [8].

4.1 Notation

Let $B(x,r)$ denote an open ball centered in $x \in \mathbb{R}^2$ and with a radius r, $D(x,r) = \operatorname{cl} B(x,r)$, $S(x,r) = \partial B(x,r)$ and $S^1 := S(0,1)$.

Proposition 4.1 (Shöenflies theorem)**.** *Any homeomorphism $h : S^1 \to h(S^1) \subset \mathbb{R}^2$ can be extended to a homeomorphism $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$.*

In particular, for any homeomorphism $h : S^1 \to h(S^1) \subset \mathbb{R}^2$ there exists a homeomorphism $\hat{h} : D(0,1) \to \hat{h}(D(0,1)) \subset \mathbb{R}^2$ such that the set $\hat{h}(S^1)$ is a boundary of $\hat{h}(D(0,1))$ and the equality $h(x) = \hat{h}(x)$ holds for all $x \in S^1$.

Let $A \subset \mathbb{T} \times \mathbb{R}^2$. Then we define:

$$A_t := \{x \in \mathbb{R}^2 \ ; \ (t,x) \in A\}.$$

Let $\Theta : \mathbb{T} \times S^1 \to \mathbb{R}^2$ be a continuous function such that:

- $\Theta_t : S^1 \to \Theta_t(S^1) \subset \mathbb{R}^2$, where $\Theta_t(x) = \Theta(t,x)$, is a homeomorphism,
- $\Theta(t,s) = \Theta(\sigma(t), s)$.

For all $t \in \mathbb{T}$ let Ω_t be a closure of a bounded open set surrounded by the curve $\Theta(t, S^1)$ and

$$\Omega := \bigcup_{t \in \mathbb{T}} \{t\} \times \Omega_t.$$

For such construction we will say that Ω *is Θ-bounded.* In particular Ω can be a constant tube $\Omega = \mathbb{T} \times \Omega_0$, where Ω_0 is homeomorphic to $D(0,1)$.

We consider the following parts of the set Ω:

$$\partial_{\mathbb{T}}\Omega := \Theta(\mathbb{T} \times S^1) = \bigcup_{t \in \mathbb{T}} \{t\} \times \partial(\Omega_t)$$

$$\partial_{\mathbb{T}}\Omega^+ := \bigcup_{t \in \mathbb{T}} \{t\} \times \operatorname{cl}\{x \in \mathbb{R}^2 \ ; \ (t,x) \in \partial_{\mathbb{T}}\Omega \wedge \exists_{r>0} \forall_{y \in B(x,r) \cap \Omega_t} \forall_{\lambda \in (0,1)} \ \lambda x + $$
$$+ (1-\lambda) y \in \operatorname{int} \Omega\}$$

It is a well kown property of planes, which is presented for example in [12, pp. 68,72].

In other words, $(\partial_{\mathbb{T}}\Omega^{+})_t$ is a closure of the set of points in $\partial(\Omega_t)$ that have strictly convex neighborhoods in Ω_t.

For any maps $f:\mathbb{T}\times\mathbb{R}^2\to\mathbb{R}^2$ and $g:\mathbb{R}^2\to\mathbb{R}^2$ we define where it makes sense:

$f_t(x) := f(t,x)$,
$\Phi_g(\cdot,\cdot)$, a local flow generated by the equation $y' = g(y)$
$w_g(x)$, a duration of a solution in a local flow Φ_g started in x

We focus our attention at the following subsets of Ω:

- Set of *egress points*:

$$\begin{aligned} E :=\{&(t,x)\in\partial_{\mathbb{T}}\Omega\ ;\ y' = f_t(y) \text{ generates a local flow } \Phi_{f_t} \\ &\text{and } \Phi_{f_t}(x,(0,s]) \not\subset \Omega_t \text{ for any } s\in(0,w_{f_t}(x))\}\end{aligned}$$

- Set of *escape points*:

$$Es := \{(t,x)\in\Omega\ ;\ \mu(t)\neq 0 \text{ and } x+\mu(t)f(t,x)\notin \mathrm{int}\Omega_{\sigma(t)}\}$$

4.2 Theorems

Now we will prove the main theorem of the paper.

