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Monogenic functions in finite-dimensional commutative associative algebras

Let $\mathbb{A}_n^m$ be an arbitrary n-dimensional commutative associative algebra over the field of complex numbers with m idempotents. Let $e_1 = 1, e_2, \ldots, e_k$, $2 \leq k \leq 2n$, are linearly independent over the field of real numbers elements of $\mathbb{A}_n^m$. We consider monogenic (i. e., continuous and differentiable in the sense of Gateaux) functions of the variable $\sum_{j=1}^{k} x_j\, e_j$, where $x_1, x_2, \ldots, x_k$ are real, and obtain a constructive description of all mentioned functions by means of holomorphic functions of complex variables. Due to this description obtain, that monogenic functions have Gateaux derivatives of all orders. The present article is a generalization of the author's paper [1], where mentioned results are obtained for $k = 3$.

1. Introduction. It seemed, W. Hamilton (1843) made the first attempts to construct an algebra associated with the three-dimensional Laplace equation

$$\Delta_3 u(x,y,z) := \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u(x,y,z) = 0 \tag{1}$$

meaning that components of hypercomplex functions satisfy the Eq. (1).

He constructed an algebra of noncommutative quaternions over the field of real numbers $\mathbb{R}$ and made a base for developing the hypercomplex analysis.

C. Segre [2] constructed an algebra of commutative quaternions over the field $\mathbb{R}$ that can be considered as a two-dimensional commutative semi-simple algebra of bicomplex numbers over the field of complex numbers

$\mathbb{C}$. M. Futagawa [3] and J. Riley [4] obtained a constructive description of analytic function of a bicomplex variable, namely, they proved that such an analytic function can be constructed with use of two holomorphic functions of complex variables.

F. Ringleb [5] and S. N. Volovel'skaya [6, 7] succeeded in developing a function theory for noncommutative algebras with unit over the real or complex fields, by pursuing a definition of the differential of a function on such an algebra suggested by Hausdorff in [8]. These definitions make the *a priori* severe requirement that the coordinates of the function have continuous first derivatives with respect to the coordinates of the argument element. Namely, F. Ringleb [5] considered an arbitrary finite-dimensional associative (commutative or not) *semi-simple* algebra over $\mathbb{R}$. For given class of functions which maps the mentioned algebra onto itself, he obtained a constructive description by means of real and complex analytic functions.

S. N. Volovel'skaya developed the Hausdorff's idea defining the *monogenic* functions on *non-semisimple* associative algebras and she generalized the Ringleb's results for such algebras. In [6], there was obtained a constructive description of monogenic functions in a special three-dimensional non-commutative algebra over the field $\mathbb{R}$. The results of paper [6] were generalized in the paper [7], where Volovel'skaya obtained a constructive description of monogenic functions in non-semisimple associative algebras of the first category over $\mathbb{R}$.

A relation between spatial potential fields and analytic functions given in commutative algebras was established by P. W. Ketchum [9], who shown that every analytic function $\Phi(\zeta)$ of the variable $\zeta = xe_1+ye_2+ze_3$ satisfies the Eq. (1) in the case where the elements e_1, e_2, e_3 of a commutative algebra satisfy the condition

$$e_1^2 + e_2^2 + e_3^2 = 0\,, \tag{2}$$

because

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \equiv \Phi''(\zeta)\ (e_1^2 + e_2^2 + e_3^2) = 0\,, \tag{3}$$

where $\Phi'' := (\Phi')'$, $\Phi'(\zeta)$ is defined by the equality $d\Phi = \Phi'(\zeta)d\zeta$.

We say that a commutative associative algebra $\mathbb{A}$ is *harmonic* (cf. [9–11]) if in $\mathbb{A}$ there exists a triad of linearly independent vectors $\{e_1, e_2, e_3\}$ satisfying the equality (2) with $e_k^2 \neq 0$ for $k = 1, 2, 3$. We say also that such a triad $\{e_1, e_2, e_3\}$ is *harmonic*.

P. W. Ketchum [9] considered the C. Segre algebra of quaternions [2] as an example of a harmonic algebra.

Further M. N. Roşculeţ establishes a relation between monogenic functions in commutative algebras and partial differential equations. He defined *monogenic* functions f of the variable w by the equality $df(w)\,dw = 0$. In [12], M. N. Roşculeţ proposed a procedure for constructing an infinite-dimensional topological vector space with commutative multiplication such that monogenic functions in it are the *all* solutions of the equation

$$\sum_{\alpha_0+\alpha_1+\ldots+\alpha_p=N} C_{\alpha_0,\alpha_1,\ldots,\alpha_p}\,\frac{\partial^N\Phi}{\partial x_0^{\alpha_0}\,\partial x_1^{\alpha_1}\,\ldots\partial x_p^{\alpha_p}} = 0, \tag{4}$$

where $C_{\alpha_0,\alpha_1,\ldots,\alpha_p} \in \mathbb{R}$. In particular, such infinite-dimensional topological vector space are constructed for the Laplace equation (3). In [13], Roşculeţ finds a certain connection between monogenic functions in commutative algebras and systems of partial differential equations.

