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Cyclic and cocyclic maps and generalized Whitehead products

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Given co-H-spaces X and Y, B. Gray [13] has defined a co-H-space $X \circ Y$ and a natural transformation $X \circ Y \to X \vee Y$ which leads to a generalized Whitehead product. We make use of that product and sketch ideas on its dual to examine cyclic and cocyclic maps. Given spaces X and Y, some results on Gottlieb sets $\mathcal{G}(X,Y)$ and dual Gottlieb sets $\mathcal{DG}(X,Y)$ are stated.

Introduction

The Gottlieb group $G_n(X)$ of a space X is the subgroup of the homotopy group $\pi_n(X)$ of X consisting of homotopy classes of maps $f : \mathbb{S}^n \to X$ such that the map $f \vee \mathrm{id}_X : \mathbb{S}^n \vee X \to X$ admits an extension $F : \mathbb{S}^n \times X \to X$. The study of the properties and structure of the Gottlieb groups represents a fundamental problem in homotopy theory dating back to their introduction by D. Gottlieb in the 1960's

[8, 10]. Connections between the Gottlieb groups and fixed point theory [8, 15, 22], transformation groups [11, 20], covering spaces [11, 16] and the homotopy theory of fibrations [9, 12, 21] have been extensively researched.

The definition of $G_n(X)$ uses the concept of cyclic homotopies. K. Varadarajan [23] studies the role of cyclic and cocyclic (dual of cyclic) maps in the set-up of Eckmann-Hilton duality. The set of homotopy classes of cyclic maps $X \to Y$, denoted by $\mathcal{G}(X,Y)$ is a group provided X carries an H-cogroup structure. Dually, the set of homotopy classes of cocyclic maps $X \to Y$, denoted by $\mathcal{DG}(X,Y)$ is a group provided Y carries an H-group structure. Relationships between these generalized Gottlieb (dual Gottlieb groups) and the generalized Whitehead product (the dual generalized Whitehead product) [1] have been considered in [14, 17, 18, 19] and other various papers.

The aim of this paper is to present those results in the context of the so called Theriault product considered by B. Gray in [13] being an extended version of the generalized Whitehead product from [1] and its dual. The first section expounds the notions and clarify results needed in next two sections. Section 2 recalls results on cyclic maps and then takes up the systematic study of these maps in the context of results from [13].

Section 3 is devoted to cocyclic maps. First, their relations with the dual generalized Whitehead product [1] are summarized. In particular, a characterization of co-H-spaces in terms of the cocyclicity of maps is concluded. Then, following *mutatis mutandis* the construction presented by B. Gray in [13] and the cotelescope concept, we sketch ideas of the dual Theriault product extending the dual generalized Whitehead [1] and relate cocyclic maps to this product. Many results and proofs on the Theriault product can be dualized. The details will be published somewhere shortly.

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supporting during his stay.

2 Prerequisites

We concentrate with connected and based spaces having the homotopy type of CW-complexes. All maps and homotopies preserve base points. For simplicity, we sometimes use the same symbol for a map and its homotopy class. Denote by $[X, Y]$ the set of homotopy classes of continuous maps $X \to Y$ and write $\mathbb{S}^n$ for the n-dimensional sphere. In particular, let $\pi_n(X) = [\mathbb{S}^n, X]$ be the nth homotopy group of a space X for $n \geq 0$.

Next, write ΣX and ΩX for the suspension and the loop space of X. Recall that ΣX and ΩX are an H-cogroup and an H-group, respectively. If $f : X \to Y$ then for every space Z, we have homomorphisms $(\Sigma f)^* : [\Sigma Y, Z] \to [\Sigma X, Z]$ and $(\Omega f)_* : [Z, \Omega X] \to [Z, \Omega Y]$. Further, there are canonical natural maps $e : \Sigma\Omega X \to X$ and $e' : X \to \Omega\Sigma X$.

The following well-known results are frequently used:

Proposition 2.1. (1) *If X is a co-H-space, then there is a map $s : X \to \Sigma\Omega X$ such that $es \simeq \mathrm{id}_X$;*

(2) *If X is an H-space, then there is a map $s' : \Omega\Sigma X \to X$ such that $s'e' \simeq \mathrm{id}_X$;*

(3) *Let X and Y be an H-cogroup and an H-group, respectively. Then, $[X, Z]$ and $[Z, Y]$ are groups for any space Z.*

Let $X \flat Y$ be the flat product and $X \wedge Y$ the smash product, that is, the fibre and the cofibre of the inclusion $X \vee Y \hookrightarrow X \times Y$. Next, write $\Delta : X \to X \times X$ and $\nabla : X \vee X \to X$ for the diagonal and folding maps, respectively.