Theorem 4.2. *Let $f:\mathbb{T}\times\mathbb{R}^2\to\mathbb{R}^2$ be a map such that:*

(H0) *equation $x^{\Delta}(t) = f(t,x(t))$ generates a local Δ-process Π,*

(H1) *for all $t\in\mathbb{T}$ a function $\Pi(\cdot,t,\sigma(t))$ preserves an orientation of $\mathbb{R}^2$,*

(H2) *Ω is Θ-bounded (see section 4.1),*

(H3) *there exists a closed set $W\subsetneq S^1$ such that W is not a retract of $D(0,1)$ and $\Theta(\mathbb{T}\times W) = E$,*

(H4) *if $\mu(t)\neq 0$, then*

(H4a) $\Pi(E_t,t,\sigma(t))\cap\Omega_{\sigma(t)} = \emptyset$,

(H4b) $\Pi(int\Omega_t,t,\sigma(t))\cap\partial_{\mathbb{T}}\Omega_{\sigma(t)}\subset E_{\sigma(t)}$.

Then, for all $t_0\in\mathbb{T}$, there exists a point $x_0\in\Omega_{t_0}$ such that the solution starting in (t_0,x_0) remains in Ω for all $t\in\mathbb{T}$ bigger than t_0.

Proof. (ad absurdum)

Let us fix for a while point $t \in \mathbb{T}$ such that $\mu(t) \neq 0$.

By assumption (H4a), we know that $E_t \subset Es_t$.

By definition of Es we also know that $\Pi(E_{\sigma(t)}, \sigma(t), t) \cap \Omega_t \subset \partial(Es_t)$.

By assumptions (H2) and (H3) we know that $\Omega_t = \Omega_{\sigma(t)}$ and $E_t = E_{\sigma(t)}$. This allows us to define the following continuous tube:

$$\tilde{E} = \{(s, x) \in \mathbb{R} \times \mathbb{R}^2 \ ; \ (sup\{t \in \mathbb{T} \ ; \ t < s\}, x) \in E\}.$$

In other words we fill the interstices caused by a time scale.

We will construct a continuous function $w_t : Es_t \to [t, \sigma(t)] \times E_t \subset \tilde{E}$ such that:

w1 $\forall_{x \in E_t} \ w_t(x) = (t, x)$

w2 $\forall_{x \in \Pi(E_{\sigma(t)}, \sigma(t), t) \cap \Omega_t} \ w_t(x) = (\sigma(t), \Pi(x, t, \sigma(t)))$

which we will use to construct a continuous function from Ω_{t_0} to $\tilde{E}$ and by that, a retraction from $D(0, 1)$ to W.

Let I be the set of indices of connected components E_t^i of $\Pi(E_t, t, \sigma(t))$ By assumption (H4a) we know that each set E_t^i is contained in a corresponding connected component γ_i of $\Pi(\partial_{\mathbb{T}}\Omega_t, t, \sigma(t)) \setminus \partial_{\mathbb{T}}\Omega_t$. The curve γ_i cuts from $\mathrm{cl}(\Omega_t^c)$ a closed bounded connected set denoted by Es_t^i.

