Збірник праць Ін-ту математики НАН України 2013, том 6, N 6, 35–55

B. Balcerzak, A. Pierzchalski

Derivatives of skew-symmetric and symmetric vector-valued tensors

Second order elliptic operator of Laplace type on bundles of vector-valued tensors on a Lie algebroid are introduced and investigated. The Weitzenböeck type formulas in the case of skew-symmetric and symmetric tensors are derived.

1. Introduction

A Lie algebroid over a manifold M is a vector bundle A over M with a homomorphism of vector bundles $\varrho_A : A \to TM$ called an *anchor*, and a real Lie algebra structure $(\Gamma(A), [\![\cdot,\cdot]\!])$ such that $[\![a, fb]\!] = f[\![a, b]\!] + \varrho_A(a)(f) \cdot b$ for all $a, b \in \Gamma(A)$, $f \in \mathcal{C}^\infty(M)$. If the anchor is constant rank [surjective] we say that the Lie algebroid is *regular* [*transitive*]. Any smooth manifold M defines a Lie algebroid, where $A = TM$ with the identity anchor and the natural Lie algebra of vector fields on M. Other examples of Lie algebroids are: Lie algebras, integrable distributions (in particular foliations), cotangent bundles of Poisson manifolds, Lie algebroids of principal bundles.

For more complete treatment of the category of Lie algebroids and its connections we refer to: [9], [6], [10], [7], [1].

This article is an extension of our paper [3] where generalized gradients in the sense of Stein and Weiss on Lie algebroids were introduced and investigated. Stein-Weiss gradients are irreducible (with respect to the action of the orthogonal group) summands of a covariant derivative (cf. [14]). The exterior derivative on skew-symmetric forms and its coderivative, the Ahlfors operator ([13]) and in particular the Cauchy-Riemann operator are the examples. A connection in a Lie algebroid A has a natural extension to the first order linear operator

$$\nabla : \Gamma(\textstyle\bigwedge^k A^*) \longrightarrow \Gamma(A^* \otimes \textstyle\bigwedge^k A^*).$$

The last bundle has the following splitting onto three irreducible summands:

$$\Gamma(A^* \otimes \textstyle\bigwedge^k A^*) = \Gamma(\textstyle\bigwedge^{k+1} A^*) \oplus \Gamma(\textstyle\bigwedge^{1,k} A^*) \oplus \Gamma(\textstyle\bigwedge^{k-1} A^*)$$

(cf. [3]). So, generalized gradients in this case are compositions of ∇ with the projections defined by the splitting. Here, we are going to focus on two gradients: exterior derivative d^a and its conjugate d^{a*} acting on skew-symmetric tensors and being—up to multiplicative constants—compositions of ∇ with the projections on the first and on the third summand respectively. In the case of the bundle of symmetric forms an analogous splitting leads to their symmetric counterparts d^s, d^{s*} acting on symmetric tensors. In the both cases a proper composition, namely

$$\triangle^a = d^{a*} d^a + d^a d^{a*}$$

in the first case and

$$\Delta^s = d^{s*} d^s - d^s d^{s*}$$

in the other, lead to important second order differential operators. Both of them are elliptic and, like the Bochner Laplacian $\nabla^* \nabla$, are of metric symbol (see sections 3 and 4). As a consequence we derive Weitzenböck type formulas in each case:

$$\Delta = \nabla^* \nabla \pm \mathcal{R} \mp \mathcal{T} - \mathcal{M}$$

(cf. theorems 3 and 8). The formulas describe exact relations of Δ to the Bochner Laplacian. The relations depends explicitly on three indicators of the connection: its curvature (the operator $\mathcal{R}$), its torsion (the operator $\mathcal{T}$) and non-compatibility of the connection and the metric (the operator $\mathcal{M}$). It is important that the two second order linear elliptic operators differ practically by a tensor. In this context deriving its explicit shape seems to be essential.

In classical differential geometry the formula enables deriving many classical results establishing the relation between the topological structure of an algebroid and its geometry. By the standard Bochner technique, from the Weitzenböck formula, one can get then information on existence or nonexistence of some important deformations like isometric, projective, conformal (cf. [15] by K. Yano). One can also get some information on cohomologies (Betti numbers, [16]) or on lower bounds for spectrum of Δ (cf. [5]). Many possible applications of Weitzenböck types formulas can be found in the paper [4] by J.-P. Bourguignon.

It seems to be interesting that the two quiet antipodal cases: the skew-symmetric and the symmetric one behave so similar. To stress this harmony we apply exactly the same arrangement of the material in the both cases. In the case of a general Lie algebroid there is no equivalent of global (integral) scalar product even if the algebroid bundle carries a Riemannian structure. The adjoint operators are then defined here as the negative traces of suitable parts of the covariant derivative. They coincide then in the particular case of the algebroid of the tangent bundle of a compact Riemannian manifold with the operators adjoint with respect to global (integral) scalar product. In contrast to [3] we consider here the tensors (forms) with values in a given vector bundle. This bundle needs not to have any additional structure like algebraic or metric. It is equipped with a connection only.

2. The exterior covariant derivative for an arbitrary connection

Let $(A, \varrho_A, [\![\cdot,\cdot]\!])$ be a Lie algebroid over a manifold M and let E be a vector bundle over M. Let $\mathscr{A}(A,E) = \bigoplus_{p\geq 0} \mathscr{A}^k(A,E)$, where $\mathscr{A}^k(A,E) = \Gamma(\bigwedge^k A^* \otimes E)$, be the $\mathcal{C}^\infty(M)$-module of skew-symmetric forms on the Lie algebroid A of values in the vector bundle E. $\mathscr{A}(A,E)$ is the module over the ring $\mathcal{C}^\infty(M)$ and the module over the algebra $\mathscr{A}(A) = \mathscr{A}(A, M\times\mathbb{R})$ with the multiplication defined in the following way:

$$\wedge : \mathscr{A}^p(A, M\times\mathbb{R}) \times \mathscr{A}^q(A,E) \longrightarrow \mathscr{A}^{p+q}(A,E),$$

$$(\omega\wedge\eta)(a_1,\ldots,a_{p+q}) = \sum_{\sigma\in S(p,q)} \operatorname{sgn}\sigma\cdot\omega\left(a_{\sigma(1)},\ldots,a_{\sigma(p)}\right)\cdot\eta\left(a_{\sigma(p+1)},\ldots,a_{\sigma(p+q)}\right),$$

where $S(p,q)$ is the set of (p,q)-shuffles.

Let

$$\nabla : A \longrightarrow \mathcal{A}(E)$$

be an A-connection in E, i.e. a homomorphism of vector bundles A and $\mathcal{A}(E)$, which commutes with anchors, and where $\mathcal{A}(E)$ is the Lie algebroid of E. We recall (cf. [9]) that the module $\mathcal{CDO}(E)$ of sections of $\mathcal{A}(E)$ is the space of all covariant differential operators in E, i.e. $\mathbb{R}$-linear operators $\ell : \Gamma(E) \to \Gamma(E)$ such that there is $X_\ell \in \mathfrak{X}(M)$

satisfying $\ell(fe) = f\ell(e) + X_\ell(f)\, e$ for all $f \in \mathcal{C}^\infty(M)$ and $e \in \Gamma(E)$. ∇ defines a $\mathcal{C}^\infty(M)$-linear operator

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$$

of modules of sections which will be denoted also by ∇ and also called an A-*connection*. One can observe that

$$\operatorname{Sec} \varrho_{\mathcal{A}(E)} \circ \nabla = \operatorname{Sec} \varrho_A,$$

where $\operatorname{Sec} \varrho_{\mathcal{A}(E)}$ and $\operatorname{Sec} \varrho_A$ are morphisms of $\mathcal{C}^\infty(M)$-modules determined by the anchor $\varrho_{\mathcal{A}(E)}$ in the Lie algebroid $\mathcal{A}(E)$ and ϱ_A, respectively. The 2-form $\mathcal{R}^\nabla \in \mathscr{A}^2(A, \operatorname{End}(E))$ defined by

$$\mathcal{R}^\nabla(a,b) = \nabla_a \circ \nabla_b - \nabla_b \circ \nabla_a - \nabla_{[\![a,b]\!]}$$

is called the *curvature* of the A-connection ∇. We say that ∇ is *flat* if $\mathcal{R}^\nabla = 0$.

Recall that the exterior derivative $d^\nabla : \mathscr{A}^k(A,E) \to \mathscr{A}^{k+1}(A,E)$ determined by ∇ is defined by

$$\begin{aligned}(2.1)\qquad \left(d^\nabla \eta\right)(a_1,\ldots,a_{k+1}) \;=\; & \sum_{j=1}^{k+1} (-1)^{j-1} \nabla_{a_j}\left(\eta\left(a_1,\ldots \widehat{a}_j \ldots, a_{k+1}\right)\right) \\ & + \sum_{i<j} (-1)^{i+j}\, \eta\left([\![a_i,a_j]\!], a_1, \ldots \widehat{a}_i \ldots \widehat{a}_j \ldots, a_{k+1}\right).\end{aligned}$$

d^∇ is a first order differential operator giving a cohomology space if ∇ is flat. In particular, if ∇ is the anchor considered as an A-connection in the vector bundle $M \times \mathbb{R}$, $d^\nabla = d^{\varrho_A}$ gives the cohomology of the Lie algebroid A (cf. [11]).

Let ∇^A be an A-connection in A. By a *torsion* of ∇^A we mean the 2-form $T^A \in \mathscr{A}^2(A,A)$ given by

$$T^A(a,b) = \nabla^A_a b - \nabla^A_b a - [\![a,b]\!], \quad a,b \in \Gamma(A).$$

Denote the vector bundle $\bigotimes^k A^*$ by $A^{*\otimes k}$ and $\bigotimes A^* = \bigoplus_{k\geq 0} A^{*\otimes k}$ by $A^{*\otimes}$. ∇ and ∇^A induce an A-connection

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}\left(A^{*\otimes} \otimes E\right)$$

in the vector bundle $A^{*\otimes} \otimes E$ by

$$(\nabla_a \zeta)(a_1,\ldots,a_p) = \nabla_a\left(\zeta(a_1,\ldots,a_p)\right) - \sum_{j=1}^p \zeta\left(a_1,\ldots,\nabla^A_a a_j,\ldots,a_p\right),$$

$a, a_1, \ldots, a_p \in \Gamma(A)$, $\zeta \in \Gamma(A^{*\otimes p} \otimes E)$.

