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Koszul complexes and Chevalley's theorems for Lie algebroids

We use Koszul complexes and Chevalley-type theorems to calculate the cohomology $\mathbf{H}(A)$ of a transitive Lie algebroid A under some assumptions on the isotropy Lie algebras.

1 Introduction

How can we calculate the cohomology $\mathbf{H}(A)$ of a transitive Lie algebroid A with the Atiyah sequence $0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0$ **?** This is one of the fundamental questions for the topology of Lie algebroids [I-K-V], [M]. A classical method is to use spectral sequences. We can use the Leray spectral sequence for the Čech–de Rham complex of transitive Lie algebroids [K-M-1] as well as the Hochschild-Serre spectral sequence for the pair of Lie algebras $(\operatorname{Sec}\boldsymbol{g}, \operatorname{Sec}A)$ and the observation that the vector bundle of the cohomology $\mathbf{H}(\boldsymbol{g})$ of the isotropy Lie algebras, $\mathbf{H}(\boldsymbol{g})_{|x} = \mathbf{H}\left(\boldsymbol{g}_{|x}\right)$, is flat, and that $E_2^{j,i} = \mathbf{H}^j_\nabla\left(M; \mathbf{H}^i(\boldsymbol{g})\right)$ where ∇ is the flat covariant derivative in $\mathbf{H}^i(\boldsymbol{g})$ [K-M-2], [K-M-3].

In this paper we propose an adaptation of the method of Koszul complexes and Chevalley-type theorems [G-H-V, Vol. III] to the calculation of $\mathbf{H}(A)$. Originally the method is based on the operation of a reductive Lie algebra in a graded differential algebra admitting an algebraic connection. A fundamental theorem of Chevalley gives a homomorphism from the corresponding Koszul complex which induces an isomorphism of cohomology. Classically, this isomorphism is applied to the cohomology

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of principal fibre bundles. Namely: the Chevalley theorem (for pfb's) says that under some assumptions, the cohomology of the total space $\mathbf{H}(P)$ of a pfb P depends uniquely on the cohomology of the base manifold M and the characteristic classes (the Chern-Weil homomorphism $h_P : \left(\bigvee \mathfrak{g}^*\right)_{I_G} \longrightarrow \mathbf{H}(M)$). It turns out that this assertion has a counterpart for Lie algebroids, but in this context we cannot use the standard operation of a Lie algebra directly. We propose some modification of this method.

2 Lie algebroid of a principal fibre bundle, Lie functor

2.1 Examples of Lie algebroids

2.1.1 Lie algebroid of a Lie group

The Lie algebroid of a Lie group G (the infinitesimal object of a Lie group G) is simply its Lie algebra $\mathfrak{g} = T_eG = TG/G$ (for example, through the right action of G on TG we obtain the "right Lie algebra of a Lie group").

2.1.2 Lie algebroid of a principal fibre bundle

The vector space $\mathbf{A}(P) := TP/G$ of cosets of the right action of G on TP (introduced by M. Atiyah in 1955) is an infinitesimal object of a principal fibre bundle $P(M,G)$. It has two extra structures: a Lie bracket in the space of global cross-sections $\operatorname{Sec}\mathbf{A}(P)$ and a linear homomorphism $\#_{\mathbf{A}(P)} : \mathbf{A}(P) \longrightarrow TM$ called the anchor. The Lie bracket in $\operatorname{Sec}\mathbf{A}(P)$ is introduced via the isomorphism $\operatorname{Sec}(\mathbf{A}(P)) \cong \mathfrak{X}^R(P)$ where $\mathfrak{X}^R(P)$ is the space of right invariant vector fields on P with the usual Lie bracket. The anchor is defined by $\#_{\mathbf{A}(P)} : \mathbf{A}(P) \longrightarrow TM$, $[v] \longmapsto \pi_*(v)$ where $\pi : P \to M$ is the projection of P. The anchor $\#_{\mathbf{A}(P)}$ is bracket-preserving: $\#_{\mathbf{A}(P)}(\llbracket \xi_1, \xi_2 \rrbracket) = \left[\#_{\mathbf{A}(P)}(\xi_1), \#_{\mathbf{A}(P)}(\xi_2)\right]$, and the Leibniz formula holds: $\llbracket \xi_1, f \cdot \xi_2 \rrbracket = f \cdot \llbracket \xi_1, \xi_2 \rrbracket + \left(\#_{\mathbf{A}(P)}(\xi_1)\right)(f) \cdot \xi_2$. The Lie algebroid of the trivial principal fibre bundle $P = M \times G$ is equal to

$$\mathbf{A}(P) = TP/G = T(M \times G)/G = TM \times (TG/G) = TM \times \mathfrak{g},$$

with the bracket $[\![(X,\sigma),(Y,\eta)]\!] = ([X,Y], X(\eta) - Y(\sigma) + [\sigma,\eta])$ for $X,Y \in \mathfrak{X}(M)$, $\sigma,\eta \in C^\infty(M,\mathfrak{g})$, and the anchor $\#_{TM\times\mathfrak{g}} = pr_1 : TM\times\mathfrak{g} \to TM$.

2.2 Pradines' definition of a Lie algebroid

Generalizing the structure $\left(\mathbf{A}(P), [\![\cdot,\cdot]\!], \#_{A(P)}\right)$ for a pfb $P(M,G)$ J. Pradines gives the definition of a Lie algebroid [P]:

Definition 1. *A Lie algebroid on a manifold M is a triple $(A, [\![\cdot,\cdot]\!], \#_A)$ where A is a vector bundle on M, $(\operatorname{Sec} A, [\![\cdot,\cdot]\!])$ is an $\mathbb{R}$-Lie algebra, $\#_A : A \to TM$ is a linear homomorphism of vector bundles and the following Leibniz condition is satisfied:*

$$[\![\xi, f\cdot\eta]\!] = f\cdot[\![\xi,\eta]\!] + \gamma_L(\xi)(f)\cdot\eta, \quad f\in C^\infty(M),\ \xi,\eta\in\operatorname{Sec} A.$$

The anchor is bracket-preserving, $\#_A\circ[\![\xi,\eta]\!] = [\#_A\circ\xi, \#_A\circ\eta]$.

The image of the anchor, $\operatorname{Im}\#_A \subset TM$, is an integrable non-constant-rank (in general) distribution whose leaves form a Stefan foliation of M. If the anchor $\#_A$ is of constant rank then the Lie algebroid A is called regular and $\operatorname{Im}\#_A$ forms a regular foliation on M. The Lie algebroid is called transitive if $\#_A$ is an epimorphism. A transitive Lie algebroid is called integrable if it is isomorphic to the Lie algebroid of a principal fibre bundle.

We deal here only with transitive Lie algebroids.

For a transitive Lie algebroid A we have the Atiyah sequence

$$0 \longrightarrow \boldsymbol{g} \hookrightarrow A \xrightarrow{\#_A} TM \longrightarrow 0.$$

The vector bundle $\boldsymbol{g}$ is a Lie algebra bundle, called the adjoint of A; in particular, all the isotropy Lie algebras $\boldsymbol{g}_{|x}$ are isomorphic.

