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The Lefschetz Theorem for multivalued maps

A subcollection of multivalued maps called s-maps is introduced. Then to a self s-map f of a finite connected CW-complex an integer $\mathcal{L}_s(f)$ is associated and an analog of the Lefschetz Fixed Point Theorem is proved.

1 Introduction

The Lefschetz Fixed Point Theorem states that if X is a sufficiently nice space and $f\colon X \to X$ is a singlevalued continuous map, then it is possible to associate to f an integer $\mathcal{L}(f)$ such that if $\mathcal{L}(f) \neq 0$, then f has a fixed point. The number $\mathcal{L}(f)$ is called the Lefschetz number of f and is a well known and very useful homotopical invariant.

In the literature one can find many tries of generalizations of the Lefschetz number to the case of multivalued maps. The paper [2] presents the Lefschetz number defined for maps for which the image of any two different points has the same finite number of elements. More general approches regarding to acyclic and admissible maps are presented in [3]. Authors of [6] consider multivalued maps with an additional algebraic structure called the weight of a map. They construct for such maps the Lefschetz number using the Darbo homology functor, but that number essentially depends on the weight as well.

The main goal of this paper is to introduce a subcollection of multivalued maps, called s-maps for which it is possibile to define the Lefschetz number in a new way. In Section 2 we recall some basic information about the ordinary Lefschetz number and the Lefschetz set of admissible maps presented in [3]. Then in Section 3 we show a categorical construction which leads to an extension of some definitions to larger categories. We apply such a construction to define a subcollection of multivalued maps called s-maps and the Lefschetz number of them as well. Moreover, we prove an analog of the Lefschetz Fixed Point Theorem for s-maps (Theorem 3.9). Next in Section 4 we compare our approach with the Lefschetz set of admissible maps. We present some examples of s-maps which are not admissible. At the end of the section we use the categorical construction again to define s-admissible maps which generalize both admissible and s-maps. Next we formulate the Lefschetz Fixed Point Theorem for such maps (Theorem 4.9).

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2 Preliminaries

First we recall some basic information about multivalued maps. More details one can find in [3].

Let X, Y be two topological spaces and assume that for every point $x \in X$ a nonempty compact subset $\varphi(x) \subseteq Y$ is given. In this case, we say that φ is a *multivalued map* from X to Y and we write $\varphi\colon X \multimap Y$.

Let $\varphi\colon X \multimap Y$ be a multivalued map and $A \subseteq X$, then the *image* of A under φ is the set

$$\varphi(A) = \bigcup_{x \in A} \varphi(x).$$

Let $\varphi\colon X \multimap Y$ be a multivalued map and $B \subseteq Y$, then the *large preimage* of B under φ is the set

$$\varphi^{-1}(B) = \{x \in X \mid \varphi(x) \cap B \neq \emptyset\}.$$

If $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$ are two multivalued maps, then for any $C \subseteq Z$ we have

$$(\psi \circ \varphi)^{-1}(C) = \varphi^{-1}(\psi^{-1}(C)).$$

A multivalued map $\varphi\colon X \multimap Y$ is called *upper semicontinuous* (*u.s.c.*), provided for every closed $B \subseteq Y$ the set $\varphi^{-1}(B)$ is closed in X. If $\varphi\colon X \multimap$

Y and $\psi\colon Y \multimap Z$ are u.s.c. maps, then the composition $\psi \circ \varphi\colon X \multimap Z$ is also u.s.c..

Remark 2.1. Let $f\colon X \to Y$ be a singlevalued continuous map onto Y. Then its inverse can be considered as a multivalued map $f^{inv}\colon Y \multimap X$ defined by $f^{inv}(y) = f^{-1}(y)$ for $y \in Y$. If f is closed, then f^{inv} is u.s.c.. Later we write f^{-1} instead of f^{inv}.