The connection ∇ determines the differential operator

$$\nabla : \Gamma\left(A^{*\otimes p} \otimes E\right) \longrightarrow \Gamma\left(A^{*\otimes p+1} \otimes E\right)$$

given by

$$(2.2)\qquad (\nabla\zeta)(a_0,a_1,\ldots,a_k) = (\nabla_{a_0}\zeta)(a_1,\ldots,a_k)$$

for $\zeta \in \Gamma(A^{*\otimes p} \otimes E)$, $a_j \in \Gamma(A)$.

Let $a \in \Gamma(A)$. The substitution operator

$$i_a : \Gamma\left(A^{*\otimes} \otimes E\right) \longrightarrow \Gamma\left(A^{*\otimes} \otimes E\right)$$

on $\Gamma(A^{*\otimes} \otimes E)$ is defined by

$$(i_a\zeta)(a_1,\ldots,a_{p-1}) = \zeta(a, a_1, \ldots, a_{p-1})$$

for all $\zeta \in \Gamma(A^{*\otimes p} \otimes E)$, $a_1, \ldots, a_{p-1} \in \Gamma(A)$.

Define the *second covariant derivative*

$$\nabla^2 = \nabla \circ \nabla : \Gamma\left(A^{*\otimes p} \otimes E\right) \longrightarrow \Gamma\left(A^{*\otimes p+2} \otimes E\right)$$

and for any $a, b \in \Gamma(A)$ the operator $\nabla^2_{a,b}$ such that

$$\nabla^2_{a,b} = i_a i_b \nabla^2,$$

i.e. $\nabla^2_{a,b}$ is a operator of the zero degree given explicitly by

$$\nabla^2_{a,b}\zeta = \nabla_a(\nabla_b \zeta) - \nabla_{\nabla^A_a b}\zeta \tag{2.3}$$

for $\zeta \in \Gamma(A^{*\otimes} \otimes E)$.

Lemma 1.

$$\nabla_a i_b = i_b \nabla_a + i_{\nabla^A_a b}$$

for any $a, b \in \Gamma(A)$.

Proof. Let $\theta \in \Gamma(A^{*\otimes p} \otimes E)$, $a_1, \ldots, a_p \in \Gamma(A)$. Then

$$\begin{aligned}
& (\nabla_a i_b \theta - i_b \nabla_a \theta)(a_1, \ldots, a_p) \\
= \;& (\nabla_a (i_b \theta))(a_1, \ldots, a_p) - (\nabla_a \theta)(b, a_1, \ldots, a_p) \\
= \;& (\nabla_a (\theta(b, a_1, \ldots, a_p))) - \sum_{s=1}^{p} \theta\left(b, a_1, \ldots, \nabla^A_a a_s, \ldots, a_p\right) \\
& - (\nabla_a(\theta(b, a_1, \ldots, a_p))) + \theta\left(\nabla^A_a b, a_1, \ldots, a_p\right) + \sum_{s=1}^{p} \theta\left(b, a_1, \ldots, \nabla^A_a a_s, \ldots, a_p\right) \\
= \;& \left(i_{\nabla^A_a b}\theta\right)(a_1, \ldots, a_p).
\end{aligned}$$

□

Lemma 2.

$$\mathcal{R}^{\nabla}_{a,b}\zeta = \nabla^2_{a,b}\zeta - \nabla^2_{b,a}\zeta + \nabla_{T^A(a,b)}\zeta$$

for $\zeta \in \Gamma(A^{*\otimes} \otimes E)$, $a, b \in \Gamma(A)$.

Proof. Use Lemma 1 to obtain:

$$\begin{aligned}
\nabla^2_{a,b}\zeta - \nabla^2_{b,a}\zeta &= i_b(\nabla_a(\nabla\zeta)) - i_a(\nabla_b(\nabla\zeta)) \\
&= \left(\nabla_a i_b - i_{\nabla^A_a b}\right)(\nabla\zeta) - \left(\nabla_b i_a - i_{\nabla^A_b a}\right)(\nabla\zeta) \\
&= \nabla_a(\nabla_b\zeta) - \nabla_{\nabla^A_a b}\zeta - \nabla_b(\nabla_a\zeta) + \nabla_{\nabla^A_b a}\zeta \\
&= \nabla_a(\nabla_b\zeta) - \nabla_b(\nabla_a\zeta) - \nabla_{[\![a,b]\!]}\zeta - \nabla_{\nabla^A_a b - \nabla^A_b a - [\![a,b]\!]}\zeta \\
&= \left(\mathcal{R}^{\nabla}_{a,b} - \nabla_{T^A(a,b)}\right)\zeta.
\end{aligned}$$

□

The curvature of $\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(A^{*\otimes} \otimes E)$ depends explicitly on curvatures of the connections $\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}(E)$ and $\nabla^A : \Gamma(A) \longrightarrow \mathcal{CDO}(A)$.

Lemma 3. *If* $\eta \in \Gamma\left(A^{*\otimes k} \otimes E\right)$, $a, b, a_1, \ldots, a_k \in \Gamma(A)$, *then*

$$\left(\mathcal{R}^{\nabla}_{a,b}\eta\right)(a_1, \ldots, a_k) = \mathcal{R}^{\nabla}_{a,b}(\eta(a_1, \ldots, a_k)) - \sum_{s=1}^{k} \eta\left(a_1, \ldots, \mathcal{R}^{\nabla^A}_{a,b} a_s, \ldots a_k\right).$$

Proof. Let $\eta \in \Gamma\left(A^{*\otimes k} \otimes E\right)$, $a, b, a_1, \ldots, a_k \in \Gamma(A)$. Then

$$\begin{aligned}
&\left(\mathcal{R}^{\nabla}_{a,b}\eta\right)(a_1,\ldots,a_k)\\
=\ &\nabla_a\left(\nabla_b\left(\eta\left(a_1,\ldots,a_k\right)\right)\right) - \sum_{s=1}^{k}\nabla_a\left(\eta\left(a_1,\ldots,\nabla^A_b a_s,\ldots,a_k\right)\right)\\
&-\sum_{s=1}^{k}\nabla_b\left(\eta\left(a_1,\ldots,\nabla^A_a a_s,\ldots,a_k\right)\right)+\sum_{s=1}^{k}\eta\left(a_1,\ldots,\nabla^A_b\left(\nabla^A_a a_s\right),\ldots,a_k\right)\\
&+\sum_{s=1}^{k}\sum_{t\neq s}\eta\left(a_1,\ldots,\nabla^A_b a_t,\ldots,\nabla^A_a a_s\ldots,a_k\right)+\sum_{s=1}^{k}\nabla_a\left(\eta\left(a_1,\ldots,\nabla^A_b a_s,\ldots,a_k\right)\right)\\
&-\sum_{s=1}^{k}\sum_{t\neq s}\eta\left(a_1,\ldots,\nabla^A_b a_t,\ldots,\nabla^A_a a_s\ldots,a_k\right)-\sum_{s=1}^{k}\eta\left(a_1,\ldots,\nabla^A_a\left(\nabla^A_b a_s\right),\ldots,a_k\right)\\
&-\nabla_{[\![a,b]\!]}\left(\eta\left(a_1,\ldots,a_k\right)\right)+\sum_{s=1}^{k}\eta\left(a_1,\ldots,\nabla^A_{[\![a,b]\!]}a_s,\ldots,a_k\right).
\end{aligned}$$

Now, by collecting similar terms we obtain that

$$\begin{aligned}
&\left(\mathcal{R}^{\nabla}_{a,b}\eta\right)(a_1,\ldots,a_k)\\
=\ &\nabla_a\left(\left(\nabla_b\eta\right)(a_1,\ldots,a_k)\right)-\nabla_b\left(\left(\nabla_a\eta\right)(a_1,\ldots,a_k)\right)-\nabla_{[\![a,b]\!]}\left(\eta\left(a_1,\ldots,a_k\right)\right)\\
&-\sum_{s=1}^{k}\eta\left(a_1,\ldots,\nabla^A_a\left(\nabla^A_b a_s\right)-\nabla^A_b\left(\nabla^A_a a_s\right)-\nabla^A_{[\![a,b]\!]}a_s,\ldots a_k\right)\\
=\ &\mathcal{R}^{\nabla}_{a,b}\left(\eta\left(a_1,\ldots,a_k\right)\right)-\sum_{s=1}^{k}\eta\left(a_1,\ldots,\mathcal{R}^{\nabla^A}_{a,b}a_s,\ldots,a_k\right).
\end{aligned}$$

□

Define the A-connection

$$\nabla : \Gamma(A) \longrightarrow \mathcal{CDO}\left(\bigwedge A^* \otimes E\right)$$

in the vector bundle $\bigwedge A^* \otimes E$ by

$$\left(\nabla_a\eta\right)(a_1,\ldots,a_p) = \nabla_a\left(\eta\left(a_1,\ldots,a_p\right)\right) - \sum_{j=1}^{p}\eta\left(a_1,\ldots,\nabla^A_a a_j,\ldots,a_p\right),$$

$a, a_1, \ldots, a_p \in \Gamma(A)$, $\eta \in \mathscr{A}^p(A,E)$. Observe that for all $\eta \in \mathscr{A}(A,E)$, $f \in \mathcal{C}^\infty(M) = \mathscr{A}^0(A,E)$, $a \in \Gamma(A)$ we have

$$\nabla_a(f\cdot\eta) = f\cdot\nabla_a\eta + \left(\varrho_A\right)_a(f)\cdot\eta, \tag{2.4}$$

where $\varrho_A : \Gamma(A) \longrightarrow \mathcal{CDO}\left(\bigwedge A^* \otimes (M\times\mathbb{R})\right)$ is the A-connection in the bundle $\bigwedge A^* \otimes (M\times\mathbb{R})$ determined by the pair of connections ϱ_A and ∇^A. So, we see that indeed, for every $a \in \Gamma(A)$, the operator ∇_a has values in $\mathcal{CDO}\left(\bigwedge A^*\otimes E\right)$.