I. P. Mel'nichenko proposed to use hypercomplex functions differentiable in the sense of Gateaux for describing solutions of the equation (4), since conditions of monogenicity are the least restrictive in this case. He started to implement this approach with respect to the 3-D Laplace equation (3) (see [10]). Mel'nichenko proved that exist exactly 3 three-dimensional harmonic algebras with unit over the field $\mathbb{C}$ (see [10, 11, 14]).

In [15], it is developed the Melnichenko's idea for the equation (4).

An investigation of partial differential equations using the hypercomplex methods is more effective if hypercomplex monogenic (in any sense) functions can be constructed explicitly. Constructive descriptions of monogenic (i. e. continuous and differentiable in the sense of Gateaux) functions taking values in the mentioned three-dimensional harmonic algebras by means of three corresponding holomorphic functions of the complex variable are obtained in [16–18]. Such descriptions make it possible to prove the infinite differentiability (in the sense of Gateaux) of monogenic functions and integral theorems for these functions, being analogous to classical theorems in Complex Analysis (see, e. g., [19, 20]).

Furthermore, constructive descriptions of monogenic functions taking values in special n-dimensional commutative algebras by means n holomorphic functions of complex variables are obtained in [21, 22].

In [1], there is obtained a constructive description of *all* monogenic functions of the variable $x_1e_1 + x_2e_2 + x_3e_3$ taking values in an arbitrary

n-dimensional commutative associative algebra with unit by means of holomorphic functions of the complex variables. It follows from this description that monogenic functions are infinitely differentiable in the sense of Gateaux.

In this paper we extend results of the paper [1] to monogenic functions of the variable $\sum\limits_{r=1}^{k} x_r e_r$, where $2 \leq r \leq 2n$.

2. The algebra $\mathbb{A}_n^m$. Let $\mathbb{N}$ be the set of natural numbers. We fix ordered numbers $m, n \in \mathbb{N}$, $m \leq n$. Let $\mathbb{A}_n^m$ be an arbitrary commutative associative algebra with unit over the field of complex number $\mathbb{C}$. E. Cartan [23, p. 33] proved that there exists a basis $\{I_r\}_{r=1}^n$ in $\mathbb{A}_n^m$ satisfying the following multiplication rules:

1. $\forall\, r, s \in [1, m] \cap \mathbb{N}:\qquad I_r I_s = \begin{cases} 0 & \text{if} \quad r \neq s, \\ I_r & \text{if} \quad r = s; \end{cases}$

2. $\forall\, r, s \in [m+1, n] \cap \mathbb{N}:\qquad I_r I_s = \sum\limits_{p=\max\{r,s\}+1}^{n} \Upsilon_{r,p}^s I_p\,;$

3. $\forall\, s \in [m+1, n] \cap \mathbb{N}\ \ \exists!\ u_s \in [1, m] \cap \mathbb{N}\ \ \forall\, r \in [1, m] \cap \mathbb{N}:$

$$I_r I_s = \begin{cases} 0 \ \text{if} \ r \neq u_s\,, \\ I_s \ \text{if} \ r = u_s\,. \end{cases} \tag{5}$$

Moreover, the structure constants $\Upsilon_{r,p}^s \in \mathbb{C}$ satisfy the associativity conditions:

(A 1). $(I_r I_s) I_p = I_r (I_s I_p) \quad \forall\, r, s, p \in [m+1, n] \cap \mathbb{N}$;

(A 2). $(I_u I_s) I_p = I_u (I_s I_p) \quad \forall\, u \in [1, m] \cap \mathbb{N}\ \ \forall\, s, p \in [m+1, n] \cap \mathbb{N}$.

Obviously, that the first m basic vectors $\{I_u\}_{u=1}^m$ are idempotents and define the basis of the semi-simple subalgebra of the algebra $\mathbb{A}_n^m$. The vectors $\{I_r\}_{r=m+1}^n$ define the basis of the nilpotent subalgebra of the algebra $\mathbb{A}_n^m$. The element $1 = \sum_{u=1}^m I_u$ is the unit of $\mathbb{A}_n^m$.

In the cases where

Consider some particular cases of $\mathbb{A}_n^m$.

Proposition 1 [1]. *If there exists the unique $u_0 \in [1, m] \cap \mathbb{N}$ such that $I_{u_0} I_s = I_s$ for all $s = m+1, \ldots, n$, then the associativity condition* (A 2) *is satisfied.*

Thus, under the conditions of Proposition 1, the associativity condition (A 1) is merely required. It means that the nilpotent subalgebra of $\mathbb{A}_n^m$ with the basis $\{I_r\}_{r=m+1}^n$ can be an arbitrary commutative associative nilpotent algebra of dimension $n-m$. Note, that such nilpotent algebras are completely described for the dimensions $1,2,3$ in the paper [24], and some four-dimensional nilpotent algebras can be found in [25, 26].