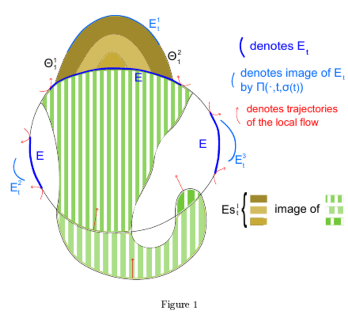

Figure 1

By assumption (H3) we know that $E_t \neq \partial_{\mathbb{T}}\Omega_t$ so a boundary of Es_t^i is a closed curve and a sum of four curves: θ_i^1, E_t^i, θ_i^2 and $\partial_{\mathbb{T}}\Omega \cap Es_t^i$ (each of them homeomorphic to line segments), so it is homeomorphic to S^1 (see Figure 1). By assumption (H1) the sets $\partial_{\mathbb{T}}\Omega_t$ and $\Pi(\partial_{\mathbb{T}}\Omega_t, t, \sigma(t))$

have the same orientation, and therefore E^i_t and $\partial_{\mathbb{T}}\Omega \cap Es^i_t$ have an opposite orientation on the boundary of Es^i_t, so we can parameterize that boundary to obtain a homeomorphism $h_i : \partial T_i \to \partial(Es^i_t)$ such that:

- T_i is a trapezoid with vertices $(0,0)$, $(1,0)$, $(a_i,1)$, $(b_i,1)$ where $[a_i,b_i] \subset [0,1]$, ,
- $h_i([0,1],0) = E^i_t$,
- $h_i([a_i,b_i],1) = \partial_{\mathbb{T}}\Omega_t \cap Es^i_t$,
- $\forall_{x\in[a_i,b_i]} h_i(x,0) = \Pi(h_i(x,1),t,\sigma(t))$.

By the Shöenflies theorem we can extend h_i to a homeomorphism $\hat{h_i} : T_i \to Es^i_t$. If $\Pi(Es_t,t,\sigma(t)) \setminus \bigcup_{i\in I} Es^i_t \neq \emptyset$, then with other con-

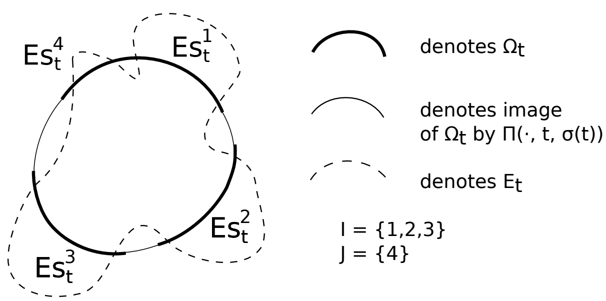

Figure 2

nected components (indexed with elements of some set J) we make similar sets Es^j_t, each of them bounded by a part of $\partial(\Omega_t)$ and a part of $\Pi(\partial(\Omega_t),t,\sigma(t))$, so bounded by a curve homeomorphic to S^1 (see Figure 2). Then we have a homeomorphism $h_j : \partial T_j \to \partial(Es^j_t)$ such that:

- T_j is a triangle with vertices $(1/2,1/2)$, $(0,1)$, $(1,1)$,
- $h_j([0,1],1) = \partial_{\mathbb{T}}\Omega_t \cap Es^j_t$.

and again by the Shöenflies theorem we can extend h_j to a homeomorphism $\hat{h_j} : T_j \to Es^j_t$.

With that construction θ^1_i and θ^2_i are the images of the side edges of T_i.
In that construction point $h_j(1/2,1/2)$ is free to choose.

Now we know that $\Pi(Es_t, t, \sigma(t)) \subset \bigcup_{i\in I} Es_t^i \cup \bigcup_{j\in J} Es_t^j$ which is the sum of disjoint sets, therefore we can define $w_t : Es_t \to [t, \sigma(t)] \times E_t \subset \tilde{E}$,

$$w_t(x) := \begin{cases} (t + \mu(t)p_2(\cdot), \Pi(\hat{h_i}(p_1(\cdot), 0), \sigma(t), t))(\hat{h_i}^{-1}(\Pi(x, t, \sigma(t)))), \\ \hfill \Pi(x, t, \sigma(t)) \in Es_t^i \ \ (i \in I) \\ (t + \mu(t)p_2(\cdot), \Pi(\hat{h_i}(p_1(\cdot), 1), \sigma(t), t))(\hat{h_i}^{-1}(\Pi(x, t, \sigma(t)))), \\ \hfill \Pi(x, t, \sigma(t)) \in Es_t^j \ \ (j \in J) \\ (\sigma(t), \Pi(x, t, \sigma(t))) \\ \hfill \Pi(x, t, \sigma(t)) \in E_{\sigma(t)} \end{cases}$$

where p_1 and p_2 are projections respectively onto the first and second variables. By construction it is a continuous function.