Lemma 4. *If $\omega \in \mathscr{A}(A, M\times\mathbb{R})$, $\nu \in \Gamma(E)$, $a \in \Gamma(A)$, then*

$$\nabla_a(\omega\otimes\nu) = \left(\varrho_A\right)_a(\omega)\otimes\nu + \omega\otimes\nabla_a\nu. \tag{2.5}$$

Proof. Let $\nu \in \Gamma(E)$, $a \in \Gamma(A)$. If $\omega \in \mathscr{A}^0(A) = \mathcal{C}^\infty(M)$, (2.5) is equivalent to (2.4). Now, let $\omega \in \mathscr{A}^p(A)$, $a_1, \ldots, a_p \in \Gamma(A)$. Then:

$$\begin{aligned} & \nabla_a(\omega \otimes \nu)(a_1, \ldots, a_p) \\ = \ & \nabla_a((\omega \otimes \nu)(a_1, \ldots, a_p)) - \sum_{j=1}^{p} (\omega \otimes \nu)\left(a_1, \ldots, \nabla_a^A a_j, \ldots, a_p\right) \\ = \ & \nabla_a(\omega(a_1, \ldots, a_p) \cdot \nu) - \sum_{j=1}^{p} \omega\left(a_1, \ldots, \nabla_a^A a_j, \ldots, a_p\right) \cdot \nu \\ = \ & \varrho_A(a)(\omega(a_1, \ldots, a_p)) \cdot \nu - \sum_{j=1}^{p} \omega\left(a_1, \ldots, \nabla_a^A a_j, \ldots, a_p\right) \cdot \nu + \omega(a_1, \ldots, a_p) \cdot \nabla_a(\nu) \\ = \ & \left((\varrho_A)_a(\omega) \otimes \nu + \omega \otimes \nabla_a \nu\right)(a_1, \ldots, a_p). \end{aligned}$$

□

Lemma 5. *If $\omega \in \mathscr{A}(A)$, $\eta \in \mathscr{A}(A, E)$, $a \in \Gamma(A)$:*

$$\nabla_a(\omega \wedge \eta) = (\varrho_A)_a(\omega) \wedge \eta + \omega \wedge \nabla_a \eta.$$

Proof. Let $\omega \in \mathscr{A}^p(A)$, $\eta \in \mathscr{A}^q(A, E)$, $a \in \Gamma(A)$. Let η be a form $\eta' \otimes \nu$ for some $\eta' \in \mathscr{A}^q(A)$ and $\nu \in \Gamma(E)$. Lemma 4 implies that

$$\begin{aligned} \nabla_a(\omega \wedge \eta) &= \nabla_a(\omega \wedge \eta' \otimes \nu) \\ &= (\varrho_A)_a(\omega \wedge \eta') \otimes \nu + (\omega \wedge \eta') \otimes \nabla_a \nu. \end{aligned}$$

Since $(\varrho_A)_a$ is a differentiation in the algebra $\mathscr{A}(A)$, from Lemma 4 we obtain:

$$\begin{aligned} \nabla_a(\omega \wedge \eta) &= ((\varrho_A)_a(\omega) \wedge \eta' + \omega \wedge (\varrho_A)_a(\eta')) \otimes \nu + (\omega \wedge \eta') \otimes \nabla_a \nu \\ &= (\varrho_A)_a(\omega) \wedge (\eta' \otimes \nu) + \omega \wedge ((\varrho_A)_a(\eta') \otimes \nu + \eta' \otimes \nabla_a \nu) \\ &= (\varrho_A)_a(\omega) \wedge \eta + \omega \wedge \nabla_a(\eta). \end{aligned}$$

□

Now, define the operator $d^a : \mathscr{A}^k(A, E) \longrightarrow \mathscr{A}^{k+1}(A, E)$ by

$$d^a \eta = (k+1) \cdot \operatorname{Alt}(\nabla \eta), \tag{2.6}$$

where for any $\zeta \in \bigotimes^p A^*$ its *alternation* $\operatorname{Alt}\zeta$ is defined by

$$\operatorname{Alt}\zeta = \frac{1}{p!} \sum_{\sigma \in S_p} \operatorname{sgn}\sigma\,(\sigma\zeta).$$

So,

$$(d^a \eta)(a_1, \ldots, a_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j-1} \left(\nabla_{a_j} \eta\right)(a_1, \ldots \widehat{a}_j \ldots, a_{k+1}), \tag{2.7}$$

where $\eta \in \mathscr{A}^k(A, E)$, $a_1, \ldots, a_{k+1} \in \Gamma(A)$. A relation between d and d^a describes the following

Lemma 6.

$$d^a = d^\nabla + d^T$$

where $d^T : \mathscr{A}^p(A, E) \longrightarrow \mathscr{A}^{p+1}(A, E)$ is the operator given by

$$\left(d^T \eta\right)(a_1, \ldots, a_{p+1}) = \sum_{i<j} (-1)^{i+j} \eta\left(T^A(a_i, a_j), a_1, \ldots \widehat{a}_i \ldots \widehat{a}_j \ldots, a_{p+1}\right)$$

for any $\eta \in \mathscr{A}^p(A, E)$, $a_1, \ldots, a_{p+1} \in \Gamma(A)$.

Proof. Let $\eta \in \mathscr{A}^k(A,E)$, $a_1,\dots,a_{p+1} \in \Gamma(A)$. Therefore

$$\begin{aligned}
&(\mathrm{Alt}(\nabla\eta))(a_1,\dots,a_{p+1}) \\
= & \sum_{j=1}^{p+1}(-1)^{j-1}\left(\nabla_{a_j}\eta\right)(a_1,\dots\widehat{a}_j\dots,a_{p+1}) \\
= & \sum_{j=1}^{p+1}(-1)^{j-1}\nabla_{a_j}\left(\eta(a_1,\dots\widehat{a}_j\dots,a_{p+1})\right) - \sum_{i<j}(-1)^{j-1}\eta\left(a_1,\dots,\nabla^A_{a_j}a_i,\dots\widehat{a}_j\dots,a_{p+1}\right) \\
& -\sum_{j<i}(-1)^{j-1}\eta\left(a_1,\dots\widehat{a}_j\dots,\nabla^A_{a_j}a_i,\dots,a_{p+1}\right) \\
= & \sum_{j=1}^{p+1}(-1)^{j-1}\nabla_{a_j}\left(\eta(a_1,\dots\widehat{a}_j\dots,a_{p+1})\right) \\
& +\sum_{i<j}(-1)^{i+j}\eta\left(\nabla^A_{a_i}a_j-\nabla^A_{a_j}a_i,a_1,\dots\widehat{a}_i\dots\widehat{a}_j\dots,a_{p+1}\right) \\
= & \sum_{j=1}^{p+1}(-1)^{j-1}\nabla_{a_j}\left(\eta(a_1,\dots\widehat{a}_j\dots,a_{p+1})\right) \\
& +\sum_{i<j}(-1)^{i+j}\eta\left([\![a_i,a_j]\!]+T^A(a_i,a_j),a_1,\dots\widehat{a}_i\dots\widehat{a}_j\dots,a_{p+1}\right) \\
= & \left(d^\nabla\eta\right)(a_1,\dots,a_{p+1})+\left(d^T\eta\right)(a_1,\dots,a_{p+1}).
\end{aligned}$$

□

Notice that if ∇^A is torsion-free, $d^a = d^\nabla$ (cf. also [2]).

3. Weitzenböck Formula for Skew-symmetric Forms

Assume that in the vector bundle A we have a Riemannian metric g. For any $k > 1$ and any $\zeta \in \Gamma(A^{*\otimes k}\otimes E)$ define the trace $\mathrm{tr}\,\zeta \in \Gamma(A^{*\otimes k-2}\otimes E)$ as the trace with respect to the first two arguments by

$$(\mathrm{tr}\,\zeta)(a_1,\dots,a_{k-2}) = \sum_{j=1}^{n}\zeta(e_j,e_j,a_1,\dots,a_{k-2}) \tag{3.1}$$

where $(e_1,\dots,e_n)$ is a local orthonormal frame of A ($n = \dim A_x$, $x \in M$). Define additionally $\mathrm{tr}\,\zeta = 0$ for $\zeta \in \Gamma(A^{*\otimes 1})$. One can see that tr do not depend on the choice of the frame.

By the *exterior coderivative* d^{a*} we mean the operator:

$$d^{a*} = -\,\mathrm{tr}\circ\nabla : \mathscr{A}^k(A,E) \longrightarrow \mathscr{A}^{k-1}(A,E). \tag{3.2}$$

Remark 1. In the case of invariantly oriented Lie algebroids we can use the integral fibre operator and a scalar product on the module $\mathscr{A}(A)$ such that d^{a*} is formally adjoint to $d^a = d^{\varrho_A}$ with respect to this product, see [8]. For a general Lie algebroid we do not have such a scalar product.