Example 2. *(1) A single Lie algebra $\mathfrak{g}$ is a Lie algebroid over a one–point set and with the zero anchor.*

(2) The tangent bundle TM of a manifold M is a Lie algebroid on M with id_{TM} as anchor and with the usual Lie bracket of vector fields.

(3) Trivial Lie algebroid: $TM\times\mathfrak{g}$ with the projection pr_1 as anchor and with the bracket given by

$$[\![\,(X,\sigma),(Y,\eta)\,]\!] = ([X,Y], X(\eta) - Y(\sigma) + [\sigma,\eta]),$$

$X, Y \in \mathfrak{X}(M)$, $\sigma, \eta \in C^{\infty}(M; \mathfrak{g})$, *is a transitive Lie algebroid, called trivial. (Each transitive Lie algebroid L over a contractible manifold is isomorphic to the trivial one).*

(4) The Lie algebroid $A(P) = TP/G$ of a G-principal fibre bundle $P = P(M, G)$.

(5) The Lie algebroid $\mathbf{A}(\mathfrak{f})$ of a vector bundle $\mathfrak{f}$: With a vector bundle $\mathfrak{f}$ we associate a transitive Lie algebroid $\mathbf{A}(\mathfrak{f})$ (isomorphic to the Lie algebroid of the principal fibre bundle of all frames of $\mathfrak{f}$, $\mathbf{A}(\mathfrak{f}) = \mathbf{A}(\mathrm{L}\,\mathfrak{f})$) whose space of global cross-sections $\operatorname{Sec}\mathbf{A}(\mathfrak{f})$ is equal to the space of all covariant differential operators for $\mathfrak{f}$. The Lie algebra bundle adjoint to $\mathbf{A}(\mathfrak{f})$ is equal to $\operatorname{End}(\mathfrak{f})$, so the Atiyah sequence reads

$$0 \longrightarrow \operatorname{End}(\mathfrak{f}) \longrightarrow \mathbf{A}(\mathfrak{f}) \longrightarrow TM \to 0.$$

Example 3 (Other examples). *(6) The Lie algebroid $A(M, \mathcal{F})$ of a transversally complete foliation $(M, \mathcal{F})$ of a connected Hausdorff paracompact manifold M, in particular:*

(6') The Lie algebroid $A(G; H)$ of a nonclosed Lie subgroup H of G: It is the Lie algebroid of the TC-foliation $\mathcal{F}_{G,H} = \{aH; a \in G\}$ of left cosets of a nonclosed Lie subgroup H in a Lie group G. These include nonintegrable Lie algebroids.

(7) Poisson manifolds yield nontransitive Lie algebroids.

Definition 4. *By a homomorphism of Lie algebroids $F : (A, [\![\cdot,\cdot]\!], \#_A) \longrightarrow (A', [\![\cdot,\cdot]\!], \#_{A'})$ on a manifold M we mean a linear homomorphism $F : A \to A'$ of vector bundles commuting with the anchors:*

$$\begin{array}{ccc} A & \xrightarrow{F} & A' \\ \downarrow \#_A & & \downarrow \#_{A'} \\ TM & = & TM \end{array}$$

and such that F is a homomorphism of the Lie algebras of global cross-sections:

$$F\left([\![\xi_1, \xi_2]\!]\right) = [\![F\xi_1, F\xi_2]\!], \quad \xi_i \in \operatorname{Sec} A.$$

A homomorphism $F : A \longrightarrow B$ of transitive Lie algebroids induces a linear homomorphism of the adjoint Lie algebra bundles $F^+ : \boldsymbol{g} \longrightarrow \boldsymbol{g}'$ and for any $x \in M$, $F_x^+ : \boldsymbol{g}_{|x} \longrightarrow \boldsymbol{g}'_{|x}$ is a homomorphism of Lie algebras.

We obtain in this way a homomorphism of Atiyah sequences,

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \boldsymbol{g} & \overset{F^{+}}{\longrightarrow} & \boldsymbol{g}' \\ \downarrow & & \downarrow \\ A & \overset{F}{\longrightarrow} & A' \\ \downarrow & & \downarrow \\ TM & = & TM \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

2.3 Lie functor

To have a Lie functor for pfb's we need to define a homomorphism of Lie algebroids induced by a homomorphism of pfb's. Let P and P' be two pfbs with structural Lie groups G and G', respectively. Assume that $\mu : G \longrightarrow G'$ is a homomorphism of Lie groups, and $F : P \longrightarrow P'$ a μ-homomorphism of pfbs, i.e. $F(z \cdot a) = F(z) \cdot a'$. Then the linear homomorphism (the Lie algebroid differential of F)

$$F_* : \mathbf{A}(P) \longrightarrow \mathbf{A}(P'), \quad [v_z] \longmapsto [dF_{*z}(v_z)],$$

is a homomorphism of the induced Lie algebroids.

2.4 Cohomology of a Lie algebroid

To a Lie algebroid A we associate the cohomology algebra $\mathbf{H}(A)$ defined via the DG-algebra of A-differential forms (with real coefficients) $(\Omega(A), d_A)$, where

$$\begin{aligned} \Omega(A) &= \mathrm{Sec} \bigwedge A^*, \\ d_A &: \Omega^*(A) \longrightarrow \Omega^{*+1}(A) \end{aligned}$$

$$\begin{aligned} (d_A z)(\xi_0, ..., \xi_k) = &\sum_{j=0}^{k} (-1)^j \; (\#_A \circ \xi_j)(z(\xi_0, ...\hat{\jmath}..., \xi_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \; z([\![\xi_i, \xi_j]\!], \xi_0, ...\hat{\imath}...\hat{\jmath}..., \xi_k), \end{aligned}$$

$z \in \Omega^k(A)$, $\xi_i \in \operatorname{Sec} A$. The exterior derivative d_A induces the cohomology algebra

$$\mathbf{H}(A) = \mathbf{H}(\Omega(A), d_A).$$

Why the differential d_A must be given by the above formula? It is easy to obtain this formula starting with the Lie algebroid of a Lie groupoid $\Phi = (\Phi, (\alpha, \beta), \cdot . M)$ on a manifold M, with source α and target β and partial multiplication $\cdot$. Let $i : M \to \Phi$ be the embedding of M onto the submanifold of units, $i(x) = u_x$, of this Lie groupoid. Then

$$\mathbf{A}(\Phi) = i^*(T^\alpha \Phi)$$

where $T^\alpha \Phi$ is the subbundle of α-vertical vectors. We see that for any $x \in M$, the submanifold $\Phi_x = \alpha^{-1}(x)$ of all elements starting at x (i.e. having x as source) forms a Φ_x^x-pfb (Φ_x^x is the Lie isotropy group at x, $\Phi_x^x = \{h \in \Phi : \alpha h = \beta h = x\}$) with the projection $\beta_x : \Phi_x \longrightarrow M$. We have $\mathbf{A}(\Phi)_{|x} = T_{u_x}(\Phi_x)$, the tangent space to the total space Φ_x at the unit x. For all pfb's Φ_x we can consider standard differential operators, like the exterior derivative of usual differential forms (or Lie derivative and substitution operator), and pass to the units u_x and "glue". By this procedure we obtain just d_A.