Notice that for $x \in E_t$ there exists a unique $i \in I$ such that $\Pi(x, t, \sigma(t)) \in E_t^i$, and consequently $y := \hat{h}_i^{-1}(\Pi(x, t, \sigma(t))) \in [0, 1] \times \{0\}$. Since $p_1(y) = y$ and $p_2(y) = 0$, we obtain that $w_t(x) = (t, \Pi(\hat{h}_i(y), \sigma(t), t)) = (t, x)$. Hence property **w1** is satisfied.

Moreover, if $x \in \Pi(E_{\sigma(t)}, \sigma(t), t) \cap \Omega_t$, then $\Pi(x, t, \sigma(t)) \in E_{\sigma(t)}$, so also property **w2** is fulfilled.

Falsity of thesis means that for every $x \in \Omega_{t_0}$ we have:

$t_e(x) := \sup\{t \in \mathbb{T} \, ; \, \forall_{s\in\mathbb{T}\cup[t_0,t]} (t, \Pi(x, t_0, t)) \in \Omega\} < \sup \mathbb{T}$.

If $\mu(t_e(x)) = 0$, then $(t_e(x), \Pi(x, t_0, t_e(x)))$ is already in $E \subset \tilde{E}$.

If $\mu(t_e(x)) \neq 0$, then $(t_e(x), \Pi(x, t_0, t_e(x))) \in \Omega$ and $(\sigma(t_e(x)), \Pi(x, t_0, \sigma(t_e(x))) \notin \Omega$. So we can use the function $w_{t_e(x)}$ to it.

Therefore, we can define $r : \Omega_{t_0} \to \tilde{E}$,

$$r(x) := \begin{cases} (t_e(x), \Pi(x, t_0, t_e(x))), & \text{if } \mu(t_e(x)) = 0, \\ w_{t_e(x)}(\Pi(x, t_0, t_e(x))), & \text{if } \mu(t_e(x)) \neq 0. \end{cases}$$

Take any point x such that $\mu(t_e(x)) = 0$.

Then for each $\varepsilon > 0$ there exists $\tau \in \mathbb{T}$ such that $0 < \tau - t_0 < \varepsilon$ and $\Pi(x, t_0, \tau) \notin \Omega_\tau$ so, by the continuity of Π, for each $\varepsilon > 0$ there exist $\delta > 0$ and $\tau \in \mathbb{T}$ such that $0 < \tau - t_0 < \varepsilon$ and $\Pi(B(x, \delta), t_0, \tau) \cap \Omega_\tau = \emptyset$.

Therefore

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{y\in B(x,\delta)} \ t_e(y) < t_e(x) + \varepsilon.$$

Similarly we show that for each point x such that $t_e(x) - \rho(t_e(x)) = 0$ we have

$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{y\in B(x,\delta)} \ t_e(y) > t_e(x) - \varepsilon.$

Using a continuity of Π we get a continuity of r in all x such that $t_e(x)$ is a dense point of $\mathbb{T}$. Furthermore, properties **w1** and **w2** guarantee a continuity of r in every point x such that $\mu(t_e(x)) \neq 0$ or $t_e(x) - \rho(t_e(x)) \neq 0$.

Hence r is continuous for all points in Ω_{t_0}.