Proposition 2 [1]. *If all u_r are distinct in the multiplication rule* 3*, then $I_sI_p=0$ for all $s,p=m+1,\dots,n$.*

Thus, under the conditions of Proposition 2, the multiplication table of the nilpotent subalgebra of $\mathbb{A}_n^m$ with the basis $\{I_r\}_{r=m+1}^n$ consists only of zeros and all associativity conditions are satisfied.

The algebra $\mathbb{A}_n^m$ contains m maximal ideals

$$\mathcal{I}_u:=\Bigg\{\sum_{r=1,\,r\neq u}^{n}\lambda_rI_r:\lambda_r\in\mathbb{C}\Bigg\},\quad u=1,2,\dots,m,$$

and their intersection is the radical

$$\mathcal{R}:=\Big\{\sum_{r=m+1}^{n}\lambda_rI_r:\lambda_r\in\mathbb{C}\Big\}.$$

Consider m linear functionals $f_u\colon\mathbb{A}_n^m\longrightarrow\mathbb{C}$ satisfying the equalities

$$f_u(I_u)=1,\quad f_u(\omega)=0\quad\forall\,\omega\in\mathcal{I}_u\,,\quad u=1,2,\dots,m.$$

Inasmuch as the kernel of functional f_u is the maximal ideal $\mathcal{I}_u$ obtain, that this functional is also continuous and multiplicative (see [27, p. 147]).

3. Monogenic functions.

Let us consider linearly independent over the field of real numbers $\mathbb{R}$ (see [22]) vectors $e_1=1, e_2,\dots,e_k$ in $\mathbb{A}_n^m$, where $2\leq k\leq 2n$. It means that the equality $\sum\limits_{j=1}^{k}\alpha_je_j=0$, $\alpha_j\in\mathbb{R}$, holds if and only if $\alpha_j=0$ for all $j=1,2,\dots,k$.

Let the vectors $\{e_1,\dots,e_k\}$ have the following decompositions in the basis $\{I_r\}_{r=1}^n$:

$$e_1=\sum_{r=1}^{m}I_r\,,\quad e_j=\sum_{r=1}^{n}a_{jr}\,I_r\,,\quad a_{jr}\in\mathbb{C},\quad j=2,3,\dots,k. \tag{6}$$

Let $\zeta := \sum\limits_{j=1}^{k} x_j\, e_j$, $x_j \in \mathbb{R}$. Evidently, that

$$\xi_u := f_u(\zeta) = x_1 + \sum_{j=2}^{k} x_j\, a_{ju}, \quad u = 1, 2, \ldots, m.$$

Let $E_k := \{\zeta = \sum\limits_{j=1}^{k} x_j e_j : x_j \in \mathbb{R}\}$ be the linear span of vectors $\{e_1, \ldots, e_k\}$ over $\mathbb{R}$.

Let Ω be a domain in E_k. With a domain $\Omega \subset E_k$ we associate the domain $\Omega_{\mathbb{R}} := \Big\{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : \zeta = \sum\limits_{j=1}^{k} x_j\, e_j \in \Omega\Big\}$.

We say that a continuous function $\Phi\colon \Omega \longrightarrow \mathbb{A}_n^m$ is *monogenic* in Ω if Φ is differentiable in the sense of Gateaux in Ω, i.e., there exists an element $\Phi'(\zeta) \in \mathbb{A}_n^m$ such that

$$\lim_{\varepsilon \to 0+0} \left(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\right) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall\, h \in E_k. \tag{7}$$

in any $\zeta \in \Omega$. The function $\Phi'(\zeta)$ is the *Gateaux derivative* of the function Φ in the point ζ.

Consider the decomposition of a function $\Phi\colon \Omega \longrightarrow \mathbb{A}_n^m$ in the basis $\{I_r\}_{r=1}^{n}$:

$$\Phi(\zeta) = \sum_{r=1}^{n} U_r(x_1, x_2, \ldots, x_k)\, I_r\, . \tag{8}$$

If functions $U_r\colon \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$ are $\mathbb{R}$-differentiable in $\Omega_{\mathbb{R}}$, i.e., for every $(x_1, x_2, \ldots, x_k) \in \Omega_{\mathbb{R}}$ the following asymptotic equality is valid: $U_r\,(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_k + \Delta x_k) - U_r(x_1, x_2, \ldots, x_k) = \sum\limits_{j=1}^{k} \frac{\partial U_r}{\partial x_j}\, \Delta x_j + o\left(\sqrt{\sum\limits_{j=1}^{k} (\Delta x_j)^2}\right)$, $\sum\limits_{j=1}^{k} (\Delta x_j)^2 \to 0$, the function Φ is monogenic in the domain Ω if and only if the following Cauchy – Riemann conditions are satisfied in Ω:

$$\frac{\partial \Phi}{\partial x_j} = \frac{\partial \Phi}{\partial x_1}\, e_j \qquad \text{for all} \quad j = \overline{2, 3, \ldots, k}. \tag{9}$$