Define three differential operators of order zero. The first, a *Ricci type operator* $\mathcal{R}^a : \mathscr{A}(A,E) \to \mathscr{A}(A,E)$ defined by

$$(\mathcal{R}^a\eta)(a_1,\dots,a_k) = \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\mathcal{R}^\nabla_{e_j,a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k), \tag{3.3}$$

the operator $\mathcal{T}^a : \mathscr{A}(A,E) \to \mathscr{A}(A,E)$ by

$$(\mathcal{T}^a\eta)(a_1,\dots,a_k) = \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla_{T^A(e_j,a_s)}\eta\right)(a_1,\dots\widehat{a}_s\dots,a_k), \tag{3.4}$$

and next, the operator $\mathcal{M}^a : \mathscr{A}(A,E) \longrightarrow \mathscr{A}(A,E)$ by

$$(\mathcal{M}^a\eta)(a_1,\dots,a_k) = \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(i_{\nabla^A_{a_s}e_j}i_{e_j} + i_{e_j}i_{\nabla^A_{a_s}e_j}\right)(\nabla\eta)(a_1,\dots\widehat{a}_s\dots,a_k), \tag{3.5}$$

where $\eta \in \mathscr{A}^k(A,E)$, $a_1,\dots,a_k \in \Gamma(A)$, $(e_1,\dots,e_n)$ is a local orthonormal frame of A, $\mathcal{R}^\nabla$ is the curvature tensor of the connection ∇. The first one $\mathcal{R}^a$ is the trace of the curvature tensor. The next $\mathcal{T}^a$ indicates a deviation of the connection from being torsion-free. The third $\mathcal{M}^a$ measures a non-compatibility of ∇ with the metric. By Lemma 2,

$$\begin{aligned}&(\mathcal{R}^a\eta - \mathcal{T}^a\eta)(a_1,\dots,a_k) \\ &= \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla^2_{e_j,a_s}\eta - \nabla^2_{a_s,e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k).\end{aligned} \tag{3.6}$$

Moreover observe that the operators $\mathcal{R}^a$, $\mathcal{T}^a\eta$, $\mathcal{M}^a\eta$ can be written in the following forms

$$\begin{aligned}(\mathcal{R}^a\eta) &= \operatorname{Alt}\left(\sum_{j=1}^{n} i_{e_j}\left(\mathcal{R}^\nabla_{e_j,\cdot}\eta\right)\right), \\ \mathcal{T}^a\eta &= -\operatorname{Alt}\left(\sum_{j=1}^{n}\nabla_{T^{\nabla^A}(e_j,\cdot)}\eta\right), \\ \mathcal{M}^a\eta &= -\operatorname{Alt}\left(\sum_{j=1}^{n}\left(i_{\nabla^A e_j}i_{e_j} + i_{e_j}i_{\nabla^A e_j}\right)\right)(\nabla\eta).\end{aligned}$$

Define the *Laplace operator* on differential forms on the Lie algebroid A by

$$\Delta^a = d^{a*}d^a + d^a d^{a*}.$$

Recall that for a linear operator $P : \Gamma(F) \to \Gamma(F)$ of order m in a vector bundle F its symbol at a given point $x \in M$ is defined by

$$\sigma_P(e,\omega) = P(f^m\eta)(x)$$

for $e \in F_x$ and such $\omega \in A^*_x$ that $\omega = (df)(x)$ for some smooth function f with $f(x) = 0$, and where $\eta \in \Gamma(F)$, $\eta(x) = e$ (cf. [12]). The definition is independent either of f nor of η.

Observe that if A is transitive, Δ^a is a second order strongly elliptic operator with the metric symbol

$$\sigma_{\Delta^a}(\omega,\eta) = |\omega|^2\eta.$$

Indeed, let $x \in M$, $\omega \in A^*_x$, $e \in \Lambda^k A^*_x \otimes E_x$ and let $f \in \mathcal{C}^\infty(M)$, $s \in \Gamma\left(\Lambda^k A^* \otimes E\right)$ satisfy $f(x) = 0$, $(df)(x) = \omega$, $s(x) = e$. Then

$$\sigma_{d^a}(\omega,e) = d^a(fs)(x) = (d^a f \wedge s + f d^a s)(x) = \omega \wedge e.$$

Moreover, since $(\varrho_A)(f) = d^a f$, the relation (2.4) implies

$$\sigma_{d^{a*}}(\omega,e) = d^{a*}(fs)(x) = \left(i_{(df)^\sharp}s\right)(x) = i_{\omega^\sharp}e$$

where $\sharp : A^* \to A$ is the musical isomorphism determined by the metric g, i.e. for an 1-form $\xi \in \mathscr{A}^k(A, M\times\mathbb{R})$

$$g\left(\xi^\sharp, b\right) = i_b\xi \quad \text{for} \quad b \in \Gamma(A).$$

Hence

$$\sigma_{d^{a*}d^a}(\omega, e) = i_{\omega^\sharp}(\omega \wedge e) = i_{\omega^\sharp}\omega \wedge e - \omega \wedge i_{\omega^\sharp} e$$

and

$$\sigma_{d^a d^{a*}}(\omega, e) = \omega \wedge i_{\omega^\sharp} e.$$

Consequently,

$$\sigma_{\Delta^a}(\omega, e) = \sigma_{d^{a*}d^a + d^a d^{a*}}(\omega, e) = i_{\omega^\sharp}\omega \wedge e = g\left(\omega^\sharp, \omega^\sharp\right) e.$$

Now we write the explicit formulas for the two terms of Δ in the case of an arbitrary Lie algebroid A.

Theorem 1.

$$d^{a*} d^a \eta = -\operatorname{trace} \nabla^2 \eta + \sum_{j=1}^{n} \operatorname{Alt}\left(i_{e_j}\left(\nabla^2_{e_j,(\cdot)}\eta\right)\right)$$

for $\eta \in \mathscr{A}(A, E)$.

Proof. Let $\eta \in \mathscr{A}^k(A, E)$, $a_1, \ldots, a_k \in \Gamma(A)$ and $(e_1, \ldots, e_n)$ be a local orthonormal frame of A. By (2.7) and the definition of d^{a*} we obtain that

$$\begin{aligned}
&\left(d^{a*} d^a \eta\right)(a_1, \ldots, a_k) \\
&= -\sum_{j=1}^{n}\left(\nabla_{e_j}\left(d^a \eta\right)(e_j, a_1, \ldots, a_k)\right) + \sum_{j=1}^{n}\left(d^a \eta\right)\left(\nabla^A_{e_j} e_j, a_1, \ldots, a_k\right) \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\left(d^a \eta\right)\left(e_j, a_1, \ldots, \nabla^A_{e_j} a_s, \ldots, a_k\right) \\
&= -\sum_{j=1}^{n}\nabla_{e_j}\left(\left(\nabla_{e_j}\eta\right)(a_1, \ldots, a_k)\right) - \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s \nabla_{e_j}\left(\left(\nabla_{a_s}\eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k)\right) \\
&+ \sum_{j=1}^{n}\left(\nabla_{\nabla^A_{e_j} e_j}\eta\right)(a_1, \ldots, a_k) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{a_s}\eta\right)\left(\nabla^A_{e_j} e_j, a_1, \ldots \widehat{a}_s \ldots, a_k\right) \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{e_j}\eta\right)\left(a_1, \ldots, \nabla^A_{e_j} a_s, \ldots, a_k\right) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla_{\nabla^A_{e_j}}\eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{s \neq t}(-1)^t\left(\nabla_{a_t}\eta\right)\left(e_j, a_1, \ldots \widehat{a}_t \ldots, \nabla^A_{e_j} a_s, \ldots, a_k\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{n} \left(\nabla_{e_j}\left(\nabla_{e_j}\eta\right)\right)(a_1,\dots,a_k) - \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{e_j}\eta\right)\left(a_1,\dots,\nabla^A_{e_j}a_s,\dots,a_k\right) \\
&- \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{e_j}\left(\nabla_{a_s}\eta\right)\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,a_k\right) \\
&- \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{a_s}\eta\right)\left(\nabla^A_{e_j}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right) \\
&- \sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{s\neq t}(-1)^s\left(\nabla_{a_s}\eta\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,\nabla^A_{e_j}a_s,\dots,a_k\right) \\
&+ \sum_{j=1}^{n}\left(\nabla_{\nabla^A_{e_j}e_j}\eta\right)(a_1,\dots,a_k) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{a_s}\eta\right)\left(\nabla^A_{e_j}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right) \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{e_j}\eta\right)\left(a_1,\dots,\nabla^A_{e_j}a_s,\dots,a_k\right) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla_{\nabla^A_{e_j}}\eta\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,a_k\right) \\
&+ \sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{s\neq t}(-1)^t\left(\nabla_{a_t}\eta\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,\nabla^A_{e_j}a_s,\dots,a_k\right)
\end{aligned}
$$

After collecting similar summands and using (2.3) one obtains

$$
\begin{aligned}
&\left(d^{a*}d^a\eta\right)(a_1,\dots,a_k) \\
&\quad = -\sum_{j=1}^{n}\left(\nabla_{e_j}\left(\nabla_{e_j}\eta\right) - \nabla_{\nabla^A_{e_j}e_j}\eta\right)(a_1,\dots,a_k) \\
&\qquad - \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^s\left(\nabla_{e_j}\left(\nabla_{a_s}\eta\right)\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,a_k\right) \\
&\qquad + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla_{\nabla^A_{e_j}}\eta\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,a_k\right) \\
&\quad = -\operatorname{trace}\nabla^2\eta\,(a_1,\dots,a_k) + \sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla^2_{e_j,a_s}\eta\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,a_k\right).
\end{aligned}
$$

Moreover observe that

$$
\begin{aligned}
&\sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla^2_{e_j,a_s}\eta\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,a_k\right) \\
= \;&\sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(i_{a_s}\left(i_{e_j}\nabla\left(\nabla\eta\right)\right)\right)\left(e_j,a_1,\dots\widehat{a}_s\dots,a_k\right) \\
= \;&\sum_{s=1}^{k}(-1)^{s-1}\left(\sum_{j=1}^{n} i_{e_j}i_{a_s}\left(i_{e_j}\nabla\left(\nabla\eta\right)\right)\right)\left(a_1,\dots\widehat{a}_s\dots,a_k\right) \\
= \;&\operatorname{Alt}\left(\sum_{j=1}^{n} i_{e_j}\left(\nabla^2_{e_j,(\cdot)}\eta\right)\right)(a_1,\dots,a_k).
\end{aligned}
$$