Example 5. *(1) If $A = \mathbf{A}(P) = TP/G$ for a G-principal fibre bundle $P \longrightarrow M$ then*

$$\Omega(A) \cong \Omega^R(P) \hookrightarrow \Omega(P),$$

$\Omega^R(P)$ are G-right invariant differential forms on P and

$$\mathbf{H}(A) \cong \mathbf{H}\left(\Omega^R(P)\right) \xrightarrow{\ i\ } \mathbf{H}_{dR}(P).$$

The homomorphism i is an isomorphism if G is compact and connected.

(2) If $A = A(M; \mathcal{F}) \longrightarrow W$ is the Lie algebroid of a TC-foliation $\mathcal{F}$ on M (W is the so called basic manifold of the foliation $\mathcal{F}$), then [K3, Th. 6.2]

$$\Omega(A) \cong \Omega_b(M; \mathcal{F}),$$

$\Omega_b(M; \mathcal{F})$ is the algebra of $\mathcal{F}$-basic differential forms, therefore $\mathbf{H}(A) \cong \mathbf{H}_b(M; \mathcal{F})$ is the algebra of basic cohomology.

Below, we will propose a calculation of $\mathbf{H}(A)$ using the old technique of Koszul complexes and the so-called Chevalley theorems known for principal fibre bundles with structural Lie groups with reductive Lie algebras.

These Chevalley theorems (for pfb's) say that under some assumptions, the cohomology of the total space $\mathbf{H}(P)$ of a pfb P depends uniquely on the cohomology of the base manifold M and the characteristic classes (the Chern-Weil homomorphism $h_P : \left(\bigvee \mathfrak{g}^*\right)_{I_G} \longrightarrow \mathbf{H}(M)$). It turns out that this assertion has a counterpart for Lie algebroids.

3 Koszul complexes and Chevalley's theorem in the framework of Lie algebroids

3.1 Representations of Lie algebroids and invariant cross-sections

Consider an arbitrary transitive Lie algebroid A on a manifold M with the Atiyah sequence $0 \longrightarrow \boldsymbol{g} \longrightarrow A \overset{\#_A}{\longrightarrow} TM \longrightarrow 0$ and a vector bundle $\mathfrak{f}$ on M.

Definition 6. *By a representation of A on $\mathfrak{f}$ we mean a homomorphism of Lie algebroids*

$$T : A \longrightarrow \mathbf{A}(\mathfrak{f}).$$

Look at the induced homomorphism of Atiyah sequences:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \boldsymbol{g} & \overset{T^+}{\longrightarrow} & \operatorname{End}(\mathfrak{f}) \\ \downarrow & & \downarrow \\ A & \overset{T}{\longrightarrow} & \mathbf{A}(\mathfrak{f}) \\ \downarrow & & \downarrow \\ TM & = & TM \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

At each point x we get a representation of the isotropy Lie algebra $\boldsymbol{g}_{|x}$ on the vector space $\mathfrak{f}_{|x}$,

$$T_x^+ : \boldsymbol{g}_{|x} \longrightarrow \operatorname{End}\left(\mathfrak{f}_{|x}\right).$$

For a cross-section $\xi \in \operatorname{Sec} A$ its image $T\xi \in \operatorname{Sec} \mathbf{A}(\mathfrak{f})$ determines a covariant differential operator

$$\mathcal{L}_{T\xi} : \operatorname{Sec} \mathfrak{f} \longrightarrow \operatorname{Sec} \mathfrak{f}.$$

Example 7. *(The representation of a Lie algebroid induced by a representation of a pfb) Let G be any Lie group and $\mu : G \longrightarrow GL(V)$ be any representation of G on a vector space V. If $F : P \longrightarrow \mathrm{L}\mathfrak{f}$ is a μ-homomorphism of pfb's (called a μ-representation of P on $\mathfrak{f}$) then its Lie algebroid's differential $F_* : \mathbf{A}(P) \longrightarrow \mathbf{A}(\mathfrak{f})$ is a representation of $\mathbf{A}(P)$ on $\mathfrak{f}$.*

Definition 8. *A cross-section $\nu \in \mathrm{Sec}\,\mathfrak{f}$ is called T-invariant (or T-parallel) if it belongs to the kernel of $\mathcal{L}_{T\xi}$ for each ξ, i.e.*

$$\mathcal{L}_{T\xi}(\nu) = 0 \quad \textit{for all} \;\; \xi \in \mathrm{Sec}\,A.$$

The space of all T-invariant cross-sections is denoted by $(\mathrm{Sec}\,\mathfrak{f})_{I_T}$. If $\nu \in \mathrm{Sec}\,\mathfrak{f}$ is invariant then its value ν_x at x is invariant with respect to $T^+_{|x} : \boldsymbol{g}_{|x} \to \mathrm{End}\left(\mathfrak{f}_{|x}\right)$, i.e.

$$\nu_x \in \left(\mathfrak{f}_{|x}\right)_{I_{T^+_x}}.$$

One can prove that for each transitive Lie algebroid A and each representation $T : A \longrightarrow \mathbf{A}(\mathfrak{f})$ the following theorem holds.

Theorem 9. *If ν_1 and ν_2 are T-invariant cross-sections of $\mathfrak{f}$ and they are equal at some point $x_0 \in M$, $\nu_1(x_0) = \nu_2(x_0)$, then they are equal globally, $\nu_1 = \nu_2$ (M is assumed to be connected), see [M], [K2].*

Therefore, the evaluation map

$$(\mathrm{Sec}\,\mathfrak{f})_{I_T} \longrightarrow \left(\mathfrak{f}_{|x}\right)_{I_{T^+_x}}, \quad \nu \longmapsto \nu(x),$$

is a monomorphism. Denote its image by

$$\left(\widetilde{\mathfrak{f}_{|x}}\right)_{I_{T^+_x}};$$

it contains all invariant vectors $u \in \left(\mathfrak{f}_{|x}\right)_{I_{T^+_x}}$ which can be extended to globally defined invariant cross-sections, i.e.

$$(\mathrm{Sec}\,\mathfrak{f})_{I_T} \cong \left(\widetilde{\mathfrak{f}_{|x}}\right)_{I_{T^+_x}} \subset \left(\mathfrak{f}_{|x}\right)_{I_{T^+_x}}.$$

Moreover, each invariant vector $u \in \left(\mathfrak{f}_{|x}\right)_{I^o\left(T^+_x\right)}$ can be extended to a locally defined (on some neighbourhood of x) invariant cross-section of the vector bundle $\mathfrak{f}$.

There is a wider class of Lie algebroids (integrable and nonintegrable) and representations where each invariant vector $u \in \left(\mathfrak{f}_{|x}\right)_{I_{T_x^+}}$ can be extended to globally defined invariant cross-sections.