4. Expansion of the resolvent. Let $b := \sum\limits_{r=1}^{n} b_r\, I_r \in \mathbb{A}_n^m$, where $b_r \in \mathbb{C}$. Note, that $f_u(b) = b_u$, $u = 1, 2, \ldots, m$. It follows form Lemmas 1, 3 in [1] that

$$b^{-1}=\sum_{u=1}^{m}\frac{1}{b_u}\,I_u+\sum_{s=m+1}^{n}\sum_{r=2}^{s-m+1}\frac{\widetilde{Q}_{r,s}}{b_{u_s}^{r}}\,I_s, \tag{10}$$

where $\widetilde{Q}_{r,s}$ are determined by the following recurrence relations:

$$\widetilde{Q}_{2,s}:=b_s\,,\qquad \widetilde{Q}_{r,s}=\sum_{q=r+m-2}^{s-1}\widetilde{Q}_{r-1,q}\,\widetilde{B}_{q,\,s}\,,\quad r=3,4,\ldots,s-m+1, \tag{11}$$

$$\widetilde{B}_{q,s}:=\sum_{p=m+1}^{s-1}b_p\Upsilon_{q,s}^{p}\,,\ \ p=m+2,m+3,\ldots,n, \tag{12}$$

and the natural numbers u_s are defined by the rule 3 of the multiplication table of the algebra $\mathbb{A}_n^m$.

In the next lemma we find an expansion of the resolvent $(te_1-\zeta)^{-1}$.

Lemma 1. *The resolvent has the following expansion*

$$(te_1-\zeta)^{-1}=\sum_{u=1}^{m}\frac{1}{t-\xi_u}\,I_u+\sum_{s=m+1}^{n}\sum_{r=2}^{s-m+1}\frac{Q_{r,s}}{(t-\xi_{u_s})^{r}}\,I_s \tag{13}$$

$$\forall\, t\in\mathbb{C}:\ t\neq\xi_u,\quad u=1,2,\ldots,m,$$

where coefficients $Q_{r,s}$ *are determined by the following recurrence relations*:

$$Q_{2,s}=T_s\,,\quad Q_{r,s}=\sum_{q=r+m-2}^{s-1}Q_{r-1,q}\,B_{q,\,s}\,,\quad r=3,4,\ldots,s-m+1, \tag{14}$$

here

$$T_s:=\sum_{j=2}^{k}x_ja_{js}\,,\quad B_{q,s}:=\sum_{p=m+1}^{s-1}T_p\Upsilon_{q,s}^{p}\,,\ \ p=m+2,m+3,\ldots,n, \tag{15}$$

and the natural numbers u_s *are defined by the rule* 3 *of the multiplication table of the algebra* $\mathbb{A}_n^m$.

Proof. Taking into account the decomposition $te_1-\zeta= \sum_{u=1}^{m}(t-\xi_u)I_u-\sum_{r=m+1}^{n}\sum_{j=2}^{k}x_ja_{js}\,I_r$, conclude, that the relation (13) follows directly from the equality (10) in which instead of b_u, $u = 1, 2, \ldots, m$, it should be used the expansion $t - \xi_u$; and instead of b_s, $s = m+1, m+2, \ldots, n$, it should be used the expansion $\sum\limits_{j=2}^{k} x_j a_{js}$. The lemma is proved.

It follows from Lemma 1 that points $(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, which are correspond to the non-invertible elements $\zeta = \sum\limits_{j=1}^{k} x_j\, e_j$, form the set

$$M_u^{\mathbb{R}} : \quad \begin{cases} x_1 + \sum\limits_{j=2}^{k} x_j \,\mathrm{Re}\, a_{ju} = 0, \\ \\ \sum\limits_{j=2}^{k} x_j \,\mathrm{Im}\, a_{ju} = 0, \end{cases} \qquad u = 1, 2, \ldots, m$$

in the k-dimensional space $\mathbb{R}^k$. Consider the set $M_u := \{\zeta \in E_k : f_u(\zeta) = 0\}$ for $u = \overline{1, 2, \ldots, m}$. It is obvious that the set $M_u^{\mathbb{R}} \subset \mathbb{R}^k$ is congruent to the set $M_u \subset E_k$.

5. A constructive description of monogenic functions.

We say that a domain $\Omega \subset E_k$ is *convex with respect to the set of directions* M_u if Ω contains the segment $\{\zeta_1 + \alpha(\zeta_2 - \zeta_1) : \alpha \in [0, 1]\}$ for all $\zeta_1, \zeta_2 \in \Omega$ such that $\zeta_2 - \zeta_1 \in M_u$.