Now we recall the Lefschetz Fixed Point Theorem for singlevalued maps. For details check [1], [4] and [5]. Denote by $\mathcal{C}$ the collection of all finite connected CW-complexes. Let $X \in \mathcal{C}$ and $f\colon X \to X$ be a singlevalued continuous map. Recall that we have a well defined integer $\mathcal{L}(f) = \sum_{k\in\mathbb{Z}} (-1)^k \mathrm{tr}(f_k)$, called the *Lefschetz number* of f, where $f_k\colon H_k(X,\mathbb{Q}) \to H_k(X,\mathbb{Q})$ are maps induced by f on the rational homology groups and $\mathrm{tr}(f_k)$ denotes the trace of the homomorphism f_k.

Remark 2.2. When we work in $\mathcal{C}$ there is no difference which homology functor we use, but if X is an arbitrary topological space, then by $H_k(X,\mathbb{Q})$ we mean the k-th Čech rational homology group of X.

Notice that the Lefschetz number has the following very useful properties, which are a simple consequences of analogous properties of the trace:

Proposition 2.3. *Let X, $Y \in \mathcal{C}$, $f\colon X \to Y$ and $g\colon Y \to X$ be two singlevalued continuous maps, then $\mathcal{L}(fg) = \mathcal{L}(gf)$.*

Corollary 2.4. *Let X, $Y \in \mathcal{C}$, $g\colon X \to Y$ be a homeomorphism and $f\colon X \to X$ be a singlevalued continuous map, then $\mathcal{L}(f) = \mathcal{L}(gfg^{-1})$.*

There is also a property which connects the Lefschetz number of a map with the Lefschetz number of its prime iteration:

Theorem 2.5 (The mod p Theorem [4])**.** *Let $X \in \mathcal{C}$, $f\colon X \to X$ be a contiuous map and p be a prime. Then $\mathcal{L}(f^p) \equiv \mathcal{L}(f) \bmod p$.*

The most famouse and important application of the Lefschetz number is:

Theorem 2.6 (Lefschetz Fixed Point Theorem [5]). *Let $X \in \mathcal{C}$ and $f\colon X \to X$ be a singlevalued continuous map. If $\mathcal{L}(f) \neq 0$, then f has a fixed point.*

To recall the construction of the Lefschetz number of multivalued admissible maps, we need some definitions. All results presented below are stated in [3].

A space X is called *acyclic*, when:
(i) $H_k(X, \mathbb{Q}) = 0$ for all $k \geq 1$;
(ii) $H_0(X, \mathbb{Q}) = \mathbb{Q}$.

A singlevalued continuous map $f\colon X \to Y$ is called *proper*, provided for every compact $K \subseteq Y$ the set $f^{-1}(K)$ is compact.

A singlevalued continuous map $f\colon X \to Y$ is called a *Vietoris map*, provided the following conditions hold:
(i) $f\colon X \to Y$ is proper;
(ii) the set $f^{-1}(y)$ is acyclic for all $y \in f(X)$.

Vietoris maps have the following important property:

Theorem 2.7 (Vietoris [3]). *If X, $Y \in \mathcal{C}$ and $f\colon X \to Y$ is a Vietoris map, then the induced homomorphism $f_k\colon H_k(X, \mathbb{Q}) \to H_k(Y, \mathbb{Q})$ is an isomorphism for all $k \geq 0$.*

A multivalued map $\varphi\colon X \multimap Y$ is called *admissible*, provided there exist a space Z and two continuous maps $p\colon Z \to X$ and $q\colon Z \to Y$ such that:
(i) p is a Vietoris map;
(ii) $q(p^{-1}(x)) \subseteq \varphi(x)$ for all $x \in X$.
We write $(p, q) \subseteq \varphi$ when maps p and q are as above.

The *Lefschetz set* of an admissible map $\varphi\colon X \multimap X$ is defined by:

$$\mathcal{L}_a(\varphi) = \left\{ \sum_{k \in \mathbb{Z}} (-1)^k \mathrm{tr}(q_k p_k^{-1}) \mid (p, q) \subseteq \varphi \right\}.$$

Remark 2.8. Let $X, Y \in \mathcal{C}$ and $\varphi\colon X \multimap Y$ be an admissible map. It is easy to see, that if $(p,q) \subseteq \varphi$, then $qp^{-1}\colon X \multimap Y$ is u.s.c. and the set $qp^{-1}(x)$ is connected for all $x \in X$.