Theorem 10 ([K1]). *Let P be a connected G-principal fibre bundle (G can be disconnected) and let $F : P \longrightarrow \mathrm{L}\,\mathfrak{f}$ be any μ-representation of P on $\mathfrak{f}$ where $\mu : G \longrightarrow GL(V)$ is a representation of G on V. Denote by $\mu_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ the differential of μ (it is a representation of the Lie algebra $\mathfrak{g}$ of G on V). Then for the induced representation $F_* : \mathbf{A}(P) \longrightarrow \mathbf{A}(\mathfrak{f})$ of the Lie algebroid $\mathbf{A}(P)$ on $\mathfrak{f}$ we have*

$$(\mathrm{Sec}\,\mathfrak{f})_{I_{F_*}} \cong V_{I(\mu)} \quad \subset \quad V_{I(\mu_*)} \cong \left(\mathfrak{f}_{|x}\right)_{I_{F_{*x}^+}} .$$

If additionally G is connected then each invariant vector $v \in \left(\mathfrak{f}_{|x}\right)_{I_{F_{*x}^+}}$ (with respect to the representation $F_{*|x}^+$) can be extended to a globally defined F_*-invariant cross-section of $\mathfrak{f}$ and

$$(\mathrm{Sec}\,\mathfrak{f})_{I_{F_*}} \cong V_{I(\mu)} = V_{I(\mu_*)} \cong \left(\mathfrak{f}_{|x}\right)_{I_{F_{*x}^+}} .$$

If G is not connected then there may be invariant vectors which sometimes extend to global cross-sections and sometimes not (the Pfaffian is a typical example).

A representation $T : A \longrightarrow \mathbf{A}(\mathfrak{f})$ extends to representations on the associated vector bundles such as the dual bundle $\mathfrak{f}^*$, the exterior and symmetric powers $\bigwedge \mathfrak{f}^*$, $\bigvee^l \mathfrak{f}^*$ and their tensor products $\bigwedge \mathfrak{f}^* \otimes \bigvee^l \mathfrak{f}^*$.

3.2 Weil algebra for Lie algebroids [K1]

A fundamental example of a representation is the adjoint representation of A on the adjoint Lie algebra bundle $\boldsymbol{g}$ defined by

$$\begin{aligned} ad_A &: A \longrightarrow \mathbf{A}(\boldsymbol{g}), \\ ad_A(\xi) &: \mathrm{Sec}\,\boldsymbol{g} \longrightarrow \mathrm{Sec}\,\boldsymbol{g}, \quad \nu \longmapsto [\![\xi, \nu]\!]. \end{aligned}$$

Clearly the induced representation at an arbitrary point x, $\left(ad_A^+\right)_{|x}$, is the adjoint representation of the Lie algebra $\boldsymbol{g}_{|x}$,

$$\left(ad_A^+\right)_{|x} = ad_{\boldsymbol{g}_{|x}} : \boldsymbol{g}_{|x} \longrightarrow \mathrm{End}\left(\boldsymbol{g}_{|x}\right).$$

The adjoint representation ad_A induces representations on the associated vector bundles $\bigwedge \boldsymbol{g}^*$, $\bigvee^l \boldsymbol{g}^*$ (the skew symmetric and symmetric powers of the dual bundle $\boldsymbol{g}^*$) and on

$$(W\boldsymbol{g})^{k,2l} := \bigwedge \boldsymbol{g}^* \otimes \bigvee^l \boldsymbol{g}^*,$$

denoted also by ad_A. Put

$$(\mathcal{W}\boldsymbol{g})^{k,2l} = \operatorname{Sec}(W\boldsymbol{g})^{k,2l},$$
$$\mathcal{W}\boldsymbol{g} = \bigoplus_{k,l} \operatorname{Sec}(W\boldsymbol{g})^{k,2l}.$$

For a point $x \in M$ we take the anticommutative (bi)graded tensor product of anticommutative graded algebras, i.e. the Weil algebra of the space $\boldsymbol{g}_{|x}$,

$$W\boldsymbol{g}_{|x} = \bigwedge \boldsymbol{g}^*_{|x} \bigotimes \bigvee \boldsymbol{g}^*_{|x},$$
$$W\boldsymbol{g}_{|x} = \bigoplus_{k.l} \left(W\boldsymbol{g}_{|x}\right)^{k,2l}, \quad \left(W\boldsymbol{g}_{|x}\right)^{k,2l} = \bigwedge^k \boldsymbol{g}^*_{|x} \bigotimes \bigvee^l \boldsymbol{g}^*_{|x}.$$

The module $\mathcal{W}\boldsymbol{g}$ is a bigraded algebra with multiplication defined pointwise, called the *Weil algebra of the Lie algebroid A.*

In the space $W\boldsymbol{g}_{|x} = \bigwedge \boldsymbol{g}^*_{|x} \bigotimes \bigvee \boldsymbol{g}^*_{|x}$ (as for an arbitrary Lie algebra) there exist three standard operators: the substitution operator, the differential, and the adjoint representation, here denoted by

$$(\iota_x)_\nu\,,\ \delta_{W_x}\,,\ (\theta_x)_\nu\,, \quad \nu \in \boldsymbol{g}_{|x}.$$

It is easy to see that the adjoint representation $\theta_x^{k,2l} : \boldsymbol{g}_{|x} \to \operatorname{End}\left(W\boldsymbol{g}_{|x}\right)^{k,2l}$ is induced by the adjoint representation ad_A of the Lie algebroid A on $(W\boldsymbol{g})^{k,2l} = \bigwedge^k \boldsymbol{g}^* \bigotimes \bigvee^l \boldsymbol{g}^*$, $k, l \geq 0$, at a point x.

We have:

(a) $(\iota_\nu)_x$ is an antiderivation of degree -1 defined by

$$(\iota_\nu)_x (\Phi \otimes \Gamma) = (\iota_\nu)_x \Phi \otimes \Gamma,$$

$\Phi \in \bigwedge \boldsymbol{g}^*_{|x}$, $\Gamma \in \bigvee \boldsymbol{g}^*_{|x}$,

(b) δ_{W_x} is an antiderivation of degree $+1$ defined by

$$\delta_{W_x}\left(h^*\otimes 1\right)=1\otimes h^*+\delta_{\boldsymbol{g}_{|x}}h^*\otimes 1,$$

where $h^*\in\boldsymbol{g}^*_{|x}$, $\delta_{\boldsymbol{g}_{|x}}$ is the Chevalley-Eilenberg differential

$$\delta_{W_x}\left(1\otimes h^*\right)\in\left(W\boldsymbol{g}\right)^{1,2}=\boldsymbol{g}^*\otimes\boldsymbol{g}^*,$$

such that

$$\left(\iota_\nu\right)_x\left(\delta_{W_x}\left(1\otimes h^*\right)\right)=\left(\theta_x\right)_\nu h^*.$$

The operators $\left(\iota_x\right)_\nu,\ \delta_{W_x},\ \left(\theta_x\right)_\nu,\ x\in M$, together give operators on smooth cross-sections

$$\iota_\nu,\ \delta_{\mathcal{W}},\ \theta_\nu:\mathcal{W}\boldsymbol{g}\longrightarrow\mathcal{W}\boldsymbol{g},\quad \nu\in\operatorname{Sec}\boldsymbol{g}.$$

The cross-section $\Theta\in\mathcal{W}\boldsymbol{g}$ is called horizontal if $\iota_\nu\Theta=0$ for all $\nu\in\operatorname{Sec}\boldsymbol{g}$. Denote by

$$\left(\mathcal{W}\boldsymbol{g}\right)_\iota$$

the space of horizontal elements.