□

Theorem 2.

$$
\begin{aligned}
& \left(d^a d^{a*}\eta\right)(a_1,\ldots,a_k) \\
= \; & -\sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(\nabla^2_{a_s,e_j}\eta\right)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k) \\
& -\sum_{j=1}^{n}\sum_{s=1}^{k}(-1)^{s-1}\left(i_{\nabla^A_{a_s}e_j}i_{e_j}+i_{e_j}i_{\nabla^A_{a_s}e_j}\right)(\nabla\eta)(a_1,\ldots\widehat{a}_s\ldots,a_k),
\end{aligned}
$$

i.e.

$$
d^a d^{a*}\eta = -\sum_{j=1}^{n}\mathrm{Alt}\left(i_{e_j}\left(\nabla^2_{(\cdot),e_j}\eta\right)\right)-\sum_{j=1}^{n}\mathrm{Alt}\left(i_{\nabla^A e_j}i_{e_j}+i_{e_j}i_{\nabla^A e_j}\right)(\nabla\eta)
$$

for $\eta\in\mathscr{A}(A,E)$*,* $a_1,\ldots,a_k\in\Gamma(A)$*.*

Proof. Let $\eta\in\mathscr{A}^k(A,E)$, $a_1,\ldots,a_k\in\Gamma(A)$ and $(e_1,\ldots,e_n)$ be a local orthonormal frame of A. By (2.7) and the definition of d^{a*} we have

$$
\begin{aligned}
& \left(d^a d^{a*}\eta\right)(a_1,\ldots,a_k) \\
= \; & \sum_{s=1}^{k}(-1)^{s-1}\left(\nabla_{a_s}(d^*\eta)\right)(a_1,\ldots\widehat{a}_s\ldots,a_k) \\
= \; & -\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(\nabla_{a_s}\left(i_{e_j}\left(\nabla_{e_j}\eta\right)\right)\right)(a_1,\ldots\widehat{a}_s\ldots,a_k) \\
= \; & -\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\nabla_{a_s}\left(\left(\nabla_{e_j}\eta\right)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k)\right) \\
& +\sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{t\neq s}(-1)^{s-1}\left(\nabla_{e_j}\eta\right)\left(e_j,a_1,\ldots,\nabla^A_{a_s}a_t,\ldots\widehat{a}_s\ldots,a_k\right) \\
= \; & -\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(\nabla_{a_s}\left(\nabla_{e_j}\eta\right)\right)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k) \\
& -\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(\nabla_{e_j}\eta\right)\left(\nabla^A_{a_s}e_j,a_1,\ldots\widehat{a}_s\ldots,a_k\right) \\
& -\sum_{s=1}^{k}\sum_{j=1}^{n}\sum_{t\neq s}(-1)^{s-1}\left(\nabla_{e_j}\eta\right)\left(e_j,a_1,\ldots,\nabla^A_{a_s}a_t,\ldots\widehat{a}_s\ldots,a_k\right) \\
& +\sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{t\neq s}(-1)^{s-1}\left(\nabla_{e_j}\eta\right)\left(e_j,a_1,\ldots,\nabla^A_{a_s}a_t,\ldots\widehat{a}_s\ldots,a_k\right).
\end{aligned}
$$

Now, collecting similar terms one concludes that

$$
\begin{aligned}
&\left(d^a d^{a*}\eta\right)(a_1,\dots,a_k)\\
=\; &-\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(\nabla_{a_s}\left(\nabla_{e_j}\eta\right)\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&-\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(\nabla_{e_j}\eta\right)\left(\nabla^A_{a_s}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right)\\
=\; &-\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(\nabla^2_{a_s,e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&-\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(\nabla_{\nabla^A_{a_s}e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&-\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(\nabla_{e_j}\eta\right)\left(\nabla^A_{a_s}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right)\\
=\; &-\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(\nabla^2_{a_s,e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&-\sum_{s=1}^{k}\sum_{j=1}^{n}(-1)^{s-1}\left(i_{e_j}i_{\nabla^A_{a_s}e_j}+i_{\nabla^A_{a_s}e_j}i_{e_j}\right)(\nabla\eta)(a_1,\dots\widehat{a}_s\dots,a_k)\,.
\end{aligned}
$$

□

As a consequence of theorems 1 and 2 we have the following

Theorem 3. (*Weitzenböck Formula for Skew-Symmetric Forms*)

$$\Delta^a=\nabla^*\nabla+\mathcal{R}^a-\mathcal{T}^a-\mathcal{M}^a \tag{3.7}$$

where $\mathcal{R}^a$, $\mathcal{T}^a$ *and* $\mathcal{M}^a$ *are the operators defined in* (3.3)—(3.5).

Observe that if there exists a local orthonormal frame of sections $(e_1,\dots,e_n)$ with the property $\nabla^A_{e_i}e_j\big|_x=0$ at a single point $x\in M$, then $\mathcal{M}^a$ is equal to zero. This condition is fulfilled in case $A=F\subset TM$ is an integrable distribution on M and ∇^A is the Levi-Civita connection. The assumption of existence of a local orthonormal frame of sections that have vanishing covariant derivatives at a single point implies that the isotropy algebra of A (i.e. $\ker\varrho_A|_x$) is abelian, and then $\mathcal{T}^a=0$.

4. d^{a*} and Δ^a in the case of a metric connection

Consider some particular cases. Assume that ∇^A is metric (is compatible with g), i.e.

$$(\varrho_A\circ a)(g(b,c))=g(\nabla_a b,c)+g(b,\nabla_a c)\quad\text{for all}\quad a,b,c\in\Gamma(A)\,.$$

We see at once that then the operator $\mathcal{M}^a$ vanishes. Consequently, the Weitzenböck Formula reduces to the form

$$\Delta^a=\nabla^*\nabla+\mathcal{R}^a-\mathcal{T}^a.$$

If ∇ is a torsion-free A-connection on A, then $d^a=d^\nabla$ is the exterior derivative on A given in (2.1) and $\mathcal{T}^a=0$. In particular, if $\nabla^A:\Gamma(A)\longrightarrow\mathcal{CDO}(A)$ is the Levi-Civita connection in A, i.e.

$$\begin{aligned} & 2g\left(\nabla^A_a b, c\right) \\ = \ & \left(\varrho_A \circ a\right)\left(g\left(b,c\right)\right) + \left(\varrho_A \circ b\right)\left(g\left(a,c\right)\right) - \left(\varrho_A \circ c\right)\left(g\left(a,b\right)\right) \\ & + g\left([\![a,b]\!],c\right) + g\left([\![c,b]\!],a\right) + g\left([\![c,a]\!],b\right) \end{aligned}$$

for any $a, b, c \in \Gamma(A)$ (then ∇^A is uniquely determined metric and torsion-free connection), the Laplacian reduces to its classical shape:

$$\Delta^a = \nabla^* \nabla + \mathcal{R}^a.$$

If ∇^A is metric, the coderivative d^{*a} we can expressed in the language of the Hodge stat operator.

Assume that A is oriented and let $\Omega \in \mathscr{A}^n(A, M \times \mathbb{R})$ be the volume form ($n = \dim A_x$, $x \in M$).

For any $a \in \Gamma(A)$ we will denote by a^* the 1-form dual to a with respect to g, i.e. $a^* = g(a, \cdot)$. We extend g to the scalar product $\langle \cdot, \cdot \rangle_g$ on $\mathscr{A}^k(A, M \times \mathbb{R})$ in the usual way putting

$$\langle a_1^* \wedge \ldots \wedge a_k^*, b_1^* \wedge \ldots b_k^* \rangle_g = \det\left(\left\langle a_i^*, b_j^* \right\rangle_g\right),$$

$a_1, \ldots, a_k, b_1, \ldots, b_k \in \Gamma(A)$.

Definition 1. *Let $(e_1, \ldots, e_n)$ be a local oriented orthonormal frame for A and $(e^{*1}, \ldots, e^{*n})$ — the dual local orthonormal frame for A^*. Let $I = (i_1, \ldots, i_p)$ and $J = (j_1, \ldots, j_{n-p})$, where $i_1 < \ldots < i_p$, $j_1 < \ldots < j_{n-p}$, be a complementary set such that (I, J) is a permutation of $\{1, \ldots, n\}$. Let*

$$\omega_I = e^{*i_1} \wedge \ldots \wedge e^{*i_p}, \quad \omega_J = e^{*j_1} \wedge \ldots \wedge e^{*j_{n-p}}, \quad \nu \in \Gamma(E).$$

Define a $\mathcal{C}^\infty(M)$-linear operator

$$* : \mathscr{A}^p(A, E) \longrightarrow \mathscr{A}^{n-p}(A, E)$$

by

$$*(\omega_I \otimes \nu) = \epsilon(I, J)\, \omega_J \otimes \nu,$$

where $\epsilon(I, J)$ is the sign of the permutation $(I, J) = (i_1, \ldots, i_p, j_1, \ldots, j_{n-p})$.

One can check that

$$\Omega \otimes (*\eta)(a_1, ..., a_{n-p}) = (-1)^{p(n-p)}\, a_1^* \wedge ... \wedge a_{n-p}^* \wedge \eta,$$

for any $a_1, ..., a_{n-p} \in \Gamma(A)$, $\eta \in \mathscr{A}^p(A, E)$.

Consequently, by properties of the star operator on scalar forms (cf. [2]) we obtain

Lemma 7. *For any $\nu \in \Gamma(E)$, $f \in \mathcal{C}^\infty(M)$, $\eta \in \mathscr{A}^p(A, E)$, $a, a_1, ..., a_{n-p+1} \in \Gamma(A)$ the following equalities are fulfilled:*

(a) $*(\Omega \otimes \nu) = \nu$, $*(f\Omega \otimes \nu) = f\nu$, $*(\nu) = \Omega \otimes \nu$,
(b) $(*\eta)(a_1, ..., a_{n-p}) = (-1)^{p(n-p)} * \left(a_1^* \wedge ... \wedge a_{n-p}^* \wedge \eta\right)$,
(c) $i_a(*\eta) = (-1)^p * (a^* \wedge \eta)$,
(d) $* * \eta = (-1)^{p(n-p)}\, \eta$.