By the Shöenflies theorem we can extend Θ_t to $\hat{\Theta}_t : D(0,1) \to \Omega_t$ for every $t \in \mathbb{T}$. Using this we define a map $R : D(0,1) \to W$:

$$R(y) := \Theta^{-1}_{t_e(\hat{\Theta}_{t_0}(y))} \circ p_2 \circ r(\hat{\Theta}_{t_0}(y)).$$

For all $y \in W$ we have $R(y) = \Theta^{-1}_{t_0} p_2 r(\Theta_{t_0}(y)) = \Theta^{-1}_{t_0}(\Pi(\Theta_{t_0}(y), t_0, t_0)) = y$ and, by the continuity of r, we get that R is a retraction, what contradicts assumption (H3).

□

The geometric assumption (H4') in the next theorem corresponds to the assumption (H4) and may occure to be easier to check.

Theorem 4.3. *Assume that:*

(H0) *equation $x^{\Delta}(t) = f(t, x(t))$ generates a local Δ-process Π,*

(H1) *for all $t \in \mathbb{T}$ a function $\Pi(\cdot, t, \sigma(t))$ preserves an orientation of $\mathbb{R}^2$,*

(H2) *Ω is Θ-bounded,*

(H3') *there exists a closed set $W \subsetneq S^1$ such that W is not a retract of $D(0,1)$ and $\Theta(\mathbb{T} \times W) = \partial_{\mathbb{T}}\Omega^+ = E$,*

(H4') *if $\mu(t) \neq 0$, then $\Omega_t \subset x + T_{\Omega_t}(x)$ for all $x \in int_{\partial_{\mathbb{T}}\Omega_t} E_t$, where $T_{\Omega_t}(x)$ is a Bouligand tangent cone of the set $\Omega_t \in \mathbb{R}^2$ in a point x ($T_K(x) = \{v \in \mathbb{X}\ ;\ \liminf_{h \to 0^+} \frac{d(x+hv, K)}{h} = 0\}$).*

Then, for all $t_0 \in \mathbb{T}$, there exists a point $x_0 \in \Omega_{t_0}$ such that a solution starting in (t_0, x_0) remains in Ω for all $t \in \mathbb{T}$ bigger than t_0.

Proof. To use Theorem 4.2 it is sufficient to show that, if $\mu(t) \neq 0$, then $\Pi(E_t, t, \sigma(t)) \cap \Omega_{\sigma(t)} = \emptyset$ and $\Pi(\mathrm{int}\Omega_t, t, \sigma(t)) \cap \partial_{\mathbb{T}}\Omega_{\sigma(t)} \subset E_{\sigma(t)}$.

Let us fix $t \in \mathbb{T}$ such that $\mu(t) \neq 0$.

We have that $\Pi(x, t, \sigma(t)) = x + \mu(t) f_t(x)$ and, for all $x \in E_t$, vectors $f_t(x)$ are directed outside the set Ω_t so a local strict convexity of points in E_t (assumption (H3')) guarantees that each connected component $E_{t,i}$

of E_t has no common points with $\Pi(E_{t,i}, t, \sigma(t))$. Moreover, assumption (H4') ensures that the set $\Pi(E_{t,i}, t, \sigma(t))$ is outside of the rest of $\Omega_{\sigma(t)}$, so the first part is fulfilled.

A connected component $I_{t,i}$ of $\partial_{\mathbb{T}}\Omega_t \setminus E_t$ has only points without strictly convex neighborhoods in Ω_t. All of that points are not egress points in a local flow, so $f_t(x)$ are directed inside the set Ω_t. If there were $y \in I_{t,i} \cap \Pi(I_{t,i}, \sigma(t), t)$ and $[y, \Pi(y, t, \sigma(t))] \not\subset \partial_{\mathbb{T}}\Omega_t$, then $I_{t,i}$ would be a part of a spiral shaped curve which end is a beginning of a part of E_t, which would contradict assumption (H4'). Therefore there exists a small enough neighborhood $O_{t,i}$ of $I_{t,i}$ such that an image of $O_{t,i} \cap \mathrm{int}\Omega_t$ has no common points with $I_{t,i}$.

