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Free $\mathcal{A}_4$-actions on products of spheres

We summarize what is known about free actions of $\mathcal{A}_4$, the alternating group on four letters, on products of spheres. New results are also included: in particular, we prove that $\mathcal{A}_4$ acts freely on $S_n \times S_n \times S_n$ if and only if $n = 1, 3, 7$.

1 Introduction

The following result first appeared in a 1979 paper of Oliver.

Proposition 1.1 ([15, p. 547]). *Let G be a finite group. For any integer $n \geq 1$, there exists an integer $k \geq 1$ such that G acts freely on $(S^{2n-1})^k$.*

Proof. Let G be a finite group, $H \subseteq G$ a subgroup and X an H-space. Then the space $\mathrm{Map}_H(G, X)$ of all H-equivariant maps $G \to X$, endowed with the compact-open topology, is a G-space in the obvious way. Note that if X is a free H-space, then $\mathrm{Map}_H(G, X)$ is also a free H-space. Furthermore,

$$\mathrm{Map}_H(G, X) \approx X^{[G\,:\,H]},$$

where $[G : H]$ denotes the index of H in G; the homeomorphism is given by the evaluation on a set of representatives of cosets of H.

For any non-trivial element $g \in G$, the cyclic group $\langle g \rangle \subseteq G$ acts freely on any odd-dimensional sphere S^{2n-1}. Then G acts on

$$M_g = \mathrm{Map}_{\langle g \rangle}(G, S^{2n-1}) \approx (S^{2n-1})^{[G\,:\,\langle g \rangle]},$$

with the subgroup $\langle g \rangle \subseteq G$ acting freely. The product $\prod_{g \in G,\, g \neq 1} M_g$ with the diagonal G-action is a free G-space. $\square$

The downside of the construction outlined in Proposition 1.1 is that it is very inefficient: the number of spheres in the resulting product is *very* unlikely to be minimal. For example, for G the elementary abelian p-group of rank r, it yields an action on $(S^{2n-1})^{p^{r-1}(p^r-1)}$, while such a group clearly acts freely on $(S^{2n-1})^r$. This raises an interesting problem:

Given a finite group G, determine the minimal number $k = k(G)$ such that G acts freely on a finite CW complex homotopy equivalent to $(S^n)^k$ for some $n \geq 1$.

A lot of effort has been put into determining k for various classes of groups. For example, the solution of the spherical space form problem asserts that $k(G) = 1$ if and only if G has periodic cohomology (see [9]).

In [5], we investigated k for the class of simple alternating groups, i.e., with $\mathcal{A}_4$ excluded; the main result is that $k(\mathcal{A}_d) > d - 1$ for many values of $d \geq 5$. Our goal for this article is twofolds. Firstly, we want to summarize what is known about and completely understand $k(\mathcal{A}_4)$, building on previous insights provided by Oliver [15] and Plakhta [16]. This is achieved in Section 3, and the main results there are that $\mathcal{A}_4$ cannot act freely on any finite-dimensional CW complex homotopy equivalent to $S^n \times S^n$ (Proposition 3.2), and that $\mathcal{A}_4$ acts freely on $S^n \times S^n \times S^n$ if and only if $n = 1, 3, 7$ (Theorem 3.4). From this point of view, this article can be seen as complementary to [5].

Secondly, in Section 4, we explain how $\mathcal{A}_4$ can act freely on $S^m \times S^n$ if $m \neq n$. Then we look at those pairs (m, n) for which there exists a free $\mathcal{A}_4$-action on $S^m \times S^n$.

Apart from that, Section 5 is dedicated to free actions of the symmetric group on three letters $\mathcal{S}_3$. This is intended mainly for the sake of completeness, but the methods presented therein also generalize the the class of dihedral groups.

Notation. All considered actions are topological, i.e., by homeomorphisms. A 'closed manifold' is taken to mean a compact and connected manifold without boundary.

2 Preliminaries

Results of Subsections 2.1 and 2.2 are indispensable to the whole Section 3. Subsection 2.3 is relevant only to Example 3.5.

2.1 Integral representations of $\mathbb{Z}_3$

Write $GL(n, \mathbb{Z})$ for the general linear group of degree n over the integers.

Lemma 2.1. (1) *Up to conjugation, there exists precisely one subgroup of order* 3 *in* $GL(2, \mathbb{Z})$*:*

$$\left\langle \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right\rangle.$$

(2) *Up to conjugation, there exist two subgroups of order* 3 *in* $GL(3, \mathbb{Z})$*:*

$$\left\langle \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\rangle.$$

Lemma 2.1 is a consequence of the general theory of integral representations of cyclic groups of prime order (see [8, §74]), but can also be derived by elementary calculations.