Lemma 11. *The space $\left(\mathcal{W}\boldsymbol{g}\right)_\iota$ of horizontal elements is a subalgebra of the Weil algebra $\mathcal{W}\boldsymbol{g}$ and contains only symmetric tensors:*

$$\left(\mathcal{W}\boldsymbol{g}\right)_\iota=\bigoplus\nolimits_l\operatorname{Sec}\bigvee\nolimits^l\boldsymbol{g}^*.$$

Denote the space of global cross-sections of the vector bundle

$$\left(W\boldsymbol{g}\right)^{k,2l}=\bigwedge\nolimits^k\boldsymbol{g}^*\bigotimes\bigvee\nolimits^l\boldsymbol{g}^*$$

invariant with respect to the adjoint representation of A on $\left(W\boldsymbol{g}\right)^{k,2l}$ (for brevity) by

$$\left(\mathcal{W}\boldsymbol{g}\right)^{k,2l}_{I^o}\subset\left(\mathcal{W}\boldsymbol{g}\right)^{k,2l}$$

and put

$$\left(\mathcal{W}\boldsymbol{g}\right)_{I^o}=\bigoplus\nolimits_{k,l}\left(\mathcal{W}\boldsymbol{g}\right)^{k,2l}_{I^o}\subset\mathcal{W}\boldsymbol{g}.$$

Proposition 12. *$\left(\mathcal{W}\boldsymbol{g}\right)_{I^o}$ is a subalgebra of the Weil algebra $\mathcal{W}\boldsymbol{g}$. Denote by $\left(\mathcal{W}\boldsymbol{g}\right)_{I^o,\iota}$ the subalgebra of invariant and horizontal elements of the Weil algebra $\mathcal{W}\boldsymbol{g}$. The operator $\delta_{\mathcal{W}}:\mathcal{W}\boldsymbol{g}\longrightarrow\mathcal{W}\boldsymbol{g}$ maps invariant elements of $\mathcal{W}\boldsymbol{g}$ into invariant ones defining an antiderivation*

$$\delta_{\mathcal{W},I^o}:\left(\mathcal{W}\boldsymbol{g}\right)_{I^o}\longrightarrow\left(\mathcal{W}\boldsymbol{g}\right)_{I^o},$$

and

$$\delta_{\mathcal{W},I^o}\mid\left(\mathcal{W}\boldsymbol{g}\right)_{I^o,\iota}=0.$$

3.3 Connections and the Chern-Weil homomorphism of Lie algebroids

Definition 13. *By a connection in a (transitive) Lie algebroid A we mean a splitting $\nabla : TM \longrightarrow A$ of the Atiyah sequence,*

$$0 \longrightarrow \boldsymbol{g} \longrightarrow A \underset{\overleftarrow{\nabla}}{\longrightarrow} TM \to 0.$$

If $\mathbf{A} = A(P)$ is the Lie algebroid of a G-principal fibre bundle $P(M,G)$ then connections in $\mathbf{A}(P)$ correspond 1-1 to usual connections in P.

Fix an arbitrary connection ∇ in A and consider:

a) the *connection form* $\omega : A \longrightarrow \boldsymbol{g}$, i.e. the 1-form on A with values in $\boldsymbol{g}$ ($\omega|\boldsymbol{g} = Id$ and $\ker\omega = \operatorname{Im}\nabla$),

$$\omega \in \Omega^1(A;\boldsymbol{g}),$$

b) the *curvature form* of ∇,

$$\Omega \in \Omega^2(A;\boldsymbol{g}),$$

defined by

$$\Omega(\xi_1,\xi_2) = \omega[\![H\xi_1, H\xi_2]\!], \quad \xi_1,\xi_2 \in \operatorname{Sec} A,$$

where $H = Id - \omega : A \longrightarrow A$ is the horizontal projection,

c) the identification $\Omega(A) = \Omega\left(M; \bigwedge \boldsymbol{g}^*\right)$.

For each point $x \in M$ the mappings $\omega_{|x} : A_{|x} \longrightarrow \boldsymbol{g}_{|x}$ and $\Omega_{|x} : \bigwedge^2 A_{|x} \longrightarrow \boldsymbol{g}_{|x}$ determine linear mappings

$$\chi_{\omega,x} : \boldsymbol{g}^*_{|x} \longrightarrow A^*_{|x} \subset \bigwedge A^*_{|x}, \quad h^* \longmapsto h^* \circ \omega_{|x},$$

and

$$\chi_{\Omega,x} : \boldsymbol{g}^*_{|x} \longrightarrow \bigwedge^2 A^*_{|x} \subset \bigwedge A^*_{|x}, \quad h^* \longmapsto h^* \circ \Omega_{|x}.$$

By the universal properties of the exterior algebra $\bigwedge \boldsymbol{g}^*_{|x}$ and the symmetric algebra $\bigvee \boldsymbol{g}^*_{|x}$ we obtain the existence and uniqueness of homomorphisms of algebras of degree 0, extending the above ones,

$$\chi^{\wedge}_{\omega,x} : \bigwedge \boldsymbol{g}^*_{|x} \longrightarrow \bigwedge A^*_{|x},$$
$$\chi^{\vee}_{\Omega,x} : \bigvee \boldsymbol{g}^*_{|x} \longrightarrow \bigwedge^{\mathrm{ev}} A^*_{|x}$$

(such that $1 \longmapsto 1$). The above morphisms define a homomorphism of algebras

$$\chi_{W,x} : W\boldsymbol{g}_{|x} = \bigwedge \boldsymbol{g}^*_{|x} \bigotimes \bigvee \boldsymbol{g}^*_{|x} \longrightarrow \bigwedge A^*_{|x},$$
$$\chi_{W,x}\left(\Phi_x \otimes \Gamma_x\right) = \chi^{\wedge}_{\omega,x}\left(\Phi_x\right) \wedge \chi^{\vee}_{\Omega,x}\left(\Gamma_x\right).$$

Passing to smooth cross-sections we obtain homomorphisms of algebras

$$\chi^{\wedge}_{\omega} : \operatorname{Sec} \bigwedge \boldsymbol{g}^* \longrightarrow \Omega\left(A\right),$$
$$\chi^{\vee}_{\Omega} : \bigoplus^{l} \operatorname{Sec} \bigvee^{l} \boldsymbol{g}^* \longrightarrow \Omega^{\mathrm{ev}}\left(A\right),$$

and

$$\chi_W : \mathcal{W}\boldsymbol{g} \rightarrow \Omega\left(A\right)$$
$$\chi_W\left(\Phi \otimes \Gamma\right) = \chi^{\wedge}_{\omega}\left(\Phi\right) \wedge \chi^{\vee}_{\Omega}\left(\Gamma\right).$$

Following [G-H-V, Vol. III, p. 341], χ_W is called the *classifying homomorphism* corresponding to the connection ∇.

One can prove that for $\Gamma \in \operatorname{Sec} \bigvee^{l} \boldsymbol{g}^*$,

$$\chi^{\vee}_{\Omega}\left(\Gamma\right) = \frac{1}{k!}\langle \Gamma, \underbrace{\Omega \vee \cdots \vee \Omega}_{l \text{ times}} \rangle$$

(the notation $\Omega \vee \cdots \vee \Omega$ comes from [G-H-V, Vol. II], it is the usual skew multiplication of differential forms whose values are multiplied according to the multilinear symmetric mapping $\vee : \boldsymbol{g} \times \cdots \times \boldsymbol{g} \longrightarrow \bigvee^{l} \boldsymbol{g}$).