$\partial_{\mathbb{T}}\Omega_t$ is homeomorphic to $\Pi(\partial_{\mathbb{T}}\Omega_t, t, \sigma(t))$ so, if $E_{t,i}$ borders on $I_{t,j}$, then their images have to border as well. Therefore an image of $I_{t,j}$ cuts out subset Ω_t^j of Ω_t that contains $I_{t,j}$. Moreover the image of $\partial_{\mathbb{T}}\Omega_t$ does not have selfintersections so, in particular, an image of $I_{t,j}$ is the only part of the image of Ω_t that can have common points with Ω_t^j. It means that

$$\Pi(\mathrm{int}\Omega_t, t, \sigma(t)) \cap (\partial_{\mathbb{T}}\Omega_{\sigma(t)} \setminus E_{\sigma(t)}) = \emptyset$$

what was needed to prove.

□

Example 4.4. Let $\mathbb{T} = \bigcup_{n\in\mathbb{N}} [2n, 2n+1]$ and $f(t,(x,y)) = \left(\frac{e^{-t}(1-|y|\sin(t\pi))-2x}{3}, \frac{2y+\sin x}{5}\right)$. We are interested in existence of trajectory convergent to $(0,0)$.

We will want to use Theorem 4.2 taking $\Omega := \bigcup_{n\in\mathbb{N}} \bigcup_{t\in[2n,2n+1]} \{t\} \times \{(x_1,x_2) \in \mathbb{R}^2 \,;\, -e^{(n-t)/3} \leqslant x_i \leqslant e^{(n-t)/3}\}$.

Firstly, for $t = 2n+1$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$ we have

$$\|f(t,(x_1,y_1)) - f(t,(x_2,y_2))\| =$$

$$= \|(2(x_2-x_1)/3, (2(y_1-y_2) + \sin(x_1) - \sin(x_2))/5)\| \leq$$

$$\leq \|(2/3, 3/5)\| \, \|(x_1-x_2, y_1-y_2)\| < \|(x_1,y_1) - (x_2,y_2)\|,$$

therefor we have $L(t)\mu(t) < 1$, so assumptions of Proposition 2.11 are met.

The set Ω is selected so that $\Omega_t = \Omega_{\sigma(t)}$ and is Θ-bounded, where $\Theta(t,(x,y)) = \frac{e^{(n-t)/3}}{\sup\{|x|,|y|\}}(x,y)$, for $t \in [2n, 2n+1]$.

We will find E.

For all $t \in [2n, 2n+1]$ and $-e^{(n-t)/3} \leq x \leq e^{(n-t)/3} = |y|$ we have $(t,(x,y)) \in \partial_{\mathbb{T}}\Omega$ and $|\sin(x)| < |y|$, therefore these are the egress points. Whereas for each $t \in [2n, 2n+1]$ i $-e^{(n-t)/3} < y < e^{(n-t)/3} = |x|$ we have $(t,(x,y)) \in \partial_{\mathbb{T}}\Omega$ and $|e^{-t}(1-|y|\sin(t\pi))| \leqslant e^{-t} < e^{(n-t)/3} = |x|$, therefore vectors $f(t,(x,y))$ areare directed to the center of set Ω_t and $\left|\frac{e^{-t}(1-|y|\sin(t\pi))-2x}{3}\right| > |\frac{1}{3}x| = |\frac{d}{dt}e^{(n-t)/3}|$, thus these points are entry points.

For $t = 2n+1$ i $(x,y) \in E_t$ we have $\Pi((x,y),t,\sigma(t)) = (x + \frac{e^{-t}-2x}{3}, y + \frac{2y+\sin(x)}{5}) \notin \Omega_{\sigma(t)}$, so the assumption (H4a) is met.