The most important application of the Lefschetz set is:

Theorem 2.9 (Lefschetz Fixed Point Theorem for admissible maps [3])**.** *Let $X \in \mathcal{C}$ and $\varphi\colon X \multimap X$ be an admissible map. If $\mathcal{L}_a(\varphi) \neq \{0\}$, then φ has a fixed point.*

Remark 2.10. In [3] a collection of spaces for which it is possible to consider admissible maps is larger than $\mathcal{C}$. Moreover, in [7] there is considered a broader class of maps for which it is possible to define the Lefschetz set.

3 The Lefschetz number of s-maps

In this section we introduce a subcollection of multivalued maps called s-maps and investigate their properties. First, we show a very usefull categorical construction which helps us in defining s-maps.

Definition 3.1. Let $\mathcal{D}$ be a category and $\mathcal{C}$ its subcategory, not necessary full. Define a category $(\mathcal{D},\mathcal{C})$ as follows:
(i) object of $(\mathcal{D},\mathcal{C})$ are quadruples (X, A, r, s), where $X \in \mathcal{D}$, $A \in \mathcal{C}$, $r\colon X \to A$ and $s\colon A \to X$ are morphisms in $\mathcal{D}$ such that $rs = \mathrm{id}_A$;
(ii) $\mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((X, A, r, s), (Y, B, t, q)) = \{(\varphi, f) \in \mathcal{D} \times \mathcal{C} \mid \varphi = qfr,\ f \in \mathrm{Mor}_{\mathcal{C}}(A, B)\}$;
(iii) a composition law in $(\mathcal{D},\mathcal{C})$ is induced from the composition laws in $\mathcal{D}$ and $\mathcal{C}$;
(iv) $\mathrm{id}_{(X,A,r,s)} = (sr, \mathrm{id}_A)$.

Observe that the composition of morphisms in the category $(\mathcal{D},\mathcal{C})$ is well defined because if $(\varphi, f) \in \mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((X, A, r, s), (Y, B, t, q))$ and $(\psi, g) \in \mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((Y, B, t, q), (Z, C, l, m))$, then $\psi\varphi = lgtqfr = lgfr$, because $tq = \mathrm{id}_Y$, so $(\psi\varphi, gf) \in \mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((X, A, r, s), (Z, C, l, m))$.

Notice that if $\varphi = qfr$, then $f = t\varphi s$, so f is uniquely determinded by φ (when suitable r, s, t and q are choosen).

Now let $\mathcal{D}$ be a category of topological spaces and multivalued u.s.c. maps and $\mathcal{C}$ its subcategory of finite connected CW-complexes and singlevalued continuous maps. Consider the category $(\mathcal{D}, \mathcal{C})$ for such $\mathcal{D}$ and $\mathcal{C}$.

Definition 3.2. Let $X \in \mathcal{D}$ and $\varphi\colon X \multimap X$ be a multivalued u.s.c. map. The map φ is called an *s-map* if there exist:
(i) $A \in \mathcal{C}$;
(ii) a singlevalued continuous map $f_\varphi\colon A \to A$;
(iii) a singlevalued continuous surjection $r\colon X \to A$;
(iv) a multivalued u.s.c. map $s\colon A \multimap X$;
such that:
(a) $(X, A, r, s) \in (\mathcal{D}, \mathcal{C})$;
(b) $(\varphi, f_\varphi) \in \mathrm{Mor}_{(\mathcal{D},\mathcal{C})}((X, A, r, s), (X, A, r, s))$.

If $\varphi\colon X \multimap X$ is an s-map and f_φ is a map like in the above definition, then we say that the morphism (φ, f_φ) represents φ in $(\mathcal{D}, \mathcal{C})$.

If $X \in \mathcal{C}$ and $\varphi\colon X \to X$ is a singlevalued continuous map, then clearly φ is an s-map, because it is enough to take $A = X$, $f_\varphi = \varphi$ and $r = s = \mathrm{id}_X$. This factorization is called *standart*. Of course for a singlevalued continuous map there can exsist factorizations different to the standart one.