2.2 Adem's results

Recall that if a group G acts on a space X, then the cohomology groups of X assume the structure of a G-module. We will need the following result due to Adem, which relates the nature of the G-action on X and the G-module structure on $H^n(X; \mathbb{Z})$ in the case when G is a cyclic group of prime order and X is a product of n-dimensional spheres.

Theorem 2.2 ([1, Corollary 4.8]). *Let k, n be positive integers, p an odd prime. If $\mathbb{Z}_p$ acts freely on a finite-dimensional CW complex X such that the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*\big((S^n)^k;\mathbb{Z}\big)$ are isomorphic, then $H^n(X;\mathbb{Z})$ splits off a trivial direct summand as a $\mathbb{Z}_p$-module.*

We will also make use of the following basic observation, again due to Adem.

Proposition 2.3 ([1, Proposition 2.1]). *Let $n \neq 1, 3, 7$. If $f\colon (S^n)^k \to (S^n)^k$ is a map such that $f^*\colon H^n\big((S^n)^k;\mathbb{Z}\big) \to H^n\big((S^n)^k;\mathbb{Z}\big)$ is an automorphism, then the modulo 2 reduction of f^* is a permutation matrix in the usual basis.*

2.3 Borel manifolds

Recall that a closed manifold M is *aspherical* if the higher homotopy groups of M vanish, i.e., if $\pi_i(M) = 0$ for $i \geq 2$, or, equivalently, if the universal cover of M is contractible. It is classically known that aspherical manifolds are classified up to homotopy by their fundamental groups: two aspherical manifolds are homotopy equivalent if and only if they have isomorphic fundamental groups.

On the geometric level, we have Borel manifolds: a closed manifold M is called a *Borel manifold* if every closed manifold homotopy equivalent to M is automatically homeomorphic to M. Crucially for us, examples of Borel manifolds include compact solvmanifolds (see [4, Chapter III, Section 4]); in particular, products of circles.

Proposition 2.4. *Let M be an n-dimensional aspherical Borel manifold. A finite group G acts freely on M if and only if there exists a group extension*

$$1 \to \pi_1(M) \to \pi_1(N) \to G \to 1,$$

where N is a closed n-dimensional aspherical manifold.

Proof. Suppose a finite group G acts freely on M. The orbit space M/G is well-known to be a closed n-dimensional manifold. Inspection of the long exact sequence of homotopy groups of the covering $M \to M/G$ reveals that M/G is aspherical and that $\pi_1(M/G)$ fits into the extension $1 \to \pi_1(M) \to \pi_1(M/G) \to G \to 1$.

Conversely, consider a group extension

$$1 \to \pi_1(M) \to \pi_1(N) \to G \to 1.$$

Let $\tilde{M}$ be the covering space of N corresponding to the subgroup $\pi_1(M) \subseteq \pi_1(N)$. Then $G \cong \pi_1(N)/\pi_1(M)$ acts freely on $\tilde{M}$. Since $\tilde{M}$ is a closed aspherical manifold with $\pi_1(\tilde{M}) \cong \pi_1(M)$, we have that $\tilde{M}$ is homotopy equivalent to M. But M is a Borel manifold by hypothesis, so $\tilde{M}$ and M are homeomorphic, and the conclusion follows. □

Remark 2.5. The famous Borel conjecture states that every closed, aspherical manifold is a Borel manifold. (See [10] for more details.)

3 Free $\mathcal{A}_4$-actions on $(S^n)^k$

Recall the following fundamental result due to Oliver:

Theorem 3.1 ([15, Theorem 1]). *Let k, n be positive integers. If the alternating group $\mathcal{A}_4$ acts freely on a finite-dimensional CW complex X such that the cohomology rings $H^*(X;\mathbb{Z}_2)$ and $H^*\big((S^n)^k;\mathbb{Z}_2\big)$ are isomorphic, then the action induced on $H^n(X;\mathbb{Z}_2)$ is non-trivial.*

Oliver combined Theorem 3.1 and the Lefschetz Fixed Point Theorem to prove that $\mathcal{A}_4$ cannot act freely on any finite CW complex X such that the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*(S^n \times S^n;\mathbb{Z})$ are isomorphic ([15, Theorem 2]). We can improve this statement in the following manner:

Proposition 3.2 ([5, Corollary 3.2]). *The alternating group $\mathcal{A}_4$ cannot act freely on any finite-dimensional CW complex X such that the cohomology rings $H^*(X;\mathbb{Z})$ and $H^*(S^n \times S^n;\mathbb{Z})$ are isomorphic, where n is any positive integer.*

Proof. Suppose that $\mathcal{A}_4$ acts freely on X as above. In view of Theorem 3.1, $H^n(X;\mathbb{Z})$ is a non-trivial $\mathcal{A}_4$-module. But $\mathcal{A}_4$ is generated by elements of order 3, so $H^n(X;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is also a non-trivial $\mathbb{Z}_3$-module for some subgroup $\mathbb{Z}_3 \subseteq \mathcal{A}_4$. By Lemma 2.1, the $\mathbb{Z}_3$-module structure on $H^n(X;\mathbb{Z})$ comes from

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix},$$

hence it does not split off a trivial direct summand. This contradicts Theorem 2.2. □

Remark 3.3. The finite-dimensionality hypothesis of Proposition 3.2 cannot be dropped: the product $E\mathcal{A}_4 \times (S^n)^k$ (k, n arbitrary) provides an example of an infinite-dimensional, free $\mathcal{A}_4$-space homotopy equivalent to $(S^n)^k$. (Here $E\mathcal{A}_4$ stands for the universal cover of the classifying space of $\mathcal{A}_4$.)

Theorem 3.4. *The alternating group $\mathcal{A}_4$ acts freely on $S^n \times S^n \times S^n$ if and only if $n = 1, 3, 7$.*

Proof. ($\Leftarrow$) As a preliminary remark, recall that $\mathcal{A}_4$ can be described twofolds: either by the presentation

$$\langle a, b \mid a^2 = b^3 = (ab)^3 = 1 \rangle,$$

or by the extension

$$0 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathcal{A}_4 \xrightarrow{\epsilon} \mathbb{Z}_3 \to 0.$$

We will make use of both.

Let $F_2 = \langle a, b \rangle$ be the free group on two generators. Define an F_2-action on $S^n \times S^n$ by setting

$$\begin{cases} a(x,y) = (-x, y) \\ b(x,y) = (y, y^{-1}x^{-1}) \end{cases} \quad \text{for } x, y \in S^n.$$

For $n = 1$ or 3, it is straightforward to verify that this action is trivial while restricted to the normal closure of a^2, b^3 and $(ab)^3$, and thus induces an $\mathcal{A}_4$-action on $S^n \times S^n$, with the subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle a, bab^2 \rangle \subseteq \mathcal{A}_4$ acting freely. The same statement is true for $n = 7$: the octonions form an alternative algebra, hence even though their multiplication is not associative in general, it is associative on any two-generated subalgebra ([19, Appendix A, Theorem 4.16]).

Now take any free action of $\mathbb{Z}_3$ on S^n (for example the one generated by the rotation $x \mapsto e^{2\pi i/3}x$, $x \in S^n$) and extend it to an action of $\mathcal{A}_4$ by means of the epimorphism ϵ. One easily checks that the product of these two actions gives rise to a free $\mathcal{A}_4$-action on $S^n \times S^n \times S^n$.

This construction should be attributed to Plakhta (cf. [16, Example 1]).

($\Rightarrow$) Suppose that $\mathcal{A}_4$ acts freely on $S^n \times S^n \times S^n$. In view of Theorem 3.1, $\mathcal{H} = H^n(S^n \times S^n \times S^n; \mathbb{Z})$ is a non-trivial $\mathcal{A}_4$-module. The only non-trivial normal subgroup of $\mathcal{A}_4$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, hence $\mathcal{H}$ is also a non-trivial $\mathbb{Z}_3$-module for any subgroup $\mathbb{Z}_3 \subseteq \mathcal{A}_4$. Since the $\mathbb{Z}_3$-module structure on $\mathcal{H}$ comes from a free $\mathbb{Z}_3$-action, $\mathcal{H}$ splits off a trivial direct summand by Theorem 2.2. Consequently, by Lemma 2.1, there exists a basis of $\mathcal{H}$ in which its $\mathbb{Z}_3$-module structure is given by

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now express the $\mathbb{Z}_3$-module structure on $\mathcal{H}$ by a matrix in the usual basis. After reducing modulo 2, the resulting matrix will be conjugate to

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

as such, it cannot be a permutation matrix. It now follows from Proposition 2.3 that $n = 1, 3, 7$. ☐

Proposition 3.2 together with Theorem 3.4 show that $k(\mathcal{A}_4) = 3$.