The Whitehead product $[-, -] : \pi_m(X) \times \pi_n(X) \to \pi_{m+n-1}(X)$, determined by the Whitehead map $w : \mathbb{S}^{m+n-1} \to \mathbb{S}^m \vee \mathbb{S}^n$ plays a crucial role in the homotopy theory. The generalized Whitehead map $w : \Sigma(X \wedge Y) \to \Sigma X \vee \Sigma Y$ constructed in [1] leads to the generalized Whitehead product

$$[-, -] : [\Sigma X, Z] \times [\Sigma Y, Z] \to [\Sigma(X \wedge Y), Z].$$

Now, let $\mathcal{CO}$ be the category of simply connected co-H-spaces and co-H-maps. In [13], a functor

$$\circ : \mathcal{CO} \times \mathcal{CO} \to \mathcal{CO}$$

(called the Theriault product) and a natural transformation $w : X \circ Y \to X \vee Y$ for co-H-spaces X, Y generalizing the Whitehead product have been defined. More precisely, in [13, Theorem 1, Theorem 2] it has been shown:

Theorem 2.2. *There is a functor*

$$\circ : \mathcal{CO} \times \mathcal{CO} \longrightarrow \mathcal{CO}$$

and equivalences in $\mathcal{CO}$:
(1) $(\Sigma X) \circ Y \cong X \wedge Y$;
(2) $\Sigma(X \circ Y) \cong X \wedge Y$;
(3) $(X_1 \vee X_2) \circ Y \cong (X_1 \circ Y) \vee (X_2 \circ Y)$
and homotopy equivalences:
(4) $X \circ Y \cong Y \circ X$;
(5) $(X \circ Y) \circ Z \cong X \circ (Y \circ Z)$.

Theorem 2.3. *There is a natural transformation*

$$w_\circ : X \circ Y \longrightarrow X \vee Y$$

which is the Whitehead product map in case X and Y are both suspensions. Furthermore, there is a homotopy equivalence

$$X \times Y \cong (X \vee Y) \cup_{w_\circ} C(X \circ Y),$$

where $(X \vee Y) \cup_{w_\circ} C(X \circ Y)$ is the mapping cone of $w_\circ : X \circ Y \longrightarrow X \vee Y$.

Notice that $w_\circ : X \circ Y \longrightarrow X \vee Y$ defines a map

$$[-,-]_\circ : [X,Z] \times [Y,Z] \to [X \circ Y, Z]$$

for any space Z.

3 Cyclic maps and evaluation groups

According to [23], a map $f : X \to Y$ is said to be *cyclic* if there exists a map $F : X \times Y \to Y$ such that the diagram

$$\begin{array}{ccc} X \vee Y & \xrightarrow{\nabla(f\vee \mathrm{id}_Y)} & Y \\ \big\downarrow & \nearrow_{F} & \\ X \times Y & & \end{array}$$

is homotopy commutative.

Write $\mathcal{G}(X,Y)$ for the set of homotopy classes of cyclic maps from X to Y called the *Gottlieb subset* of $[X,Y]$. If X is an H-cogroup then by [23, Theorem 1.5] the subset $\mathcal{G}(X,Y) \subseteq [X,Y]$ is a subgroup of $[X,Y]$. If $X = \mathbb{S}^n$, the n-dimensional sphere then $\mathcal{G}(\mathbb{S}^n,Y) = G_n(Y)$ is called the nth *evaluation subgroup* of Y or the nth *Gottlieb group* defined in [8] for $n = 1$ and then in [10] for any $n \geq 1$. Then, $G_{n+k}(\mathbb{S}^n)$ and $G_{n+k}(\mathbb{F}P^n)$ have been extensively studied in [6] and [7], respectively, where $\mathbb{F}P^n$ is the projective space over $\mathbb{F}$ being the reals $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ or the Cayley algebra $\mathbb{K}$.

To show the existence of cyclic maps, we recall:

Proposition 3.1 ([23, Lemmas 1.3 and 1.4]). *Let $f : X \to Y$ be a cyclic map and $g : Z \to X$ an arbitrary map. Then:*

(1) *$fg : Z \to Y$ is a cyclic map;*

(2) *if a map $g : Y \to Y'$ has a right homotopy inverse then $gf : X \to Y'$ is a cyclic map.*

In particular, let X be a co-H-space, $f : X \to Y$ and $e : \Sigma\Omega X \to X$ the usual map. Then f is cyclic if and only if $fe : \Sigma\Omega X \to Y$ is cyclic.