Example 3.3. A map $\varphi\colon [0,2] \to [0,2]$ given by $\varphi(x) = 0$ for all $x \in [0,2]$ is singlevalued, so we have the standard factorization. On the other hand, we can choose a different morphism in $(\mathcal{D}, \mathcal{C})$ which represet φ, for example $r\colon [0,2] \to [0,1]$ is given by

$$r(x) = \begin{cases} x & \text{for } x \in [0,1]; \\ 1 & \text{for } x \in (1,2]; \end{cases}$$

$s\colon [0,1] \multimap [0,2]$ is the inverse of r, so $s(x) = r^{-1}(x)$ for all $x \in [0,1]$ and $f_\varphi\colon [0,1] \to [0,1]$ is defined by $f_\varphi(x) = 0$ for all $x \in [0,1]$.

For selfmorphisms in category $\mathcal{C}$ we have a well defined Lefschetz number. We can extend this definition to the category $(\mathcal{D}, \mathcal{C})$ by taking $\mathcal{L}(\varphi, f_\varphi) = \mathcal{L}(f_\varphi)$. Our goal is to show that if an s-map $\varphi\colon X \multimap X$ is represented by two different morphisms (φ, f_φ) and (φ, g_φ), then $\mathcal{L}(f_\varphi) = \mathcal{L}(g_\varphi)$. To show that we need first to prove some lemmas:

Lemma 3.4. *If an s-map $\varphi\colon X \to X$ is represented by pairs (φ, f_φ) and (φ, g_φ) are such that $\varphi = sf_\varphi r$ and $\varphi = tg_\varphi q$ for suitable s, r and q, t, then:*
(i) $f_\varphi r = rtg_\varphi q$;
(ii) $qsf_\varphi = g_\varphi qs$;
(iii) $g_\varphi qsrt = g_\varphi$;
(iv) rtg_φ *is a singlevalued continuous map;*
(v) qsf_φ *is a singlevalued continuous map;*
(vi) $g_\varphi qs$ *is a singlevalued continuous map.*

Proof. (i), (ii) and (iii) are easy consequences of equalities $sf_\varphi r = tg_\varphi q$, $qt = \mathrm{id}$ and $rs = \mathrm{id}$. Let now prove (iv). Using (i) we have that $rtg_\varphi q$ is a singlevalued continuous map. Moreover, q is a singlevalued continuous surjection, so (iv) follows. Property (v) is analogous to (iv) and (vi) is a consequence of (ii) and (v). □

Lemma 3.5. *If an s-map $\varphi\colon X \to X$ is represented in $(\mathcal{D}, \mathcal{C})$ by pairs (φ, f_φ) and (φ, g_φ), then $\mathcal{L}(f_\varphi^n) = \mathcal{L}(g_\varphi^n)$ for $n \geq 2$.*

Proof. Let $\varphi = sf_\varphi r$ and $\varphi = tg_\varphi q$, then using Proposition 2.3 and Lemma 3.4 we obtain:
$\mathcal{L}(f_\varphi^n) = \mathcal{L}((rtg_\varphi qs)^n) = \mathcal{L}(rtg_\varphi^n qs) = \mathcal{L}(rtg_\varphi^{n-1} g_\varphi qs) = \mathcal{L}(g_\varphi qsrtg_\varphi^{n-1}) = \mathcal{L}(g_\varphi^n)$. □

As an easy consequence of Theorem 2.5 and Lemma 3.5 we get:

Proposition 3.6. *If an s-map $\varphi\colon X \to X$ is represented in $(\mathcal{D}, \mathcal{C})$ by pairs (φ, f_φ) and (φ, g_φ), then $\mathcal{L}(f_\varphi) = \mathcal{L}(g_\varphi)$.*

Remark 3.7. Example 3.3 shows that we cannot prove the above proposition directly as Lemma 3.5 because the composition $sr\colon X \multimap X$ does not have to be singlevalued.