Example 3.5. Let us describe another way of seeing that $\mathcal{A}_4$ acts freely on $S^1 \times S^1 \times S^1$. In view of Proposition 2.4, it suffices to produce a group extension

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(N) \to \mathcal{A}_4 \to 1,$$

where N is a closed 3-dimensional aspherical manifold.

Think of S^1 as the additive group of real numbers modulo 1. Let N be the torus bundle over S^1 given by the mapping torus of the homeomorphism $h\colon S^1 \times S^1 \to S^1 \times S^1$, $h(t_1, t_2) = (-t_2, t_1 - t_2)$ for any t_1, $t_2 \in S^1$. (N is the manifold (1.5) of [13].) It is well-known that N is a closed 3-dimensional manifold, and its asphericity follows from the long exact sequence of homotopy groups of the corresponding fiber bundle. Furthermore,

$$\pi_1(N) \cong (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{h_*} \mathbb{Z} \cong \langle a, b, c \mid [a, b] = 1,\ cac^{-1} = b,\ cbc^{-1} = a^{-1}b^{-1} \rangle.$$

Let $\mathcal{Z} \subseteq \pi_1(N)$ be the subgroup generated by a^2, b^2 and c^3. Since c^3 commutes with both a and b, it is straightforward to see that $\mathcal{Z}$ is a normal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, and that the quotient $\pi_1(N)/\mathcal{Z}$ is isomorphic to $\mathcal{A}_4$.

This approach makes it clear that the orbit space $S^1 \times S^1 \times S^1/\mathcal{A}_4$ of the arising $\mathcal{A}_4$-action is homeomorphic to N.

Remark 3.6. After passing the threshold, there is much more flexibility: $\mathcal{A}_4$ acts freely on $(S^{2n-1})^4$ for any integer $n \geq 1$. To see this, define an F_2-action on $S^n \times S^n \times S^n$ by setting

$$\begin{cases} a(x,y,z) = (-x,y,-z) \\ b(x,y,z) = (y,z,x) \end{cases} \quad \text{for } x,\, y,\, z \in S^n$$

and proceed as in the proof of Theorem 3.4.

Note that there is no point in considering free $\mathcal{A}_4$-action on products of even-dimensional spheres: if a finite group G acts freely on $X = S^{2n_1} \times S^{2n_2} \times \cdots \times S^{2n_k}$, then G is a 2-group because of the equality $2^k = \chi(X) = |G| \cdot \chi(X/G)$. Here χ denotes the Euler characteristic.

The reader interested in free actions of arbitrary alternating groups on products of equidimensional spheres is invited to consult [5].

4 Free $\mathcal{A}_4$-actions on $S^m \times S^n$

The requirement of equidimensionality of spheres in the product is actually crucial for Proposition 3.2 to hold.

Example 4.1 ([15, p. 543])**.** We will show that $\mathcal{A}_4$ acts freely on $S^2 \times S^3$. Write $SO(n)$ for the special orthogonal group of degree n. Consider the twisted product $SO(3) \times_{S^1} S^3$, with $S^1 \cong SO(2)$ acting as a subgroup on both $SO(3)$ and S^3. This, as usual, is a fiber bundle over $SO(3)/SO(2) \approx S^2$, with fiber S^3 and structure group S^1.

Observe that the S^1-action on S^3 is contained in the group action of S^3, and consequently $SO(3) \times_{S^1} S^3$ can be thought of as a principal S^3-bundle. Since

$$\pi_2(BS^3) \cong \pi_1(S^3) = 0,$$

the bundle is trivial, thus $SO(3) \times_{S^1} S^3 \approx S^2 \times S^3$. The conclusion follows from the fact that $\mathcal{A}_4$ is a subgroup of $SO(3)$.

It would be interesting to determine all pairs (m,n) for which there exists a free $\mathcal{A}_4$-action on $S^m \times S^n$. Let us summarize what is known in this direction:

- By Proposition 3.2, $m \neq n$.
- It follows from the discussion included in Remark 3.6 that m or n has to be odd.
- We will prove in Proposition 4.3 that $\mathcal{A}_4$ cannot act freely on $S^1 \times S^n$ for any $n \geq 1$.