Proposition 3.2 ([17, Proposition 3.3]). *Let Y be a space. Then the following are equivalent:*

(1) *Y is an H-space;*

(2) *id_Y is cyclic;*

(3) *$\mathcal{G}(X,Y) = [X,Y]$ for any space X.*

Another way in which cyclic maps arise naturally is by fibrations. Suppose $F \to E \to B$ is a fibration. Then we have an operation $\rho : F \times \Omega B \to F$ and the restriction $\partial = \rho|_{\Omega B}$ is cyclic.

Now, we make use of Theorem 2.3 to deduce results being key ones in sequel.

Corollary 3.3. *Let X, Y be spaces. Then:*

(1) *the map $w_\circ : \Sigma\Omega X \circ \Sigma\Omega Y \to \Sigma\Omega X \vee \Sigma\Omega Y$ coincides with the generalized Whitehead map $w : \Sigma(\Omega X \wedge \Omega Y) \to \Sigma\Omega X \vee \Sigma\Omega Y$;*

(2) *there is the commutative diagram*

$$\begin{array}{ccc} X \circ Y & \xrightarrow{\;w_\circ\;} & X \vee Y \\ {\scriptstyle e\circ e}\uparrow & & \uparrow{\scriptstyle e\vee e} \\ \Sigma\Omega X \circ \Sigma\Omega Y & \xrightarrow{\;w_\circ\;} & \Sigma\Omega X \vee \Sigma\Omega Y. \end{array}$$

Then, the result [18, Proposition 4.6] leads to:

Proposition 3.4. *Let X be a co-H-space and $f : X \to Y$ a cyclic map. Then $[f, g]_\circ = 0$ for any map $g : Z \to Y$ provided Z is a co-H-space.*

Proof. Let $f : X \to Y$ be a cyclic map. Then by Proposition 3.1 the map $fe : \Sigma\Omega X \to Y$ is cyclic as well. Hence, in view of [18, Proposition 4.6], we get $[fe, ge] = 0$. Because X and Z are co-H-spaces, Corollary 3.3 leads to $[f, g]_\circ = 0$ and the proof is complete. □

Further [5, Proposition 2.3] and Proposition 2.1 yield:

Proposition 3.5. *For a map $f : X \to Y$ of H-groups, the following are equivalent:*

(1) *f_* maps $[Z, X]$ into the center of $[Z, Y]$;*

(2) *$\nabla(f \vee \mathrm{id}_Y)i \simeq \star$, where $i : X\flat Y \hookrightarrow X \vee Y$ is the inclusion map.*

If one of the conditions above is fulfilled, T. Ganea [5] says that f *maps X into the center of Y*.

The proof of the result below is a direct consequence of Corollary 3.3 and [14, Corollary 3].

Theorem 3.6. *Let X, Y be co-H-spaces and $f : X \to Y$. Then the following are equivalent:*

(1) *f is cyclic;*

(2) *f maps ΩX into the center of ΩY;*

(3) $[f, \mathrm{id}_Y]_\circ = 0$.

Theorem 3.6 generalized the result known to spheres: $f \in \mathcal{G}(\mathbb{S}^{n+k}, \mathbb{S}^n) = G_{n+k}(\mathbb{S}^n)$ *if and only if the Whitehead product* $[f, \mathrm{id}_{\mathbb{S}^n}] = 0$ which has been applied in [6] to find $G_{n+k}(\mathbb{S}^n)$ for $k \leq 13$. Certainly, the computations depend on the Whitehead product on spheres.