Now we are in a position to define the Lefschetz number of s-maps:

Definition 3.8. Let $\varphi\colon X \multimap X$ be an s-map. The Lefschetz number of φ is a number $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi)$, where (φ, f_φ) represents φ in $(\mathcal{D}, \mathcal{C})$. According to Proposition 3.6 this number is well defined.

If $f\colon X \to X$ is a singlevalued continuous map then $\mathcal{L}_s(f) = \mathcal{L}(f)$. To show this it is enough to take the standart factorization.

Now we prove an analog of the Lefschetz Fixed Point Theorem for s-maps:

Theorem 3.9 (Lefschetz Fixed Point Theorem)**.** *Let $\varphi\colon X \multimap X$ be an s-map and $\mathcal{L}_s(\varphi) \neq 0$, then φ has a fixed point.*

Proof. The map φ is an s-map. Let suitable A, f_φ, r and s be choosen. Acoording to the definition $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi)$, where $\varphi = sf_\varphi r$. The map $f_\varphi\colon A \to A$ is singlevalued continuous and $A \in \mathcal{C}$, so the ordinary Lefschetz Fixed Point Theorem implies that f_φ has a fixed point $a = r(x)$ for some $x \in X$. Let $z \in s(r(x))$, then $\varphi(z) = \varphi(x)$, because $r(z) = r(x)$. Therefore, we have $z \in sr(x) = sf_\varphi r(x) = \varphi(x) = \varphi(z)$, so z is a fixed point of φ. □

The easiest way to show that $\varphi\colon X \multimap X$ is an s-map, it is to find an equivalence relation R on X such that $A = X/R$ and $r\colon X \to A$ is the canonical projection. If $\varphi\colon X \multimap X$, then

$$R = \{(x, y) \in X \times X \mid \varphi(x) = \varphi(y)$$

and

$$\varphi^{-1}(x) = \varphi^{-1}(y) \neq \emptyset\} \cup \{(x, x) \mid x \in X\}$$

is called a *canonical relation* for φ. If we use the canonical relation, then we write X_R istead of A.

Now we present some examples of s-maps and find their Lefschetz numbers:

Example 3.10. Let S^n be the n-sphere and $\varphi\colon S^n \to S^n$ be such that $\varphi(x) = S^n$ for all $x \in S^n$. Let R be the canonical relation for φ. Then $X_R = \{*\}$, where $\{*\}$ denotes the one point space. We have $\varphi = sf_\varphi r$, where $r\colon S^n \to \{*\}$ is given by $r(x) = \{*\}$ for every $x \in S^n$, $s\colon \{*\} \multimap S^n$ is defined by $s(*) = S^n$ and $f_\varphi = \mathrm{id}_{\{*\}}$. Therefore φ is an s-map and $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi) = \mathcal{L}(\mathrm{id}_{\{*\}}) = 1$.

Example 3.11. Let $X \in \mathcal{C}$ and $f\colon S^1 \to S^1$ be a continuous singlevalued map. Define $\varphi\colon S^1 \times X \multimap S^1 \times X$ by $\varphi(b,x) = \{f(b)\} \times X$. Let R be the canonical relation for φ. Then $X_R = S^1$, $r\colon S^1 \times X \to S^1$ is given by $r(b,x) = b$ for all $(b,x) \in S^1 \times X$, $s\colon S^1 \multimap S^1 \times X$ is given by $s(b) = \{b\} \times X$ and $f_\varphi = f$. We have $\mathcal{L}_s(\varphi) = \mathcal{L}(f)$.

4 Comparision of the Lefschetz numbers

In this section we compare the Lefschetz number of s-maps with the Lefschetz set of admissible maps. One may expect that there is some coincidence between those two conceptions, but as we see in some examples they give different results. This leads to a definition of s-admissible maps which generalizes both admissible and s-maps. We start this section with analysing some examples.