Theorem 14. *(a) The classifying homomorphism χ_W commutes with the substitution operators ι_ν, $\nu \in \operatorname{Sec}\boldsymbol{g}$:*

$$\iota_\nu\left(\chi_W \Theta\right) = \chi_W\left(\iota_\nu \Theta\right).$$

(b) The homomorphism $\chi_{W,I^o} : \left(\mathcal{W}\boldsymbol{g}\right)_{I^o} \longrightarrow \boldsymbol{\Omega}\left(A\right)$, the restriction of χ_W to the invariant elements, commutes with the differentials $\delta_{\mathcal{W},I^o}$ and d_A:

$$d_A\left(\chi_{W,I^o}\Theta\right) = \chi_{W,I^o}\left(\delta_{\mathcal{W},I^o}\Theta\right).$$

As a simple consequence we obtain the Chern-Weil homomorphism of the Lie algebroid A. Consider the restriction $\chi_{W,I^o,\iota}$ of $\chi_W : \mathcal{W}\boldsymbol{g} \longrightarrow \Omega\left(A\right)$ to the horizontal invariant elements. Since

$\delta_{\mathcal{W},I^o} \mid (\mathcal{W}\boldsymbol{g})_{I^o,\iota} = 0$ we see that all differential forms in $\operatorname{Im}\chi_{W,I^o,\iota}$ are closed and horizontal:

$$\chi_{W,I^o,\iota} : (\mathcal{W}\boldsymbol{g})_{I^o,\iota} \to Z_\iota(A)\,;$$

on the other hand, $(\mathcal{W}\boldsymbol{g})_\iota = \bigoplus_l \operatorname{Sec}\bigvee^l \boldsymbol{g}^*$, therefore

$$(\mathcal{W}\boldsymbol{g})_{I^o,\iota} = \bigoplus_l \left(\operatorname{Sec}\bigvee^l \boldsymbol{g}^*\right)_{I^o}$$

and $\Omega(M) \overset{f}{\cong} \boldsymbol{\Omega}_\iota(A)$ (via the anchor $f(\Psi)_x(v_1,...,v_k) = (\Psi)_x(\#v_1,...,\#v_k)$) and

$$\begin{array}{rcll}
\chi_{W,I^o,\iota} : (\mathcal{W}\boldsymbol{g})_{I^o,\iota} & \longrightarrow & Z_\iota(A) & \\
\| & & \| & \\
h_A : \bigoplus_l \left(\operatorname{Sec}\bigvee^l \boldsymbol{g}^*\right)_{I^o} & \longrightarrow & Z(M) & \longrightarrow \mathbf{H}(M)
\end{array}$$

3.4 Koszul complexes and Chevalley's theorem in the framework of Lie algebroids

We now apply the technique of Koszul complexes and Chevalley's theorem [G-H-V, Vol. III] to Lie algebroids. We recall that the adjoint representation ad_A of A on $(W\boldsymbol{g})^{k,2l} = \bigwedge^k \boldsymbol{g}^* \bigotimes \bigvee^l \boldsymbol{g}^*$ determines at each point x the adjoint representation $\theta_x^{k,2l} : \boldsymbol{g}_{|x} \longrightarrow \operatorname{End}\left(W\boldsymbol{g}_{|x}\right)^{k,2l}$, which together determine the representation on the Weil algebra $\theta_x : \boldsymbol{g}_{|x} \longrightarrow \operatorname{End}\left(W\boldsymbol{g}_{|x}\right)$. Denote by $\left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$ the subspace of $\left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$ consisting of all vectors whose homogeneous parts can be extended to globally defined cross-sections of $(W\boldsymbol{g})^{k,2l} = \bigwedge^k \boldsymbol{g}^* \bigotimes \bigvee^l \boldsymbol{g}^*$ invariant with respect to the adjoint representation of the Lie algebroid A,

$$(\mathcal{W}\boldsymbol{g})_{I^o} \cong \left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} \subset \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}.$$

We assume the following (rather strong) assumptions:

(A1) the isotropy Lie algebras $\boldsymbol{g}_{|x}$ are reductive, (A2) each homogeneous invariant element

$$\Theta_x \in \left(W\boldsymbol{g}_{|x}\right)^{k,2l}_{I_{\theta_x}} = \left(\bigwedge^k \boldsymbol{g}^*_{|x} \bigotimes \bigvee^l \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}$$

can be extended to a globally defined invariant cross-section of the vector bundle $\bigwedge^k \boldsymbol{g}^* \bigotimes \bigvee^l \boldsymbol{g}^*$, i.e.

$$(\mathcal{W}\boldsymbol{g})_{I^\circ} \cong \left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} = \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}.$$

(In particular, the cohomology vector bundle $\mathbf{H}(\boldsymbol{g})$, $\mathbf{H}(\boldsymbol{g})_x = \mathbf{H}\left(\boldsymbol{g}_{|x}\right)$, is trivial).

Now we return to an arbitrarily chosen connection ∇ in the Lie algebroid A, $0 \longrightarrow \boldsymbol{g} \longrightarrow A \underset{\overset{\longleftarrow}{\nabla}}{\longrightarrow} TM \longrightarrow 0$, and take $\chi_W : \mathcal{W}\boldsymbol{g} \longrightarrow \Omega(A)$, the classifying homomorphism corresponding to the connection ∇, and its restriction to the invariant elements,

$$\chi_{W,I^\circ} : (\mathcal{W}\boldsymbol{g})_{I^\circ} \cong \left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} = \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} \longrightarrow \Omega(A).$$

Now we use the assumed reductivity of the isotropy Lie algebras $\boldsymbol{g}_{|x}$. Let

$$P_x \subset \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}$$

be the graded primitive subspace. We recall that homogeneous primitive elements have odd degree (which implies that $\Phi \wedge \Phi = 0$ when $\Phi \in P_x$), therefore the inclusion $P_x \subset \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}$ extends to a homomorphism of algebras

$$\varkappa_x : \bigwedge P_x \longrightarrow \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}.$$

The Hopf-Samelson theorem [G-H-V, Vol. III, 5.18, Theorem III] says that if $\boldsymbol{g}_{|x}$ is reductive then $\varkappa_x$ is an isomorphism of graded algebras.

Further

$$\tau_x : P_x \longrightarrow \left(\bigvee^+ \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}}$$

denotes a fixed transgression in $\left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$, i.e. a linear mapping such that

(1) τ_x is homogeneous of degree $+1$, $\tau_x : P_x^{2r-1} \longrightarrow \left(\bigvee \boldsymbol{g}_{|x}^*\right)^{2r}_{I_{\theta_x}} = \left(\bigvee^r \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}}$,

(2) for each $\Phi \in P_x$ there exists $\Omega \in W^+\left(\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$ such that

$$\delta_{W_x}\Omega = 1 \otimes \tau_x\Phi \quad \text{and} \quad \Omega - \Phi \otimes 1 \in \left(\bigwedge \boldsymbol{g}_{|x}^* \otimes \bigvee^{j\geq 1} \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}}.$$

It turns out that we can demand that Ω depends linearly on Φ, and Φ and Ω are of the same degree, i.e. that there exists a linear mapping

$$\alpha_x : P_x \longrightarrow W^+\left(\boldsymbol{g}_{|x}\right)_{I_{\theta_x}},$$

homogeneous of degree 0, such that

(*) $\delta_{W_x}\left(\alpha_x\Phi\right) = 1 \otimes \tau_x\left(\Phi\right)$,

(**) $\alpha_x\left(\Phi\right) - \Phi \otimes 1 \in \left(\bigwedge \boldsymbol{g}_{|x}^* \otimes \bigvee^{j\geq 1} \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}}$.