Now we are going to show that $*$ and a metric connection ∇ commute.

Theorem 4. *If ∇^A is a metric connection,*

$$*(\nabla_a \eta) = \nabla_a(*\eta) \tag{4.1}$$

for all $\eta \in \mathscr{A}(A, E)$, $a \in \Gamma(A)$.

Proof. Let $a \in \Gamma(A)$, $\omega \in \mathscr{A}^p(A, M \times \mathbb{R})$, $\nu \in E$. From Theorem 3.2 [2] we have

$$(\varrho_A)_a(*\omega) = *((\varrho_A)_a\,\omega).$$

Therefore, by (2.5) we obtain

$$\begin{aligned}\nabla_a(*(\omega\otimes\nu)) &= \nabla_a(*\omega\otimes\nu)\\ &= (\varrho_A)_a(*\omega)\otimes\nu + (*\omega)\otimes\nabla_a\nu\\ &= *((\varrho_A)_a\,\omega)\otimes\nu + (*\omega)\otimes\nabla_a\nu\\ &= *((\varrho_A)_a\,\omega\otimes\nu + \omega\otimes\nabla_a\nu)\\ &= *(\nabla_a(\omega\otimes\nu)).\end{aligned}$$

□

Lemma 8. *If $(e_1, \ldots, e_n)$ is a local frame of A and $(e_1^*, \ldots, e_n^*)$ is the dual local frame of A^*, then*

$$d^a\eta = \sum_{s=1}^{n} e_s^* \wedge (\nabla_{e_s}\eta)$$

for $\eta \in \mathscr{A}(A, E)$.

Proof. Let $\eta \in \mathscr{A}^k(A, E)$, $a_1, \ldots, a_{k+1} \in \Gamma(A)$. Then

$$\begin{aligned}(d^a\eta)(a_1, \ldots, a_{k+1}) &= \sum_{j=1}^{k+1}(-1)^{j-1}\left(\nabla_{a_j}\eta\right)(a_1, \ldots \widehat{a}_j \ldots, a_{k+1})\\ &= \sum_{\sigma\in S(1,p)} \operatorname{sgn}\sigma\ \left(\nabla_{a_{\sigma(1)}}\eta\right)\left(a_{\sigma(2)}, \ldots, a_{\sigma(k+1)}\right)\\ &= \sum_{\sigma\in S(1,p)} \operatorname{sgn}\sigma\ \left(\nabla_{\sum_{s=1}^{n} g\left(e_s, a_{\sigma(1)}\right)e_s}\eta\right)\left(a_{\sigma(2)}, \ldots, a_{\sigma(k+1)}\right)\\ &= \sum_{s=1}^{n}\sum_{\sigma\in S(1,p)} \operatorname{sgn}\sigma\ e_s^*\left(a_{\sigma(1)}\right)(\nabla_{e_s}\eta)\left(a_{\sigma(2)}, \ldots, a_{\sigma(k+1)}\right)\\ &= \left(\sum_{s=1}^{n} e_s^*\wedge\nabla_{e_s}(\eta)\right)(a_1, \ldots, a_{k+1}).\end{aligned}$$

□

As a conclusion from lemmas 8, 7 (e) and 7 (c) we obtain the following expression of the exterior coderivative.

Theorem 5. *If ∇^A is a metric connection,*

$$d^{a*}\eta = (-1)^{n(p+1)+1} * d^a * \eta \tag{4.2}$$

for $\eta \in \mathscr{A}^p(A, E)$.

As a conclusion we obtain

Corollary 1. *If ∇^A is metric, then $d^a(\omega\wedge *\eta) = (d^{\varrho_A}\omega)\wedge *\eta + (-1)^{m+p}\omega\wedge(*d^{a*}\eta)$ for $\omega \in \mathscr{A}^m(A)$, $\eta \in \mathscr{A}^p(A, E)$.*

Proof. Observe

$$\begin{aligned}\omega\wedge(*d^{*a}\eta) &= (-1)^{n(p+1)+1}\,\omega\wedge(**\,d^a*\eta)\\ &= (-1)^{n(p+1)+1}\,\omega\wedge\left((-1)^{(n-p+1)(n-(n-p+1))}\,d^a(*\eta)\right)\\ &= (-1)^{n(p+1)+1}(-1)^{(n-p+1)(p-1)}\,\omega\wedge(d^a(*\eta))\\ &= (-1)^{np+n+1+np-n+p(-p+1)+p-1}\,\omega\wedge(d^a(*\eta))\\ &= (-1)^p\,\omega\wedge(d^a(*\eta)).\end{aligned}$$

Hence

$$\begin{aligned}d^a(\omega\wedge*\eta) &= (d^{\varrho_A}\omega)\wedge*\eta+(-1)^m\,\omega\wedge d^a(*\eta)\\ &= (d^{\varrho_A}\omega)\wedge*\eta+(-1)^{m+p}\,\omega\wedge(*d^{a*}\eta).\end{aligned}$$

□

5. Weitzenböck Formula for Symmetric Forms

Let $\mathscr{S}^k(A,E)$ be the $\mathcal{C}^\infty(M)$-module of all symmetric differential forms of values in the vector bundle E, i.e. the module of sections of $\mathsf{S}^kA^*\otimes E\subset A^{*\otimes k}\otimes E$ and $\mathscr{S}(A,E)=\bigoplus_{k\geq 0}\mathscr{S}^k(A,E)$.

Define the A-connection

$$\nabla:\Gamma(A)\longrightarrow\mathcal{CDO}(\mathsf{S}A^*\otimes E)$$

in the vector bundle $\mathsf{S}A^*\otimes E$ by

$$(\nabla_a\zeta)(a_1,\dots,a_p)=\nabla_a(\zeta(a_1,\dots,a_p))-\sum_{j=1}^{p}\zeta\left(a_1,\dots,\nabla_a^Aa_j,\dots,a_p\right), \tag{5.1}$$

$a,a_1,\dots,a_p\in\Gamma(A)$, $\zeta\in\mathscr{S}^p(A,E)$. Observe that—like in the skew-symmetric case—we have

$$\nabla_a(f\cdot\zeta)=f\cdot\nabla_a\zeta+(\varrho_A)_a(f)\cdot\zeta \tag{5.2}$$

for all $\zeta\in\mathscr{S}(A,E)$, $f\in\mathcal{C}^\infty(M)=\mathscr{S}^0(A,E)$, $a\in\Gamma(A)$, where (ϱ_A) denote here the A-connection in $\mathsf{S}A^*\otimes(M\times\mathbb{R})$ determined by the pair of connections ϱ_A and ∇^A. So, indeed the operator ∇_a has values in $\mathcal{CDO}(\mathsf{S}A^*\otimes E)$ for every $a\in\Gamma(A)$. Moreover, if $\lambda\in\mathscr{S}(A,M\times\mathbb{R})$, $\nu\in\Gamma(E)$, $a\in\Gamma(A)$, then

$$\nabla_a(\lambda\otimes\nu)=((\varrho_A)_a\lambda)\otimes\nu+\lambda\otimes\nabla_a\nu. \tag{5.3}$$

The $C^\infty(M)$-module $\mathscr{S}(A,E)$ is equipped with the structure of the module over the algebra $\mathscr{S}(A,M\times\mathbb{R})$ with the multiplication

$$\odot:\mathscr{S}^p(A,M\times\mathbb{R})\times\mathscr{S}^q(A,E)\longrightarrow\mathscr{S}^{p+q}(A,E)$$

defined by

$$(\lambda\odot\zeta)(a_1,\dots,a_{p+q})=\sum_{\sigma\in S(p,q)}\lambda\left(a_{\sigma(1)},\dots,a_{\sigma(p)}\right)\cdot\zeta\left(a_{\sigma(p+1)},\dots,a_{\sigma(p+q)}\right).$$

Observe that if $\lambda\in\mathscr{S}(A,M\times\mathbb{R})$, $\zeta\in\mathscr{S}(A,E)$, $a\in\Gamma(A)$:

$$\nabla_a(\lambda\odot\zeta)=((\varrho_A)_a\lambda)\odot\eta+\lambda\odot(\nabla_a\zeta).$$

Define the *symmetric derivative* $d^s:\mathscr{S}^k\longrightarrow\mathscr{S}^{k+1}$ by

$$(d^s\eta)(a_1,\dots,a_{k+1})=\sum_{j=1}^{k+1}\left(\nabla_{a_j}\eta\right)(a_1,\dots\widehat{a}_j\dots,a_{k+1}) \tag{5.4}$$

for $\eta \in \mathscr{S}^k$, $a_1, \ldots, a_{k+1} \in \Gamma(A)$.

One can observe that

$$d^s = (k+1) \cdot (\mathrm{Sym} \circ \nabla) \quad \text{on} \quad \mathscr{S}^k(A, E) \tag{5.5}$$

where Sym is the *symmetrizer* given by

$$(\mathrm{Sym}\, \vartheta)(a_1, \ldots, a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \vartheta\left(a_{\sigma(1)}, \ldots, a_{\sigma(k)}\right) \text{ for all } \vartheta \in \Gamma\left(A^{*\otimes k} \otimes E\right).$$

By the *symmetric coderivative* d^{s*} we mean the operator

$$d^{s*} = -\,\mathrm{tr} \circ \nabla|_{\mathscr{S}^k(A,E)} : \mathscr{S}^k(A, E) \longrightarrow \mathscr{S}^{k-1}(A, E) \tag{5.6}$$

where $\nabla : \Gamma\left(A^{*\otimes k} \otimes E\right) \longrightarrow \Gamma\left(A^{*\otimes k+1} \otimes E\right)$ is defined in (2.2), i.e. explicitly

$$(d^{s*}\zeta)(a_1, \ldots, a_{k-2}) = \sum_{j=1}^{n} \zeta(e_j, e_j, a_1, \ldots, a_{k-2})$$

for $\zeta \in \mathscr{S}^k(A, E)$, $a_1, \ldots, a_{k-2} \in \Gamma(A)$.