Our first example shows a situation when $\mathcal{L}_a(\varphi) \neq \{\mathcal{L}_s(\varphi)\}$:

Example 4.1. Let $\varphi\colon S^n \multimap S^n$ be such that $\varphi(x) = S^n$ for all $x \in S^n$. We have shown in Example 3.10, that this is an s-map and $\mathcal{L}_s(\varphi) = 1$. On the other hand, the map φ is admissible and $\mathcal{L}_a(\varphi) = \mathbb{Z}$, because $(\mathrm{id}_{S^n}, f) \subset \varphi$ for all singlevalued continuous maps $f\colon S^1 \to S^1$.

In the previous example we have $\{\mathcal{L}_s(\varphi)\} \subseteq \mathcal{L}_a(\varphi)$, but that is not true in general.

Example 4.2. Let $\varphi\colon [0,2] \multimap [0,2]$ be given by:

$$\varphi(x) = \begin{cases} x & \text{for } x \in (0,2); \\ \{0,2\} & \text{for } x \in \{0,2\}. \end{cases}$$

Then φ is both admissible and an s-map. We have

$$\mathcal{L}_a(\varphi) = \left\{ \sum_{k\in\mathbb{Z}} (-1)^k \mathrm{tr}(q_k p_k^{-1}) \mid (p,q) \subseteq \varphi \right\} = \{\mathcal{L}(\mathrm{id}_{[0,2]})\} = \{1\},$$

because we cannot choose p and q such that $qp^{-1} \neq \mathrm{id}_{[0,2]}$ (see Remark 2.8). On the other hand, we have $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi) = \mathcal{L}(\mathrm{id}_{S^1}) = 0$, because X_R is homeomorphic to S^1 and using Proposition 2.4 we can replace in our calculations the map f_φ by id_{S^1}.

In next two examples we show s-maps which are not admissible. As a consequence for those maps only the Lefschetz number of s-maps is possible to define.

Example 4.3. Let $\varphi\colon [0,2] \multimap [0,2]$ be given by:

$$\varphi(x) = \begin{cases} x+1 & \text{for } x \in [0,1); \\ \{0,2\} & \text{for } x = 1; \\ x-1 & \text{for } x \in (1,2]. \end{cases}$$

This map is not admissible (see Remark 2.8). On the other hand, φ is an s-map and we have $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi) = \mathcal{L}(\mathrm{id}_{S^1}) = 0$, because X_R is homeomorphic to S^1 and using Proposition 2.4 we can think that f_φ is a rotation by an angle π which is homotopic to the identity map on S^1.

Example 4.4. Let $\varphi\colon [0,2] \to [0,2]$ be given by:

$$\varphi(x) = \begin{cases} -x+1 & \text{for } x \in [0,1); \\ \{0,2\} & \text{for } x = 1; \\ -x+3 & \text{for } x \in (1,2]. \end{cases}$$

Then φ is not admissible, but φ is an s-map and we have $\mathcal{L}_s(\varphi) = \mathcal{L}(f_\varphi) = 2$, because we can think that $f_\varphi\colon S^1 \to S^1$ and has a degree equal -1.

Remark 4.5. After studing two previous examples one can easy see that for any integer n it is possible to find a multivalued map φ which is not admissible, but is an s-map and $\mathcal{L}_s(\varphi) = n$. Namely, it is enough to take a multivalued map $\varphi\colon [0,2] \multimap [0,2]$ which is not admissible and a suitable map $f_\varphi\colon S^1 \to S^1$ has a degree $1-n$.

Remark 4.6. The maps from the last two examples can be considered as the multivalued weighted maps (check [6] for a definition), but only trivial weight is possible for those maps.

Now we formulate a definition which generalizes both admissible and s-maps. Let $\mathcal{D}$ be a category of spaces and multivalued maps and $\mathcal{C}$ be its subcategory of finite connected CW-complexes and admissible maps.

Definition 4.7. Let $X \in \mathcal{D}$ and $\varphi\colon X \multimap X$ be a multivalued map. The map φ is called an *s-admissible map* if there exist:
(i) $A \in \mathcal{C}$;
(ii) an admissible map $\beta_\varphi\colon A \multimap A$;
(iii) a singlevalued continuous surjection $r\colon X \to A$;
(iv) a multivalued u.s.c. map $s\colon A \multimap X$;
such that:
(a) $s\beta_\varphi r(x) \subseteq \varphi(x)$ for all $x \in X$;
(b) $(X, A, r, s) \in (\mathcal{D}, \mathcal{C})$.