Denote $f_u(E_k) := \{f_u(\zeta) : \zeta \in E_k\}$. In what follows, we make the following essential assumption: $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Obviously, it holds if and only if for every fixed $u \in \{1, 2, \ldots, m\}$ at least one of the numbers $a_{2u}, a_{3u}, \ldots, a_{ku}$ belongs to $\mathbb{C} \setminus \mathbb{R}$.

Further in this section, we suppose that a domain $\Omega \subset E_k$ is convex with respect to the set of directions M_u and $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$.

Lemma 2. *Suppose that a function $\Phi \colon \Omega \longrightarrow \mathbb{A}_n^m$ is monogenic in the domain Ω. If points $\zeta_1, \zeta_2 \in \Omega$ such that $\zeta_2 - \zeta_1 \in M_u$, then*

$$\Phi(\zeta_2) - \Phi(\zeta_1) \in \mathcal{I}_u\,. \tag{16}$$

Proof. Inasmuch as $f_u(E_k) = \mathbb{C}$ then exists an element $e_2^* \in E_k$ such that $f_u(e_2^*) = i$. Consider the lineal span $E^* := \{\zeta = x e_1^* + y e_2^* + z e_3^* : x, y, z \in \mathbb{R}\}$ of the vectors $e_1^* := 1, e_2^*, e_3^* := \zeta_2 - \zeta_1$.

Now, the relations (16) can be proved in such a way as Lemma 2.1 [16], in the proof of which one must take $\Omega \cap E^*$, $f_u, \{\alpha e_3^* : \alpha \in \mathbb{R}\}$ instead of Ω_ζ, f, L, respectively. Lemma 2 is proved.

Let a domain $\Omega \subset E_k$ be convex with respect to the set of directions M_u , $u = 1, 2, \ldots, m$, $D_u := f_u(\Omega) \subset \mathbb{C}$.

We introduce linear operators A_u , $u = 1, 2, \ldots, m$, which assign holomorphic functions $F_u : D_u \longrightarrow \mathbb{C}$ to monogenic functions $\Phi : \Omega \longrightarrow \mathbb{A}_n^m$ by the formula

$$F_u(\xi_u) = f_u(\Phi(\zeta)), \tag{17}$$

where $\xi_u = f_u(\zeta) \equiv x_1 + \sum\limits_{j=2}^{k} x_j\, a_{ju}$ and $\zeta \in \Omega$. It follows from Lemma 2 that the value $F_u(\xi_u)$ does not depend on a choice of a point ζ for which $f_u(\zeta) = \xi_u$.

Now, similar to proof of Lemma 5 [1] it can be proved the following statement.

Lemma 3. *Suppose that for any fixed $u = 1, 2, \ldots, m$, a function $F_u : D_u \longrightarrow \mathbb{C}$ is holomorphic in a domain D_u and Γ_u is a closed Jordan rectifiable curve in D_u which surrounds the point ξ_u and contains no points ξ_q, $q = 1, 2, \ldots, m$, $q \neq u$. Then the function*

$$\Psi_u(\zeta) := I_u \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1}\, dt \tag{18}$$

is monogenic in the domain Ω.

Lemma 4. *Suppose that a function $V : \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$ satisfies the equalities*

$$\frac{\partial V}{\partial x_2} = \frac{\partial V}{\partial x_1}\, a_{2u}\,, \quad \frac{\partial V}{\partial x_3} = \frac{\partial V}{\partial x_1}\, a_{3u}\,, \quad \ldots, \quad \frac{\partial V}{\partial x_k} = \frac{\partial V}{\partial x_1}\, a_{ku} \tag{19}$$

in $\Omega_{\mathbb{R}}$. Then V is a holomorphic function of the variable $\xi_u = f_u(\zeta) = x_1 + \sum\limits_{j=2}^{k} x_j\, a_{ju}$ in the domain D_u.

Proof. We first separate the real and the imaginary part of the expression

$$\xi_u = x_1 + \sum_{j=2}^{k} x_j \operatorname{Re} a_{ju} + i \sum_{j=2}^{k} x_j \operatorname{Im} a_{ju} =: \tau_u + i\eta_u \tag{20}$$

and note that the equalities (19) yield

$$\frac{\partial V}{\partial \eta_u}\,\mathrm{Im}\,a_{2u}=i\,\frac{\partial V}{\partial \tau_u}\,\mathrm{Im}\,a_{2u}\,,\quad \ldots,\quad \frac{\partial V}{\partial \eta_u}\,\mathrm{Im}\,a_{ku}=i\,\frac{\partial V}{\partial \tau_u}\,\mathrm{Im}\,a_{ku}\,. \tag{21}$$

It follows from the condition $f_u(E_k)=\mathbb{C}$ that at least one of the numbers $\mathrm{Im}\,a_{2u}\,,\ \mathrm{Im}\,a_{3u}\,,\ldots,\mathrm{Im}\,b_u$ is not equal to zero. Therefore, using (21), we get

$$\frac{\partial V}{\partial \eta_u}=i\,\frac{\partial V}{\partial \tau_u}\,. \tag{22}$$