Define the Laplace-type operator on symmetric tensors by

$$\Delta^s = d^{s*} d^s - d^s d^{s*}.$$

Example 1. Consider the Lie algebroid $A = T\mathbb{R}^n$ and the trivial bundle $E = M \times \mathbb{R}$. Take

$$\omega = \sum_{|\alpha|=k} \omega_\alpha dx_1^{\alpha_1} \odot dx_2^{\alpha_2} \odot \cdots \odot dx_n^{\alpha_n} \in \mathscr{S}^k(A, M \times \mathbb{R})$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\omega_\alpha \in C^\infty(M)$. Observe that

$$\nabla\omega = \sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} dx_j \otimes dx_1^{\alpha_1} \odot dx_2^{\alpha_2} \odot \cdots \odot dx_n^{\alpha_n}.$$

and

$$\begin{aligned} d^s\omega &= \sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} dx_j \odot dx_1^{\alpha_1} \odot dx_2^{\alpha_2} \odot \cdots \odot dx_n^{\alpha_n} \\ &= \sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} dx_1^{\alpha_1} \odot dx_2^{\alpha_2} \odot \cdots \odot dx_j^{\alpha_j+1} \odot \cdots \odot dx_n^{\alpha_n} \end{aligned}$$

So,

$$\begin{aligned} d^{s*}\omega &= -\,\mathrm{tr}\, \nabla\omega \\ &= \sum_{s=1}^{n} i_{e_s} \left(\sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} \delta_j^s \otimes dx_1^{\alpha_1} \odot \cdots \odot \alpha_s dx_s^{\alpha_s - 1} \odot \cdots \odot dx_n^{\alpha_n} \right) \\ &= \sum_{j=1}^{n} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x_j} \alpha_j dx_1^{\alpha_1} \odot \cdots \odot dx_j^{\alpha_j - 1} \odot \cdots \odot dx_n^{\alpha_n}. \end{aligned}$$

Consequently,

$$\begin{aligned}\Delta^s\omega &= d^{s*}d^s\omega - d^s d^{s*}\omega \\ &= -\sum_{|\alpha|=k}\frac{\partial^2\omega_\alpha}{\partial x_j^2}\, dx_1^{\alpha_1}\odot dx_2^{\alpha_2}\odot\cdots\odot dx_n^{\alpha_n} \\ &= -\sum_{|\alpha|=k}\left(\Delta^s\omega_\alpha\right)\, dx_1^{\alpha_1}\odot dx_2^{\alpha_2}\odot\cdots\odot dx_n^{\alpha_n}\end{aligned}$$

where $\Delta^s\omega_\alpha = \Delta^a\omega_\alpha$ is the classical Laplacian on the smooth function ω_α.

Notice that if A is transitive, Δ^s is a second order strongly elliptic operator with the metric symbol

$$\sigma_{\Delta^s}(\omega,\eta) = |\omega|^2\eta, \quad \omega\in\mathscr{S}^k(A, M\times\mathbb{R}),\ \eta\in\mathscr{S}^k(A,E).$$

Indeed, take $x\in M$, $e\in \mathsf{S}^k A_x^*\otimes E_x$, $\zeta\in\mathscr{S}^k(A,E)$ and $\omega\in A_x^*$ such that $\omega=(df)(x)$ for some smooth function f satisfying $f(x)=0$ and $\zeta(x)=e$. Since $(\varrho_A)(f) = d^s f = d^a f$, the relation (5.2) implies that

$$\sigma_{d^s}(\omega,e) = d^s(f\zeta)(x) = (d^s f\odot\zeta + f d^s\zeta)(x) = \omega\odot e$$

and

$$\sigma_{d^{s*}}(\omega,e) = d^{s*}(f\zeta)(x) = \left(i_{(df)^\sharp}\zeta\right)(x) = i_{\omega^\sharp}e;$$

hence

$$\sigma_{d^{s*}d^s}(\omega,e) = i_{\omega^\sharp}(\omega\odot e) = i_{\omega^\sharp}\omega\odot e + \omega\odot i_{\omega^\sharp}e$$

and

$$\sigma_{d^s d^{s*}}(\omega,e) = \omega\odot i_{\omega^\sharp}e.$$

Consequently,

$$\sigma_{\Delta^s}(\omega,e) = \sigma_{d^{s*}d^s + d^s d^{s*}}(\omega,e) = i_{\omega^\sharp}\omega\odot e = g\left(\omega^\sharp,\omega^\sharp\right)e.$$

Define the *symmetric Ricci type operator*

$$\mathcal{R}^s : \mathscr{S}(A,E)\longrightarrow\mathscr{S}(A,E)$$

by

$$(\mathcal{R}^s\zeta)(a_1,\ldots,a_k) = \sum_{j=1}^n\sum_{s=1}^k\left(\mathcal{R}^\nabla_{e_j,a_s}\zeta\right)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k),$$

the operator

$$\mathcal{T}^s : \mathscr{S}(A,E)\longrightarrow\mathscr{S}(A,E)$$

by

$$(\mathcal{T}^s\zeta)(a_1,\ldots,a_k) = \sum_{j=1}^n\left(\nabla_{T^A(e_j,a_s)}\zeta\right)(a_1,\ldots,\widehat{a}_s,\ldots,a_k),$$

and next,

$$\mathcal{M}^s : \mathscr{S}(A,E)\longrightarrow\mathscr{S}(A,E)$$

by

$$(\mathcal{M}^s\zeta)(a_1,\ldots,a_k) = \sum_{j=1}^n\sum_{s=1}^k\left(i_{\nabla^A_{a_s}e_j}i_{e_j} + i_{e_j}i_{\nabla^A_{a_s}e_j}\right)(\nabla\zeta)(a_1,\ldots\widehat{a}_s\ldots,a_k),$$

where $\zeta \in \mathscr{S}^k(A,E)$, $a_1,\ldots,a_k \in \Gamma(A)$, $(e_1,\ldots,e_n)$ is a local orthonormal frame of A, $\mathcal{R}^\nabla$ is the curvature tensor of the connection $\nabla : \Gamma(A) \to \mathcal{CDO}(\mathsf{S}^k A^* \otimes E)$ defined in (5.1). Hence, by Lemma 2,

$$
\begin{aligned}
(5.7)\qquad & (\mathcal{R}^s\zeta)(a_1,\ldots,a_k) \\
= & \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla^2_{e_j,a_s}\zeta - \nabla^2_{a_s,e_j}\zeta\right)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k) + (\mathcal{T}^s\zeta)(a_1,\ldots,a_k).
\end{aligned}
$$

Theorem 6.

$$
-(d^{s*}d^s\eta)(a_1,\ldots,a_k) = \left(\operatorname{tr}\nabla^2\eta\right)(a_1,\ldots,a_k) + \sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla^2_{e_j,a_s}\eta\right)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k)
$$

for $\eta \in \mathscr{S}^k(A,E)$.

Proof. Let $\eta \in \mathscr{S}^k(A,E)$, $a_1,\ldots,a_k \in \Gamma(A)$. Then

$$
\begin{aligned}
& -((d^s)^* d^s\eta)(a_1,\ldots,a_k) \\
= & (\operatorname{tr}\nabla d^s\eta)(a_1,\ldots,a_k) \\
= & \sum_{j=1}^{n}\left(\nabla_{e_j}(d^s\eta)\right)(e_j,a_1,\ldots,a_k) \\
= & \sum_{j=1}^{n}\nabla_{e_j}((d^s\eta)(e_j,a_1,\ldots,a_k)) - \sum_{j=1}^{n}(d^s\eta)\left(\nabla^A_{e_j}e_j,a_1,\ldots,a_k\right) \\
& -\sum_{j=1}^{n}\sum_{s=1}^{k}(d^s\eta)\left(e_j,a_1,\ldots,\nabla_{e_j}a_s,\ldots,a_k\right) \\
= & \sum_{j=1}^{n}\nabla_{e_j}\left((\nabla_{e_j}\eta)(a_1,\ldots,a_k)\right) + \sum_{j=1}^{n}\sum_{s=1}^{k}\nabla_{e_j}((\nabla_{a_s}\eta)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k)) \\
& -\sum_{j=1}^{n}\left(\nabla_{\nabla^A_{e_j}e_j}\eta\right)(a_1,\ldots,a_k) - \sum_{j=1}^{n}\sum_{s=1}^{k}(\nabla_{a_s}\eta)\left(\nabla_{e_j}e_j,a_1,\ldots\widehat{a}_s\ldots,a_k\right) \\
& -\sum_{j=1}^{n}\sum_{s=1}^{k}(\nabla_{e_j}\eta)\left(a_1,\ldots,\nabla^A_{e_j}a_s,\ldots,a_k\right) \\
& -\sum_{j=1}^{n}\sum_{s=1}^{k}\left(\nabla_{\nabla^A_{e_j}a_s}\eta\right)(e_j,a_1,\ldots\widehat{a}_s\ldots,a_k) \\
& -\sum_{j=1}^{n}\sum_{s=1}^{k}\sum_{t\neq s}(\nabla_{a_t}\eta)\left(e_j,a_1,\ldots,\nabla^A_{e_j}a_s,\ldots\widehat{a}_t\ldots,a_k\right).
\end{aligned}
$$