Now, let $i_1 : Y_1 \hookrightarrow Y_1 \vee Y_2$ and $i_2 : Y_2 \hookrightarrow Y_1 \vee Y_2$ be the inclusion maps. Then, Theorem 3.6 leads to the following generalization of [3, Proposition 2.3]:

Corollary 3.7. *Let X, Y_1, Y_2 be co-H-spaces and $f : X \to Y_1 \vee Y_2$. Then, f is cyclic if and only if $[f, i_1]_\circ = [f, i_2]_\circ = 0$.*

If A is an abelian group and $n \geq 2$ then the Moore space $M(A, n)$ is a co-H-space as a suspension of some space. Because $M(A_1 \oplus A_2, n) \cong M(A_1, n) \vee M(A_2, n)$ for some abelian groups A_1, A_2 [3, Proposition 2.3] has been applied to compute $G_n(M(A, n))$ provided A is a finitely generated abelian group. The paper [2] considers the set of homotopy classes of co-structures on a Moore space $M(A, n)$, where A is an abelian group and $n \geq 2$ is an integer. It is shown that for $n > 2$ the set has one element and for $n = 2$ the set is in one-to-one correspondence with $\mathrm{Ext}(A, A \otimes A)$. Further, a detailed investigation of the co-H-structures on $M(A, 2)$ in the case $A = \mathbb{Z}_m$, the integers mod m has been considered. It has been shown that all co-H-structures on $M(\mathbb{Z}_m, 2)$ are associative and commutative if m is odd, and all co-H-structures on $M(\mathbb{Z}_m, 2)$ are associative and non-commutative if m is even. Therefore, Corollary 3.7 should be useful to describe $G_2(M(A, 2))$ with respect to all possible co-H-structures on $M(A, 2)$ provided A is a finitely generated group or more generally, $A = \bigoplus_{i \in I} \mathbb{Z} \oplus \bigoplus_{j \in J} \mathbb{Z}_{m_j}$.

Let Y be an H-group and $f : X \to Y$. Recall that f is called *central* if $c(\mathrm{id}_Y \times f) \simeq \star$, where $c : Y \times Y \to Y$ is the basic commutator map. If

Y is an H-space then, in view of Proposition 2.1, the map $\Omega : [X,Y] \to [\Omega X, \Omega Y]$ given by $f \mapsto \Omega f$ is injective. Write $[\Omega X, \Omega Y]_{\mathcal{C}\Omega}$ for the subset of $[\Omega X, \Omega Y]$ consisting of those homotopy classes of maps Ωf which are central. Following [18, Definition 4.1], we set $\mathcal{C}(X,Y) = \Omega^{-1}[\Omega X, \Omega Y]_{\mathcal{C}\Omega}$. By [18, Propositions 4.6 and 5.1], it holds:

Proposition 3.8. *Let X, Y and Z be spaces.*

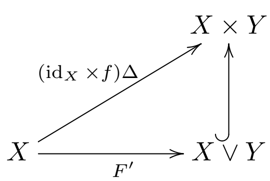
(1) *If $f \in \mathcal{C}(\Sigma X, Z)$ then $[f,g] = 0$ for any $g \in [\Sigma Y, Z]$.*

(2) *$\mathcal{C}(X,Y)$ is a subgroup contained in the center of $[X,Y]$ if X is a co-H-space with a right homotopy inverse and Y is any space.*

It follows that if X is a co-H-space with a right homotopy inverse, then for every space Y, $\mathcal{G}(X,Y) \subseteq \mathcal{C}(X,Y) \subseteq$ center of $[X,Y]$ as subgroups. In particular, $\mathcal{G}(X,Y)$ and $\mathcal{C}(X,Y)$ are abelian groups provided X is a co-H-space. This generalizes Gottlieb's result from [8] that the Gottlieb group $G_1(Y)$ lies in the center of the homotopy group $\pi_1(Y)$.

4 Cocyclic maps and coevaluation groups

According to [23], a map $f : X \to Y$ is said to be *cocyclic* if there is a map $F' : X \to X \vee Y$ such that the diagram



is homotopy commutative.

Write $\mathcal{DG}(X,Y)$ for the set of homotopy classes of cocyclic maps from X to Y called the *dual Gottlieb subset* of $[X,Y]$. If Y is an H-group then by [23, Theorem 1.5] the subset $\mathcal{DG}(X,Y) \subseteq [X,Y]$ is a subgroup of $[X,Y]$.

Certainly, every map $f : X \to Y$ is cocyclic provided X is a co-H-space.

Another way in which cocyclic maps arise naturally is by cofibrations (cf. [19]). Suppose $A \to B \to C$ is a cofibration. Then we have a cooperation $\phi : C \to C \vee \Sigma A$. Then the map $s = p_2\phi : C \to \Sigma A$ is cocyclic, where $p_2 : C \vee \Sigma A \to \Sigma A$ is the projection map.