Let $\varphi\colon X \multimap X$ be s-admissible. Denote by $(\mathcal{D}, \mathcal{C})_\varphi$ the set of all maps β_φ which are like in the above definition.

Definition 4.8. The *Lefschetz set* of s-admissible map is a set:

$$\mathcal{L}_s(\varphi) = \bigcup_{\beta_\varphi \in (\mathcal{D},\mathcal{C})_\varphi} \mathcal{L}_a(\beta_\varphi).$$

Theorem 4.9 (Lefschetz Fixed Point Theorem)**.** *Let $X \in \mathcal{C}$ and $\varphi\colon X \multimap X$ be an s-admissible map. If $\mathcal{L}_s(\varphi) \neq \{0\}$, then φ has a fixed point.*

Proof. We choose suitable s, r and β_φ such that $\mathcal{L}_a(\beta_\varphi) \neq \{0\}$. Than we use Theorem 2.9 and following the proof of Theorem 3.9 we obtain that $s\beta_\varphi r$ has a fixed point, which is also a fixed point of φ. □

Remark 4.10. The easiest way to show that $\varphi\colon X \multimap X$ is s-admissible is to find an admissible map $\psi\colon X \multimap X$ such that $\psi(x) \subseteq \varphi(x)$ for all $x \in X$ or an s-map $\eta\colon X \multimap X$ such that $\eta(x) \subseteq \varphi(x)$ for all $x \in X$.

Now we present an example of an s-admissible map which is neither admissible nor an s-map:

Example 4.11. Let $\varphi\colon [0,3] \multimap [0,3]$ be given by:

$$\varphi(x) = \begin{cases} [-x+1, -x+2] & \text{for } x \in [0,1); \\ [0, -x+2] \cup [-x+4, 3] & \text{for } x \in [1,2]; \\ [-x+4, -x+5] & \text{for } x \in (2,3]. \end{cases}$$

The map φ is not an s-map. Moreover, φ is not admissible, because the graph of φ has two connected components and neither of them is a graph of a multivalued map from $[0,3]$ to $[0,3]$. On the other hand we have an s-map $\eta\colon [0,3] \multimap [0,3]$ given by:

$$\eta(x) = \begin{cases} -x+1 & \text{for } x \in [0,1); \\ \{0,3\} & \text{for } x = 1; \\ -x+4 & \text{for } x \in (1,3] \end{cases}$$

such that $\eta(x) \subseteq \varphi(x)$ for all $x \in X$. Consequently φ is s-admissible. We have $\mathcal{L}_s(\eta) = 2$, so $2 \in \mathcal{L}_s(\varphi)$. Moreover, it can be shown that $\mathcal{L}_s(\varphi) = \{2\}$.

References

[1] R.F.Brown, *The Lefschetz fixed point theorem.* Scott, Foresman and Co., Glenview, Ill.-London 1971.

[2] R.F.Brown, *The Lefschetz number of an n-valued multimap.* JP J. Fixed Point Theory Appl. 2 (2007), no. 1, 53-60.

[3] L.Górniewicz, *Topological fixed point theory of multivalued mappings.* Second edition. Topological Fixed Point Theory and Its Applications, 4. Springer, Dordrecht, 2006.

[4] A.Granas, J.Dugundji, *Fixed point theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[5] J. Jezierski, W. Marzantowicz, *Homotopy methods in topological fixed and periodic points theory.* Topological Fixed Point Theory and Its Applications, 3. Springer, Dordrecht, 2006.

[6] J.Pejsachowicz, R.Skiba, *Fixed point theory of multivalued weighted maps.* Handbook of topological fixed point theory, 217–263, Springer, Dordrecht, 2005.

[7] M.Ślosarski, *Generalized Lefschetz sets.* Fixed Point Theory Appl. 2011, Art. ID 216146, 11 p.