One can see that

$$\begin{aligned}
& \left(\operatorname{tr} \nabla^2 \eta\right)(a_1, \ldots, a_k) \\
= & \sum_{j=1}^{n} \left(\nabla^2_{e_j, e_j} \eta\right)(a_1, \ldots, a_k) \\
= & \sum_{j=1}^{n} \nabla_{e_j} \left(\left(\nabla_{e_j} \eta\right)(a_1, \ldots, a_k)\right) - \sum_{j=1}^{n} \sum_{s=1}^{k} \left(\nabla_{e_j} \eta\right)\left(a_1, \ldots, \nabla^A_{e_j} a_s, \ldots, a_k\right) \\
& - \sum_{j=1}^{n} \left(\nabla_{\nabla^A_{e_j} e_j} \eta\right)(a_1, \ldots, a_k)
\end{aligned}$$

and

$$\begin{aligned}
& \left(\nabla^2_{e_j, a_s} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k) \\
= & \nabla_{e_j} \left(\left(\nabla_{a_s} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k)\right) - \left(\nabla_{a_s} \eta\right)\left(\nabla^A_{e_j} e_j, a_1, \ldots \widehat{a}_s \ldots, a_k\right) \\
& - \sum_{t \neq s} \left(\nabla_{a_t} \eta\right)\left(e_j, a_1, \ldots, \nabla^A_{e_j} a_s, \ldots \widehat{a}_t \ldots, a_k\right) - \left(\nabla_{\nabla^A_{e_j} a_s} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k).
\end{aligned}$$

Hence

$$\begin{aligned}
& -\left(\left(d^s\right)^* d^s \eta\right)(a_1, \ldots, a_k) \\
= & \left(\operatorname{tr} \nabla^2 \eta\right)(a_1, \ldots, a_k) + \sum_{j=1}^{n} \sum_{s=1}^{k} \left(\nabla^2_{e_j, a_s} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k).
\end{aligned}$$

□

Theorem 7.

$$\left(d^s d^{s*} \eta\right)(a_1, \ldots, a_k) = \left(\mathcal{M}^s \eta\right)(a_1, \ldots, a_k) - \sum_{s=1}^{k} \sum_{j=1}^{n} \left(\nabla^2_{a_s, e_j} \eta\right)(e_j, a_1, \ldots \widehat{a}_s \ldots, a_k)$$

for $\eta \in \mathscr{S}^k(A, E)$.

Proof. Let $\eta \in \mathscr{S}^k(A, E)$, $a_1, \ldots, a_k \in \Gamma(A)$. Since

$$\begin{aligned}
& \left(\operatorname{tr} \nabla^2 \eta\right)(a_1, \ldots, a_k) \\
= & \sum_{j=1}^{n} \left(\nabla^2_{e_j, e_j} \eta\right)(a_1, \ldots, a_k) \\
= & \sum_{j=1}^{n} \nabla_{e_j} \left(\left(\nabla_{e_j} \eta\right)(a_1, \ldots, a_k)\right) - \sum_{j=1}^{n} \sum_{s=1}^{k} \left(\nabla_{e_j} \eta\right)\left(a_1, \ldots, \nabla^A_{e_j} a_s, \ldots, a_k\right) \\
& - \sum_{j=1}^{n} \left(\nabla_{\nabla^A_{e_j} e_j} \eta\right)(a_1, \ldots, a_k)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\nabla^2_{a_s,e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
=\;&\nabla_{a_s}\left(\nabla_{e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)-\left(\nabla_{\nabla^A_{a_s}e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
=\;&\nabla_{a_s}\left(\left(\nabla_{e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\right)-\left(\nabla_{e_j}\eta\right)\left(\nabla^A_{a_s}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right)\\
&-\textstyle\sum_{t\neq s}\left(\nabla_{e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots\nabla^A_{a_s}a_t\dots,a_k)-\left(\nabla_{\nabla^A_{a_s}e_j}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k),
\end{aligned}$$

by (5.4) and (5.6) we have

$$\begin{aligned}
&-\left((d^s)^*d^s\eta\right)(a_1,\dots,a_k)\\
=\;&\left(\operatorname{tr}\nabla d^s\eta\right)(a_1,\dots,a_k)\\
=\;&\sum_{j=1}^n\left(\nabla_{e_j}(d^s\eta)\right)(e_j,a_1,\dots,a_k)\\
=\;&\sum_{j=1}^n\nabla_{e_j}\left((d^s\eta)(e_j,a_1,\dots,a_k)\right)-\sum_{j=1}^n(d^s\eta)\left(\nabla^A_{e_j}e_j,a_1,\dots,a_k\right)\\
&-\sum_{j=1}^n\sum_{s=1}^k(d^s\eta)\left(e_j,a_1,\dots,\nabla^A_{e_j}a_s,\dots,a_k\right)\\
=\;&\sum_{j=1}^n\nabla_{e_j}\left(\left(\nabla_{e_j}\eta\right)(a_1,\dots,a_k)\right)+\sum_{j=1}^n\sum_{s=1}^k\nabla_{e_j}\left((\nabla_{a_s}\eta)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\right)\\
&-\sum_{j=1}^n\left(\nabla_{\nabla^A_{e_j}e_j}\eta\right)(a_1,\dots,a_k)-\sum_{j=1}^n\sum_{s=1}^k(\nabla_{a_s}\eta)\left(\nabla^A_{e_j}e_j,a_1,\dots\widehat{a}_s\dots,a_k\right)\\
&-\sum_{j=1}^n\sum_{s=1}^k\left(\nabla_{e_j}\eta\right)\left(a_1,\dots,\nabla^A_{e_j}a_s,\dots,a_k\right)-\sum_{j=1}^n\sum_{s=1}^k\left(\nabla_{\nabla^A_{e_j}a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k)\\
&-\sum_{j=1}^n\sum_{s=1}^k\sum_{t\neq s}(\nabla_{a_t}\eta)\left(e_j,a_1,\dots,\nabla^A_{e_j}a_s,\dots\widehat{a}_t\dots,a_k\right)\\
=\;&\left(\operatorname{tr}\nabla^2\eta\right)(a_1,\dots,a_k)+\sum_{j=1}^n\sum_{s=1}^k\left(\nabla^2_{e_j,a_s}\eta\right)(e_j,a_1,\dots\widehat{a}_s\dots,a_k).
\end{aligned}$$

□

As a consequence of theorems 6, 7, definitions of $\mathcal{T}^s$, $\mathcal{M}^s$ and (5.7) we obtain the following formula on symmetric tensors.

Theorem 8. (*Weitzenböck-type Formula for Symmetric Forms*)

$$\Delta^s=\nabla^*\nabla-\mathcal{R}^s-\mathcal{M}^s+\mathcal{T}^s.$$

Notice that if ∇^A is a metric A-connection, then $\mathcal{M}^s=0$, and then $\Delta^s-\nabla^*\nabla=-\mathcal{R}^s+\mathcal{T}^s$. In the case where ∇^A is the Levi-Civita connection, the Weitzenböck formula for symmetric forms reduces to the shape:

$$\Delta^s=\nabla^*\nabla-\mathcal{R}^s.$$

References

[1] B. Balcerzak, J. Kubarski, W. Walas, *Primary characteristic homomorphism of pairs of Lie algebroids and Mackenzie algebroid*, Banach Center Publ. 54 (2001), 135–173.
[2] B. Balcerzak, J. Kalina, A. Pierzchalski, *Weitzenböck Formula on Lie algebroids*, Bull. Polish Acad. Sci. Math. 60 (2012), 165–176.
[3] B. Balcerzak, A. Pierzchalski, *Generalized Gradients on Lie Algebroids*, to appear.
[4] J.-P. Bourguignon, *The "magic" of Weitzenböck formulas*, Variational Methods, Proceedings of a Conference Paris, June 1988 (H. Berestycki, J.-M. Coron, I. Ekeland eds.), in: Progress in Nonlinear Differential Equations and Their Applications, Volume 4, Birkhäuser Boston, 1990, pp. 251–271.
[5] S. Gallot, D. Meyer, *Opérateur de courbure et laplacien des formes différentielles d'une variété riemannienne*, J. Math. Pures Appl. (9) 54 (1975), no. 3, 259–284.
[6] Ph. J. Higgins, K. C. H. Mackenzie, *Algebraic constructions in the category of Lie algebroids*, J. Algebra 129 (1990), 194–230.
[7] Y. Kosmann-Schwarzbach, C. Laurent-Gengoux, A. Weinstein, *Modular classes of Lie algebroid morphisms*, Transform. Groups 13 (2008), 727–755.
[8] J. Kubarski, *Hirzebruch signature operator for transitive Lie algebroids*, in: Differential Geometry and its Applications, Proc. Conf., in Honour of Leonhard Euler, Olomouc, August 2007, World Sci. Publ. Co., 2008, 317–328.
[9] K. C. H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, London Math. Soc. Lecture Note Ser. 213, Cambridge Univ. Press, 2005.
[10] C.-M. Marle, *Calculus on Lie algebroids, Lie groupoids and Poisson manifolds*, Dissertationes Mathematicae 457 (2008), 57 pp.
[11] L. Maxim-Raileanu, *Cohomology of Lie algebroids*, An. Sti. Univ. "Al. I. Cuza" Iasi Sect. I a Mat. (N.S.) 22 (2) (1976), 197–199.
[12] R. Narasimhan, *Analysis on Real and Complex Manifolds. Second Edition*, North-Holland, 1985.
[13] B. Ørsted, A. Pierzchalski, *The Ahlfors Laplacian on a Riemannian manifold with boundary*, Michigan Math. J. 43 (1) (1996), 99–122
[14] E. Stein, G. Weiss, *Generalization of the Cauchy-Riemann equations and representations of the rotation group*, Amer. J. Math. 90 (1968), 163–196..
[15] K. Yano, *Integral Formulas in Riemannian Geometry*, Marcel Dekker, Inc., 1970.
[16] K. Yano and S. Bochner, *Curvature and Betti Numbers*, Princeton Univ. Press, Princeton, 1953.

• Bogdan Balcerzak, Institute of Mathematics, Lodz University of Technology, Wólczańska 215, 90-924 Łódź, e-mail: bogdan.balcerzak@p.lodz.pl

• Antoni Pierzchalski, Faculty of Mathematics and Computer Science, University of Lodz, Banacha 22, 90-238 Łódź, e-mail: antoni@math.uni.lodz.pl