Notice that if $f : X \to Y$ is a cocyclic map and $g : X' \to X$ has a left homotopy inverse then $fg : X' \to Y$ is also a cocyclic map. Then, in view of [23, Lemma 7.2], Proposition 3.1 can be dualized as follows:

Proposition 4.1. *Let $f : X \to Y$ be a cocyclic map. Then:*

(1) *$gf : X \to Z$ is a cocyclic map for an arbitrary map $g : Y \to Z$;*

(2) *if a map $g : X' \to X$ has a left homotopy inverse then $fg : X' \to Y$ is a cocyclic map.*

In particular, let Y be an H-space, $f : X \to Y$ and $e' : Y \to \Omega\Sigma Y$ the usual map. Then f is cocyclic if and only if $e'f : X \to \Omega\Sigma Y$ is cocyclic. Further, [19, Proposition 3.2] provides a characterization of a co-H-space in terms of the cocyclicity of maps.

Proposition 4.2. *Let X be a space. Then the following are equivalent:*

(1) *X is a co-H-space;*

(2) id_X *is cocyclic;*

(3) *$\mathcal{DG}(X, Y) = [X, Y]$ for any space Y.*

Recall from [1] that given spaces X and Y, there is a dual Whitehead map $w' : \Omega X \times \Omega Y \to \Omega(X\flat Y)$. This leads to the dual generalized Whitehead product

$$[-,-]' : [Z, \Omega X] \times [Z, \Omega Y] \to [Z, \Omega(X\flat Y)]$$

for any space Z.

Now, let $\mathcal{CO}'$ be the category of simply connected H-spaces and H-maps. Following *mutatis mutandis* the construction presented by B. Gray in [13] and the cotelescope construction, we get a functor

$$\circ' : \mathcal{CO}' \times \mathcal{CO}' \longrightarrow \mathcal{CO}'$$

(called the dual Theriault product) and a natural transformation

$$w' : X \times Y \longrightarrow X \circ' Y$$

which leads to a map

$$[-,-]_{\circ'} : [Z,X] \times [Z,Y] \to [Z, X \circ' Y]$$

for H-spaces X, Y and any space Z. Many results and proofs of $[-,-]_{\circ}$ can be dualized. We mention only that the products $[-,-]'$ and $[-,-]_{\circ'}$ coincide provided X, Y are loop spaces. However, many cannot since $[-,-]_{\circ'}$ is not precise a dual of $[-,-]_{\circ}$. The details and dual version of Theorem 2.2 and Theorem 2.3 will be published somewhere shortly.

The dual version of Corollary 3.3 and the result [18, Proposition 4.6] yield:

Proposition 4.3. *Let Y be an H-space and $f : X \to Y$ a cocyclic map. Then $[f,g]_{\circ'} = 0$ for any map $g : X \to Z$ provided Z is an H-space.*

$>$From this a dual version of Corollary 3.7 follows:

Corollary 4.4. *Let X_1, X_2, Y be H-spaces and $f : X_1 \times X_2 \to Y$. Then, f is cocyclic if and only if $[f,p_1]_{\circ'} = [f,p_2]_{\circ'} = 0$ for the projection maps $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$.*

Let A be an abelian group and $n \geq 2$. Then the associated Eilenberg-MacLane space $K(A,n)$ inherits an H-structure. Because $K(A_1 \times A_2, n) \cong K(A_1,n) \times K(A_2,n)$ for any abelian groups A_1, A_2, Corollary 4.4 should be very useful to compute $\mathcal{DG}(K(A,n),Y)$ provided that A is an abelian finitely generated group and Y is an H-space.

The dual version of Proposition 3.5 and [5, Proposition 2.3] lead to:

Proposition 4.5. *For a map $f : X \to Y$ of H-cogroups, the following are equivalent:*

(1) *f^* maps $[Y,Z]$ into the center of $[X,Z]$;*

(2) *$j(\mathrm{id}_X \times f)\Delta \simeq \star$, where $j : X \times Y \to X \wedge Y$ is the quotient map.*

If one of the conditions above is fulfilled, we follow T. Ganea [5] to say that f *maps X into the cocenter of Y*. Let X be an H-cogroup and $f : X \to Y$. Recall that f is called *cocentral* if $(\mathrm{id}_X \vee f)c \simeq \star$, where $c : X \to X \vee X$ is the basic cocommutator map.