Now we prove that $V(x_1',x_2',\ldots,x_k')=V(x_1'',x_2'',\ldots,x_k'')$ for points $(x_1',x_2',\ldots,x_k'),(x_1'',x_2'',\ldots,x_k'')\in\Omega$ such that the segment connecting these points is parallel to a straight line $L_u\subset M_u^{\mathbb{R}}$. We use considerations with the proof of Lemma 2. Since $f_u(E_k)=\mathbb{C}$, then there exists an element $e_2^*\in E_k$ such that $f_u(e_2^*)=i$. Consider the lineal span $E^*:=$ $:=\{\zeta=xe_1^*+ye_2^*+ze_3^* : x,y,z\in\mathbb{R}\}$ of the vectors $e_1^*:=1$, e_2^*, $e_3^*:=\zeta'-\zeta''$, where $\zeta':=\sum\limits_{j=1}^{k}x_j'\,e_j\,,\quad \zeta'':=\sum\limits_{j=1}^{k}x_j''\,e_j$.

Now, the relation $V(x_1',x_2',\ldots,x_k')=V(x_1'',x_2'',\ldots,x_k'')$ can be proved in such a way as Lemma 6 [1], in the proof of which one must take $\Omega\cap E^*$,$\{\alpha e_3^* : \alpha\in\mathbb{R}\}$ instead of $\Omega_\zeta\,, L$, respectively. The lemma is proved.

Thus, a function $V\colon\Omega_{\mathbb{R}}\longrightarrow\mathbb{C}$ of the type $V(x_1,x_2,\ldots,x_k):=F(\xi_u)$, where $F(\xi_u)$ is an arbitrary function holomorphic in the domain D_u, is a general solution of the system (19). The lemma is proved.

Theorem 1. *Every monogenic function $\Phi:\Omega\to\mathbb{A}_n^m$ can be expressed in the form*

$$\Phi(\zeta)=\sum_{u=1}^{m}I_u\,\frac{1}{2\pi i}\int\limits_{\Gamma_u}F_u(t)(te_1-\zeta)^{-1}\,dt+$$

$$+\sum_{s=m+1}^{n}I_s\,\frac{1}{2\pi i}\int\limits_{\Gamma_{u_s}}G_s(t)(te_1-\zeta)^{-1}\,dt, \tag{23}$$

where F_u and G_s are certain holomorphic functions in the domains D_u and D_{u_s}, respectively; Γ_q is a closed Jordan rectifiable curve in D_q which surrounds the point ξ_q and contains no points ξ_ℓ, $\ell,q=1,2,\ldots,m,\ell\neq q$.

Proof. We set

$$F_u:=A_u\Phi,\ u=1,2,\ldots,m. \tag{24}$$

Let us show that the values of monogenic function

$$\Phi_0(\zeta) := \Phi(\zeta) - \sum_{u=1}^{m} I_u \, \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} \, dt \tag{25}$$

belong to the radical $\mathcal{R}$, i.e., $\Phi_0(\zeta) \in \mathcal{R}$ for all $\zeta \in \Omega$. As a consequence of the equality (13), we have the equality

$$I_u \, \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} \, dt = I_u \, \frac{1}{2\pi i} \int\limits_{\Gamma_u} \frac{F_u(t)}{t - \xi_u} \, dt +$$

$$+ \frac{1}{2\pi i} \sum_{s=m+1}^{n} \sum_{r=2}^{s-m+1} \int\limits_{\Gamma_u} \frac{F_u(t) Q_{r,s}}{(t - \xi_{u_s})^r} \, dt \, I_s \, I_u \, ,$$

from which we obtain the equality

$$f_u \left(\sum_{u=1}^{m} I_u \, \frac{1}{2\pi i} \int\limits_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} \, dt \right) = F_u(\xi_u). \tag{26}$$

Acting the functional f_u onto the equality (25) and taking into account the relations (17), (24), (26), we get the equality $f_u(\Phi_0(\zeta)) = = F_u(\xi_u) - F_u(\xi_u) = 0$ for all $u = 1, 2, \ldots, m$, i.e., $\Phi_0(\zeta) \in \mathcal{R}$.