As for the existence results:

- As explained in Example 4.1, $\mathcal{A}_4$ acts freely on $S^2 \times S^3$.
- Using the notion of fixity of a group, Adem–Davis–Ünlü proved that $\mathcal{A}_4$ acts freely on $S^{2n-1} \times S^{4n-5}$ for any $n \geq 3$ (see [2, Theorem 3.1]).

In order to prove that $\mathcal{A}_4$ cannot act freely on $S^1 \times S^n$ for any $n \geq 1$, we need the following basic lemma.

Lemma 4.2 ([12, Lemma 2.7]). *Let $0 \to A' \to A \xrightarrow{\kappa} A'' \to 0$ be a central extension of groups. If A' is a torsionfree abelian group and A'' is a torsion abelian group, then A is an abelian group.*

Proof. Let $a, b \in A$. Clearly, $\kappa(a)^n = 0$ for some $n > 0$ and, consequently, $a^n \in A'$. Thus $[a^n, b] = [a, b]^n = 1$. But $[a, b] \in A'$, which is torsionfree, so the conslusion follows. $\square$

Proposition 4.3. *The alternating group $\mathcal{A}_4$ cannot act freely on $S^1 \times S^n$ for any $n \geq 1$.*

Proof. In view of Proposition 3.2, we can assume without loss of generality that $n \geq 2$. If $\mathcal{A}_4$ acted freely on $S^1 \times S^n$, then by the theory of covering spaces, $\Gamma = \pi_1(S^1 \times S^n/\mathcal{A}_4)$ would act properly discontinuously on $S^n \times \mathbb{R}$. By [7, Lemma 4.2], a necessary condition for this to be possible is periodicity of Farrell cohomology of Γ. This in turn is equivalent to Γ having elementary abelian subgroups of rank at most 1 (see [6, Chapter X, Theorem 6.7]). We will show, however, that Γ contains $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a subgroup.

Consider the following commutative diagram which arises from the long exact sequence of homotopy groups of the covering $S^1 \times S^n \to S^1 \times S^n/\mathcal{A}_4$:

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \mathbb{Z}_3 \arrow[u] \arrow[r, equal] & \mathbb{Z}_3 \arrow[u] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \Gamma \arrow[r] \arrow[u] & \mathcal{A}_4 \arrow[r] \arrow[u] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & \Gamma' \arrow[r] \arrow[u] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The top horizontal extension is central, hence the same is true for the bottom one. By Lemma 4.2, Γ' is abelian, and therefore it suffices to find two elements of order 2 in Γ' to conclude the proof. In order to do so, choose a copy of $\mathbb{Z}_2 \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ to obtain:

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & \mathbb{Z}_2 \arrow[u] \arrow[r, equal] & \mathbb{Z}_2 \arrow[u] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \Gamma' \arrow[r] \arrow[u] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] \arrow[u] & 1 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u, equal] & \Gamma'' \arrow[r] \arrow[u] & \mathbb{Z}_2 \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

The bottom horizontal extension is *a fortiori* central, hence Γ'' is either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$. If the first possibility holds, then Γ' is $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$ (the only other extension of $\mathbb{Z}_2$ by $\mathbb{Z}$, the infinite dihedral group, is non-abelian). Both these choices imply that Γ is abelian, which is impossible. Thus $\Gamma'' \cong \mathbb{Z} \oplus \mathbb{Z}_2$, and consequently Γ' contains an element of order 2 for every copy of $\mathbb{Z}_2 \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

Remark 4.4. (1) The above argument works equally well if $\mathcal{A}_4$ is replaced with any finite nonabelian group which contains $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a normal subgroup and does admit an epimorphism onto $\mathbb{Z}_2$.

(2) See [11, Corollary 2.3] for an alternative proof of Proposition 4.3 for n even.

Remark 4.5. Propositions 3.2 and 4.3 show that the action presented in Example 4.1 gives the lowest-dimensional possibility, i.e., if $\mathcal{A}_4$ acts freely on $S^m \times S^n$, then $m + n \geq 5$.

It is also worth mentioning that any free $\mathcal{A}_4$-action on $S^m \times S^n$ has to be exotic, in the sense that it cannot come from a product of two actions on single spheres. For if $\mathcal{A}_4$ acted freely on $S^m \times S^n$ via a product action, then by taking an appropriate number of joins of each sphere (recall that a k-fold join of an n-sphere is a $(k(n+1)-1)$-sphere), one would obtain a free $\mathcal{A}_4$-action on $S^{(m+1)(n+1)-1} \times S^{(m+1)(n+1)-1}$, which contradicts Proposition 3.2. This was first observed by Adem–Smith ([3, Theorem 5.1]).