In the following we fix such a mapping α_x. Now we can define a Koszul complex for the Lie algebroid. To this end we recall the homomorphism

$$\chi_{W,I^o,\iota} := \left(\bigvee \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}} \cong \bigoplus_l \left(\operatorname{Sec} \bigvee^l \boldsymbol{g}^*\right)_{I^o} \longrightarrow Z_\iota\left(A\right) \cong Z\left(M\right)$$

($Z\left(M\right)$ = closed differential forms on M),

(after passing to cohomology, this yields the Chern-Weil homomorphism of A). Composing it with the transgression $\tau_x : P_x \longrightarrow \left(\bigvee^+ \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}}$ we obtain

$$\tau_A : P_x \longrightarrow \left(\bigvee \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}} \longrightarrow Z\left(M\right) \subset \Omega\left(M\right).$$

Definition 15. *In the skew tensor product of the graded algebras*

$$\Omega\left(M\right) \otimes \left(\bigwedge \boldsymbol{g}_{|x}^*\right)_{I_{\theta_x}} = \Omega\left(M\right) \otimes \bigwedge P_x$$

we introduce the operator

$$\nabla_A : \Omega\left(M\right) \otimes \bigwedge P_x \to \Omega\left(M\right) \otimes \bigwedge P_x$$

uniquely determined by the conditions:

(1) $\nabla_A(z\otimes 1)=d(z)\otimes 1,$ */d the de Rham differential*

(2) $\nabla_A(z\otimes(\Phi_0\wedge\ldots\wedge\Phi_p)) = dz\otimes(\Phi_0\wedge\ldots\wedge\Phi_p) + (-1)^q\sum_{i=0}^{p}(-1)^i\tau_A(\Phi_i)\wedge z\otimes\left(\Phi_0\wedge\ldots\widehat{i}\ldots\wedge\Phi_p\right),\quad z\in\Omega^q(M),$ $\Phi_i\in P_x$. *In particular* $\nabla_A(z\otimes\Phi)=dz\otimes\Phi+(-1)^q\tau_A(\Phi)\wedge z\otimes 1$ *and* $\nabla_A(1\otimes\Phi)=\tau_A(\Phi)\otimes 1.$

Lemma 16. *The operator* ∇_A *is an antiderivation of square* 0*, homogeneous of degree* $+1$.

Definition 17. *The pair* $\left(\Omega(M)\otimes\bigwedge P_x,\nabla_A\right)$ *is called the Koszul complex of the Lie algebroid A.*

We see that the Koszul complex for a Lie algebroid depends only on the base manifold and the Chern-Weil homomorphism of A.

Now we define a Chevalley homomorphism. Take the restriction of the classifying homomorphism $\chi_W:\mathcal{W}\boldsymbol{g}\longrightarrow\Omega(A)$ to the invariant tensors,

$$\chi_{W,I^o}:(\mathcal{W}\boldsymbol{g})_{I^o}\cong\left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}=\left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}\longrightarrow\Omega(A).$$

Composing it with the mapping $\alpha_x:P_x\to W^+\left(\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}\subset\left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$,

$$P_x\xrightarrow{\alpha_x}W^+\left(\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}\subset\left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}\cong(\mathcal{W}\boldsymbol{g})_{I^o}\xrightarrow{\chi_{W,I^o}}\Omega(A,)$$

we obtain a linear mapping homogeneous of degree 0,

$$\vartheta_A:P_x\longrightarrow\Omega(A).$$

Hence, since $\Omega(A)$ is anticommutative and $P_x^k=0$ for even k, ϑ_A extends to a homomorphism of graded algebras

$$\vartheta_A^\wedge:\bigwedge P_x\longrightarrow\Omega(A).$$

Finally, we extend $\vartheta_A^\wedge:\bigwedge P_x\longrightarrow\Omega(A)$ to a homomorphism of graded algebras

$$\vartheta_A:\Omega(M)\otimes\bigwedge P_x\longrightarrow\Omega(A)$$

by setting

$$\vartheta_A(z\otimes\Phi)=\#_A^*(z)\wedge\vartheta_A^\wedge(\Phi)$$

($\#_A^*(z)$ is the pull back, via the anchor $\#_A$, of the differential form $z\in\Omega(M)$ to a horizontal one on the Lie algebroid A).

Definition 18. *The homomorphism* $\vartheta_A : \Omega(M) \otimes \bigwedge P_x \longrightarrow \Omega(A)$ *is called the Chevalley homomorphism of* A *associated with the connection* ∇ *and the linear map* α_x.

Theorem 19 (The fundamental theorem)**.** *(A) The Chevalley homomorphism* ϑ_A *is a homomorphism of graded differential algebras*

$$\vartheta_A : \left(\Omega(M) \otimes \bigwedge P_x, \nabla_A\right) \longrightarrow (\Omega(A), d_A).$$

(B) Under the assumptions (A1) and (A2), i.e. that the isotropy Lie algebras $\boldsymbol{g}_{|x}$ *are reductive, and* $\left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} = \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$, *the induced homomorphism in cohomology*

$$\vartheta_A^{\#} : \mathbf{H}\left(\Omega(M) \otimes \bigwedge P_x, \nabla_A\right) \longrightarrow \mathbf{H}(A)$$

is an isomorphism of graded algebras.

Proof. (A) It is sufficient to check the equality $d_A \circ \vartheta_A = \vartheta_A \circ \nabla_A$ on simple tensors $z \otimes 1$ and $1 \otimes \Phi$ $(\Phi \in P_x)$ only. We have

$$d_A \circ \vartheta_A(z \otimes 1) = d_A(\#_A^* z) = \#_A^*(dz) = \vartheta_A(dz \otimes 1) = \vartheta_A \circ \nabla_A(z \otimes 1),$$

and

$$\begin{aligned} d_A \circ \vartheta_A(1 \otimes \Phi) &= d_A(\vartheta_A^{\wedge}(\Phi)) = d_A(\chi_{W,I^o}(\alpha_x(\Phi))) \\ &\overset{\text{Th }(14)}{=} \chi_{W,I^o}(\delta_{\mathcal{W},I^o}(\alpha_x(\Phi))) = \chi_{W,I^o}(\delta_{W_x}(\alpha_x(\Phi))) \\ &\overset{(*)}{=} \chi_{W,I^o}(1 \otimes \tau_x(\Phi)) = \chi_{W,I^o}(\tau_x(\Phi)) = \#_A^*(\tau_A \Phi) \\ &= \vartheta_A(\tau_A \Phi \otimes 1) = \vartheta_A \circ \nabla_A(1 \otimes \Phi). \end{aligned}$$

(B) The proof is analogous to that in the classical case for principal fibre bundles [G-H-V, Vol. III, 9.3-4, p. 359]: we use some spectral sequences and the comparison theorem for the first terms (the mapping induced on the first terms is an isomorphism).