Therefore, the function Φ_0 is of the type

$$\Phi_0(\zeta) = \sum_{s=m+1}^{n} V_s(x_1, x_2, \ldots, x_k) \, I_s \, , \tag{27}$$

where functions V_s, $s = m+1, \ldots, n$, are of the type $V_s \colon \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$. Cauchy–Riemann conditions (9) are satisfied with $\Phi = \Phi_0$. Substituting the expressions (6), (27) into the equality (9), we obtain

$$\begin{gathered} \sum_{s=m+1}^{n} \frac{\partial V_s}{\partial x_2} \, I_s = \sum_{s=m+1}^{n} \frac{\partial V_s}{\partial x_1} \, I_s \sum_{r=1}^{n} a_{2r} \, I_r \, , \\ \vdots \\ \sum_{s=m+1}^{n} \frac{\partial V_s}{\partial x_k} \, I_s = \sum_{s=m+1}^{n} \frac{\partial V_s}{\partial x_1} \, I_s \sum_{r=1}^{n} a_{kr} \, I_r \, . \end{gathered} \tag{28}$$

Equating the coefficients near I_{m+1} in these equalities, we obtain the following system of equations with unknown function $V_{m+1}(x_1, x_2, \ldots, x_k)$:

$$\frac{\partial V_{m+1}}{\partial x_2} = \frac{\partial V_{m+1}}{\partial x_1}\, a_{2\,u_{m+1}}\,, \quad \ldots, \quad \frac{\partial V_{m+1}}{\partial x_k} = \frac{\partial V_{m+1}}{\partial x_1}\, a_{k\,u_{m+1}}\,.$$

It follows from Lemma 4 that $V_{m+1}(x_1, x_2, \ldots, x_k) \equiv G_{m+1}(\xi_{u_{m+1}})$, where G_{m+1} is a function holomorphic in the domain $D_{u_{m+1}}$. Therefore,

$$\Phi_0(\zeta) = G_{m+1}(\xi_{u_{m+1}})\, I_{m+1} + \sum_{s=m+2}^{n} V_s(x_1, x_2, \ldots, x_k)\, I_s\,. \tag{29}$$

Due to the expansion (13), we have the representation

$$I_{m+1}\, \frac{1}{2\pi i} \int\limits_{\Gamma_{u_{m+1}}} G_{m+1}(t)(te_1 - \zeta)^{-1}\, dt = G_{m+1}(\xi_{u_{m+1}})\, I_{m+1} + \Psi(\zeta), \tag{30}$$

where $\Psi(\zeta)$ is a function with values in the set $\left\{ \sum_{s=m+2}^{n} \alpha_s\, I_s : \alpha_s \in \mathbb{C} \right\}$.

Now, consider the function

$$\Phi_1(\zeta) := \Phi_0(\zeta) - I_{m+1}\, \frac{1}{2\pi i} \int\limits_{\Gamma_{u_{m+1}}} G_{m+1}(t)(te_1 - \zeta)^{-1}\, dt.$$

In view of the relations (29), (30), Φ_1 can be represented in the form

$$\Phi_1(\zeta) = \sum_{s=m+2}^{n} \widetilde{V}_s(x_1, x_2, \ldots, x_k)\, I_s\,,$$

where functions $\widetilde{V}_s$, $s = m+2, \ldots, n$, are of the type $\widetilde{V}_s : \Omega_{\mathbb{R}} \longrightarrow \mathbb{C}$.

Inasmuch as Φ_1 is a monogenic function in Ω, the functions $\widetilde{V}_{m+2}, \widetilde{V}_{m+3}, \ldots, \widetilde{V}_n$ satisfy the system (28) with $V_{m+1} \equiv 0$, $V_s = \widetilde{V}_s$ for $s = m+2, m+3, \ldots, n$. Therefore, similarly to the function $V_{m+1}(x_1, x_2, \ldots, x_k) \equiv G_{m+1}(\xi_{u_{m+1}})$, the function $\widetilde{V}_{m+2}$ satisfies the equations

$$\frac{\partial \widetilde{V}_{m+2}}{\partial x_2} = \frac{\partial \widetilde{V}_{m+2}}{\partial x_1}\, a_{2\,u_{m+2}}\,, \quad \ldots, \quad \frac{\partial \widetilde{V}_{m+2}}{\partial x_k} = \frac{\partial \widetilde{V}_{m+2}}{\partial x_1}\, a_{k\,u_{m+2}}$$

and is of the form $\widetilde{V}_{m+2}(x_1, x_2, \ldots, x_k) \equiv G_{m+2}(\xi_{u_{m+2}})$, where G_{m+2} is a function holomorphic in the domain $D_{u_{m+2}}$.

In such a way, step by step, considering the functions

$$\Phi_j(\zeta) := \Phi_{j-1}(\zeta) - I_{m+j}\,\frac{1}{2\pi i}\int\limits_{\Gamma_{u_{m+j}}} G_{m+j}(t)(te_1-\zeta)^{-1}\,dt$$

for $j = 2, 3, \ldots, n-m-1$, we get the representation (23) of the function Φ. The theorem is proved.