5 Free $\mathcal{S}_3$-actions on $S^m \times S^n$

Let us back up a little and look at free actions of the symmetric group $\mathcal{S}_3$. The story starts with Milnor, who proved that $\mathcal{S}_3$ cannot act freely on any sphere ([14, Corollary 1]). On the other hand, Swan constructed a finite, 3-dimensional CW complex homotopy equivalent to S^3 which admits a free $\mathcal{S}_3$-action ([17, Appendix]). Thus $k(\mathcal{S}_3) = 1$, but it is nevertheless worthwhile to inquire about actions on actual products spheres. We have:

Proposition 5.1. *The symmetric group $\mathcal{S}_3$ acts freely on $S^m \times S^n$ if and only if m or n is odd. In particular, $\mathcal{S}_3$ acts freely on $S^n \times S^n$ if and only if n is odd.*

Proof. Because of an Euler characteristic argument (see Remark 3.6), it suffices to construct a free $\mathcal{S}_3$-action on $S^m \times S^n$ whenever m or n is odd.

Assume that m is odd, and think of S^m as a subspace of $\mathbb{C}^{(m+1)/2}$. We will proceed similarly as in the proof of Theorem 3.4. Let $F_2 = \langle a, b \rangle$ be the free group on two generators. Define an F_2-action on S^m by

$$\begin{cases} ax = \bar{x} \\ bx = e^{2\pi i/3} x \end{cases} \quad \text{for } x \in S^m.$$

Since $\mathcal{S}_3$ can be presented as $\langle a, b \mid a^2 = b^3 = (ab)^2 = 1 \rangle$, it is straightforward to verify that this action induces an $\mathcal{S}_3$-action on S^m, which clearly is free while restricted to the subgroup $\mathbb{Z}_3 = \langle b \rangle \subseteq \mathcal{S}_3$.

Now consider the antipodal action on S^n and extend it to an $\mathcal{S}_3$-action via the epimorphism ϵ coming from the extension $0 \to \mathbb{Z}_3 \to \mathcal{S}_3 \xrightarrow{\epsilon} \mathbb{Z}_2 \to 0$. The product of these two actions gives rise to a free $\mathcal{S}_3$-action on $S^m \times S^n$. □

Remark 5.2. For any $n \geq 1$, the orbit space $S^1 \times S^n/\mathcal{S}_3$ of the action constructed in the proof of Proposition 5.1 is homeomorphic to the connected sum $\mathbb{R}P^{n+1}\#\mathbb{R}P^{n+1}$ of projective spaces. Indeed,

$$\begin{aligned} S^1 \times S^n/\mathcal{S}_3 &\approx (S^1 \times S^n/\mathbb{Z}_3)/\mathbb{Z}_2 \approx \big((S^1/\mathbb{Z}_3) \times S^n\big)/\mathbb{Z}_2 \approx S^1 \times S^n/\mathbb{Z}_2 \\ &\approx \mathbb{R}P^{n+1}\#\mathbb{R}P^{n+1}, \end{aligned}$$

because the last $\mathbb{Z}_2$-action on $S^1 \times S^n$ is given by $(x, y) \mapsto (\bar{x}, -y)$ for $x \in S^1$, $y \in S^n$.

If $n = 1$ or 2, the same statement is true for an arbitrary free $\mathcal{S}_3$-action on $S^1 \times S^n$. This is clear for $n = 1$; for $n = 2$, it is a consequence of [18, Corollary 2]. In general, if n is even, it follows from [11, Corollary 2.3] that $S^1 \times S^n/\mathcal{S}_3$ is homotopy equivalent to $\mathbb{R}P^{n+1}\#\mathbb{R}P^{n+1}$ for any free $\mathcal{S}_3$-action.

Remark 5.3. The argument of Proposition 5.1 can be applied, *mutatis mutandis*, to produce a free action of any dihedral group on $S^m \times S^n$ provided m or n is odd.

In general, $k(\mathcal{A}_d) \leq k(\mathcal{S}_d) \leq 2k(\mathcal{A}_d) + 1$ for any $d \geq 1$. The second inequality follows from the "coinduction" presented in the proof of Proposition 1.1 and the "piecewise" method of building actions, as given in the proofs of Theorem 3.4 and Proposition 5.1. Apart from that, not much else can be said about the number k for the class of symmetric groups.

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