If X is a co-H-space then the map $\Sigma : [X,Y] \to [\Sigma X, \Sigma Y]$ given by $f \mapsto \Sigma f$ is injective. A subset $\mathcal{DC}(X,Y)$ of $[X,Y]$ which is the dual of $\mathcal{C}(X,Y)$ has been studied in [19]. If Y is an H-space then the map $\Sigma :$ $[X,Y] \to [\Sigma X, \Sigma Y]$ given by $f \mapsto \Sigma f$ is injective. Let $[\Sigma X, \Sigma Y]_{\mathcal{C}\Sigma}$ denote the subset of $[\Sigma X, \Sigma Y]$ consisting of those homotopy classes of maps Σf which are cocentral. Following [19, Definition 4.7], we set $\mathcal{DC}(X,Y) = \Sigma^{-1}[\Sigma X, \Sigma Y]_{\mathcal{C}\Sigma}$.

In view of [19, Propositions 4.8 and 5.2], it holds:

Proposition 4.6. *Let X, Y and Z be spaces.*

(1) *If $f \in \mathcal{DC}(Z, \Omega X)$ then $[f,g]' = 0$ for any $g \in [Z, \Omega Y]$;*

(2) *the set $\mathcal{DC}(X,Y)$ is a subgroup contained in the center of $[X,Y]$ if Y is an H-space with a left homotopy inverse and X is any space.*

It follows that if Y is an H-space with a right homotopy inverse, then for every space X there are inclusions $\mathcal{DG}(X,Y) \subseteq \mathcal{DC}(X,Y) \subseteq$ center of $[X,Y]$ of subgroups. In particular, $\mathcal{DG}(X,Y)$ and $\mathcal{DC}(X,Y)$ are abelian groups provided X is an H-space.

References

[1] M. Arkowitz, *The generalized Whitehead product*, Pacific J. Math. **12** (1962), 7–23.

[2] M. Arkowitz, M. Golasiński, *Co-H-structures on Moore spaces of type* $(G,2)$, Canad. J. Math. **46** (1994), 673–686.

[3] M. Arkowitz, K.-I. Maruyama, *The Gottlieb group of a wedge of suspensions*, (preprint).

[4] W. D. Barcus, M. G. Barratt, *On the homotopy classification of the extensions of a fixed map*, Trans. Amer. Math. Soc. **88** (1958), 57–74.

[5] T. Ganea, *Induced fibrations and cofibrations*, Trans. Amer. Math. Soc. 127 (1967), 442–459.

[6] M. Golasiński, J. Mukai, *Gottlieb groups of spheres*, Topology **47** (2008), 399–430.

[7] M. Golasiński, J. Mukai, *Gottlieb and Whitehead center groups of projective spaces*, (submitted).

[8] D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840–856.

[9] D. H. Gottlieb, *On fibre spaces and the evaluation map*, Ann. of Math. **87** (1968), 42–55.

[10] D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729–756.

[11] D. H. Gottlieb, *Covering transformations and universal fibrations*, Illinois J. Math. **13** (1969), 432–437.

[12] D. H. Gottlieb, *Applications of bundle map theory*, Trans. Amer. Math. Soc. **171** (1972), 23–50.

[13] B. Gray, *On generalized Whitehead products*, Trans. Amer. Math. Soc. **11** (2011), 6143–6158.

[14] C. S. Hoo, *Cyclic maps from suspensions to suspensions*, Canad. J. Math. **24** (1972), 789–791.

[15] B.-J. Jiang, *Estimation of the Nielsen numbers*, Acta Math. Sinica **14** (1964), 330–339.

[16] G. E. Lang, *Evaluation subgroups of factor spaces*, Pacific J. Math. **42** (1972), 701–709.

[17] K. L. Lim, *On cyclic maps*, J. Austral. Math. Soc. Ser. A **32** (1982), 349–357.

[18] K. L. Lim, *On evaluation subgroups of generalized homotopy groups*, Canad. Math. Bull. **27** (1) (1984), 78–86.

[19] K. L. Lim, *Cocyclic maps and coevaluation subgroups*, Canad. Math. Bull. **30** (1) (1987), 63–71.

[20] G. Lupton, J. Oprea, *Cohomologically symplectic spaces: toral actions and the Gottlieb group*, Trans. Amer. Math. Soc. **347** (1) (1995), 261–288.

[21] J. Oprea, *The Samelson space of a fibration*, Michigan Math. J. **34** (1) (1987), 127–141.

[22] J. Oprea, *Gottlieb groups, group actions, fixed points and rational homotopy*, Lecture Notes Series **29**, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1995.

[23] K. Varadarajan, *Generalized Gottlieb groups*, J. Indian Math. Soc. **33** (1969), 141–164.