For $t = 2n+1$ i $(x,y) \in \Omega_t$ we have similarly: $\Pi((x,y),t,\sigma(t)) = (x + \frac{e^{-t/3}-2x}{3}, y + \frac{2y+\sin(x)}{5})$, therefore the first coordinate is inside the segment $[\frac{e^{-t/3}-e^{n/3-t/3}}{3}, \frac{e^{-t/3}+e^{n/3-t/3}}{3}]$, which means that there are no common points with $\partial_{\mathbb{T}}\Omega_t \setminus E_t$, so assumption (H4b) is met too.

All assumptions are satisfied, therefore there exists trajectory remaining in the set Ω, which is convergent to $(0,0)$ (from the selection of the set Ω).

4.3 Remarks

At first we notice that holes homeomorphic to balls in Ω_t are available in Theorem 4.2. Indeed, for n holes we can consider: $\Theta : \mathbb{T} \times \bigoplus_{j=0}^{n} S^1 \to \mathbb{R}^2$ such that $\Theta(\{t\} \times \bigoplus_{k=0}^{n} S^1)$ is homeomorphic to $S(0,1) \cup \bigcup_{k=1}^{n} S((0,(k-1)/n), 1/3n))$.

In the second remark we observe that properties of a local Δ-process Π are essential, not of f itself, so in all approaches we can change our understanding of a solution of $x^{\Delta}(t) = f(t,x(t))$ and treat it as a function that fulfills the equation $\mathbb{T}$-almost everywhere (in a Sobolev space on a time scale). Moreover, we can change assumptions to a $\mathbb{T}$-almost everywhere form. It is important when we look for possible generalizations to differential inclusions or multivalued Δ-processes.

A proof technique presented in Section 4 cannot be repeated in higher dimensions because the Shöenflies theorem does not raise up to them. The following open problem appears:

That concept had been introduced in [11].

Open problem:

Is it possible to use in higher dimensional spaces the geometric idea presented in the proof of Theorem 4.2 under some additional restrictions to Θ or Π?

Nevertheless, this geometric idea opens new perspectives in the Ważewski retract method on time scales and allows us to study more classes of systems (for example hyperbolic systems).

References

[1] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles ordinaires*, Ann. Soc. Polon. Math. **20** (1947), 279-313.

[2] R. Srzednicki, *Ważewski method and Conley index*, Handbook of differential equations, 591-684, Elsevier/North-Holland, Amsterdam, 2004.

[3] G. Gabor and M. Quincampoix, *On existence of solutions to differential inclusions remaining in a prescribed closed subset of a finite-dimensional space*, J. Differential Equations 185 (2002), No. 2, 483-512.

[4] C.V. Coffman, *Asymptotic behavior of solution of ordinary difference equations*, Trans. Amer. Math. Soc., 110 (1964), 22-51.

[5] J. Diblik, *Anti-Lyapunov method for systems of discrete equations*, Nonlinear Anal. 57 (2004), no. 7-8, 1043-1057.

[6] J. Diblik, M.Růžičková, Z. Šmarda, *Ważewski's metod for systems of dynamic equations on time scales*, Nonlinear Analysis 71 (2009), e1124-e1131.

[7] L. Adamec, *A theorem of Ważewski and dynamic equations on time scales*, J. Difference Equ. Appl. 13 (2007), no. 1, 63-78.

[8] G. Gabor, S. Ruszkowski *Ważewski theorem on time scales with a set of egress points that is not the whole boundary*, Nonlinear Anal., Vol. 75 no. 18, 6541-6549.

[9] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[10] M. Bohner, A. Peterson, *Advances in dynamic equations on time scales*, Birkhäuser, Boston, 2003.

[11] A. Cabada, D.R. Vivero, *Expression of the Lebesgue Δ-integral on time scales as a usual Lebesgue integral; application to the calculus of Δ-antiderivatives*, Math. Comput. Modelling 43 (2006), no. 1-2, 194-207.

[12] E.E. Moise, *Geometric Topology in Dimensions 2 and 3*, Springer-Verlag, New York, Berlin, Heidelberg, 1977