Step 1. Filtrations: For a given Lie algebroid A with the Atiyah sequence $0 \longrightarrow \boldsymbol{g} \longrightarrow A \longrightarrow TM \longrightarrow 0$ we consider the pair of real (infinite dimensional) Lie algebras $(\Gamma(A), \Gamma(\boldsymbol{g}))$ of global cross-sections of A and $\boldsymbol{g}$.

Following [H-S], [K-M-2], we introduce the Hochschild-Serre filtration in $\Omega(A)_j$ in $\Omega(A)$ as follows:

$$\Omega(A)_j = \left\{ \begin{bmatrix} \Omega(A) & \text{for} & j \leq 0, \\ \bigoplus_{k\geq j} \ \Omega(A)_j^k & \text{for} & j > 0. \end{bmatrix} \right.$$

where $\Omega(A)_j^k$ consists of all those k-differential forms $z \in \Omega^k(A)$ for which

$$z(\xi_1, ..., \xi_k) = 0$$

whenever $k - j + 1$ of the arguments $\xi_i \in \Gamma(A)$ belong to $\Gamma(\boldsymbol{g})$. In this way we obtain a graded filtered differential space and its spectral sequence $\left(E_{A,s}^{j,i}, d_{A,s}\right)$.

Analogously, following [G-H-V, Vol. III] we introduce in the space $\Omega(M) \otimes \bigwedge P_x$ the filtration

$$\left(\Omega(M) \otimes \bigwedge P_x\right)_j = \bigoplus_{k\geq j} \ \Omega(M)^k \otimes \bigwedge P_x.$$

We obtain a graded filtered differential space and its spectral sequence $\left(E_s^{j,i}, d_s\right)$.

Step 2. We show that the Chevalley homomorphism ϑ_A is filtration preserving. Firstly we notice that $\vartheta_A(z \otimes 1) = \#_A^*(z)$ and $\vartheta_A(1 \otimes \Phi) - \chi_{W,I^o}(\Phi \otimes 1) \in \Omega(A)_1$. The first statement is obvious. To prove the second, it is sufficient to consider the case $\Phi \in P_x$. According to (**) above it follows that

$$\begin{aligned} \vartheta_A(1 \otimes \Phi) - \chi_{W,I^o}(\Phi \otimes 1) &= \chi_{W,I^o}(\alpha_x \Phi) - \chi_{W,I^o}(\Phi \otimes 1) \\ &= \chi_{W,I^o}(\alpha_x \Phi - \Phi \otimes 1) \in \Omega(A)_1. \end{aligned} \tag{1}$$

By definition, $\vartheta_A\left[\Omega(M)^k \otimes 1\right] \subset \Omega(A)_k$. Since $\left(\Omega(M) \otimes \bigwedge P_x\right)_j$ is the ideal generated by $\bigoplus_{k\geq j} \ \Omega(M)^k \otimes 1$, and since $\Omega(A)_j$ is an ideal, this implies that ϑ_A preserves filtrations.

Step 3. We show that the mapping of the first terms of the spectral sequences,

$$\vartheta_{A,1} : E_1 \longrightarrow E_{A,1},$$

is an isomorphism. In view of the Comparison Theorem the induced homomorphism in cohomology $\vartheta_A^\# : \mathbf{H}\left(\Omega(M) \otimes \bigwedge P_x, \nabla_A\right) \longrightarrow \mathbf{H}(A)$ is an isomorphism.

We start by calculating the differential operators d_0 in E_0 and $d_{A,0}$ in $E_{A,0}$. It is immediate from the definitions that

$$\nabla_A : \Omega(M)^k \otimes \bigwedge^l P_x \longrightarrow \left(\Omega(M) \otimes \bigwedge P_x\right)_{k+1}, \quad k,l \geq 0.$$

It follows that $d_0 = 0$. On the other hand, recall from [K-M-2, Conclusion 5.2] that $E^j_{A,0} = \Omega^j\left(M; \bigwedge \boldsymbol{g}^*\right)$ and that the differential $d_{A,0}$ becomes the Chevalley-Eilenberg differential of values at each point.

Now, we show that $\vartheta_{A,0} : E_0 \longrightarrow E_{A,0}$ simply comes from the inclusion map

$$j : \Omega(M) \otimes \bigwedge P_x = \Omega(M) \otimes \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}} \longrightarrow \Omega\left(M; \bigwedge \boldsymbol{g}^*\right)$$

and its values are $d_{A,0}$-closed. In fact, j is homogeneous of bidegree zero. Thus we need only show that

$$\vartheta_A - j : \Omega^k(M) \otimes \bigwedge P_x \longrightarrow \Omega(A)_{k+1}.$$

But $j(z \otimes \Phi) = \#^*_A \wedge \chi_{W,I^o}(\Phi \otimes 1)$, and so property (1) yields, for $z \in \Omega^k(M)$,

$$(\vartheta_A - j)(z \otimes \Phi) = \#^*_A z \wedge (\vartheta_A(1 \otimes \Phi) - \chi_{W,I^o}(\Phi \otimes 1)) \in \Omega(A)_{k+1}.$$

To prove Step 3 we need only show that $(\vartheta_{A,0})^{\#} : \mathbf{H}(E_0, d_0) \longrightarrow \mathbf{H}(E_{A,0}, d_{A,0})$ is an isomorphism. In view of the formulae for d_0 and $d_{A,0}$ it remains to show that the inclusion map j induces an isomorphism

$$j^{\#} : \Omega(M) \otimes \bigwedge P_x = \Omega(M) \otimes \left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}} \longrightarrow \left(\Omega\left(M; \bigwedge \boldsymbol{g}^*\right), d_{A,0}\right).$$

Since the Lie algebras $\boldsymbol{g}_{|x}$ are reductive (assumption (A1)), by the structural theorem for reductive Lie algebras [G-H-V, Vol. III, s. 5.12, Theorem 1] we have $\left(\bigwedge \boldsymbol{g}^*_{|x}\right)_{I_{\theta_x}} = \mathbf{H}\left(\boldsymbol{g}_{|x}\right)$. Therefore, the isomorphism property of $j^{\#}$ follows immediately from assumption (A2): $\left(\Omega\left(M; \bigwedge \boldsymbol{g}^*\right), d_{A,0}\right) = \Omega(M; \mathbf{H}(\boldsymbol{g})) = \Omega\left(M; \mathbf{H}\left(\boldsymbol{g}_{|x}\right)\right)$. The proof of the fundamental theorem is now complete. $\square$

Problem 20. *What can we do in the case when* $\left(\tilde{W}\boldsymbol{g}_{|x}\right)_{I_{\theta_x}} \subsetneq \left(W\boldsymbol{g}_{|x}\right)_{I_{\theta_x}}$ *to calculate* $\mathbf{H}(A)$ *? [The simplest examples of this case come from connected pfb's with disconnected structural Lie groups].*

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