Taking into account the expansion (13), one can rewrite the equality (23) in the following equivalent form:

$$\Phi(\zeta) = \sum_{u=1}^{m} F_u(\xi_u)I_u + \sum_{s=m+1}^{n}\sum_{r=2}^{s-m+1}\frac{1}{(r-1)!}\,Q_{r,s}\,F_{u_s}^{(r-1)}(\xi_{u_s})\,I_s +$$

$$+\sum_{q=m+1}^{n} G_q(\xi_{u_q})I_q + \sum_{q=m+1}^{n}\sum_{s=m+1}^{n}\sum_{r=2}^{s-m+1}\frac{1}{(r-1)!}\,Q_{r,s}\,G_q^{(r-1)}(\xi_{u_q})\,I_q\,I_s\,. \tag{31}$$

Thus, the equalities (23) and (31) rebuild any monogenic functions $\Phi : \Omega \to \mathbb{A}_n^m$ by n corresponding holomorphic functions of the complex variables in the explicit form.

The following statement follows immediately from the equality (31) due to its right-hand side is the monogenic function in the domain $\Pi :=$ $:= \{\zeta \in E_k : f_u(\zeta) = D_u,\ u = 1, 2, \ldots, m\}$.

Theorem 2. *Every monogenic function $\Phi\colon \Omega \longrightarrow \mathbb{A}_n^m$ can be continued to a monogenic function in the domain Π.*

The next statement is a fundamental consequence of the equality (31). It is true for any domain Ω.

Theorem 3. *Let $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Then for every monogenic function $\Phi\colon \Omega \longrightarrow \mathbb{A}_n^m$ in an arbitrary fixed domain Ω, the Gateaux r-th derivatives $\Phi^{(r)}$ are monogenic functions in Ω for all r.*

The proof is completely analogous to the proof of Theorem 4 [16].

Using the integral expression (23) of monogenic function $\Phi : \Omega \to \mathbb{A}_n^m$ in the case where a domain Ω is convex with respect to the set of directions M_u , $u = 1, 2, \ldots, m$, we obtain the following expression for the Gateaux r-th derivative $\Phi^{(r)}$:

$$\Phi^{(r)}(\zeta) = \sum_{u=1}^{m} I_u\,\frac{r!}{2\pi i}\int\limits_{\Gamma_u} F_u(t)\Big((te_1-\zeta)^{-1}\Big)^{r+1}\,dt +$$

$$+\sum_{s=m+1}^{n} I_s\,\frac{r!}{2\pi i}\int\limits_{\Gamma_{u_s}} G_s(t)\Big((te_1-\zeta)^{-1}\Big)^{r+1}dt \qquad \forall\,\zeta\in\Omega\,.$$

6. Remarks. We note that in the cases where the algebra $\mathbb{A}_n^m$ has some specific properties (for instance, properties described in Propositions 1 and 2), it is easy to simplify the form of the equality (31).

1. Under conditions of Proposition 1 the following equalities hold: $u_{m+1} = u_{m+2} = \ldots = u_n =: \eta$, the representation (31) gets the form

$$\Phi(\zeta)=\sum_{u=1}^{m}F_u(\xi_u)I_u+\sum_{s=m+1}^{n}\sum_{r=2}^{s-m+1}\frac{1}{(r-1)!}\,Q_{r,s}\,F_\eta^{(r-1)}(\xi_\eta)\,I_s+$$

$$+\sum_{s=m+1}^{n}G_s(\xi_\eta)I_s+\sum_{q=m+1}^{n}\sum_{s=m+1}^{n}\sum_{r=2}^{s-m+1}\frac{1}{(r-1)!}\,Q_{r,s}\,G_q^{(r-1)}(\xi_\eta)\,I_s\,I_q\,. \qquad (32)$$

The formula (32) generalizes representations of monogenic functions in both three-dimensional harmonic algebras (see [16–18]) and specific n-dimensional algebras (see [21, 22]) to the case of algebras of more general form and to a variable of more general form.

2. Under conditions of Proposition 2 the representation (23) gets the form

$$\Phi(\zeta)=\sum_{u=1}^{m}F_u(\xi_u)I_u+\sum_{s=m+1}^{n}G_s(\xi_{u_s})I_s+\sum_{s=m+1}^{n}T_sF_{u_s}^{'}(\xi_{u_s})I_s\,. \qquad (33)$$

The formula (33) generalizes representations of monogenic functions in both a three-dimensional harmonic algebra with one-dimensional radical (see [17]) and semi-simple algebras (see [18, 22]) to the case of algebras of more general form and to a variable of more general form.

3. Let $n = m$. Then the algebra $\mathbb{A}_n^n$ is semi-simple and contains no nilpotent subalgebra. Then the formulae (32), (33) combine to the form $\Phi(\zeta) = \sum\limits_{u=1}^{n} F_u(\xi_u)I_u$, because there are no vectors $\{I_k\}_{k=m+1}^{n}$. This formula is obtained in the paper [22].

7. Relations between monogenic functions and partial differential equations. Consider the following linear partial differential equation with constant coefficients:

$$\mathcal{L}_N U(x_1, x_2, \ldots, x_k) := \sum_{\alpha_1+\alpha_2 \ldots+\alpha_k=N} C_{\alpha_1,\alpha_2,\ldots,\alpha_k} \frac{\partial^N \Phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_k^{\alpha_k}} = 0, \tag{34}$$

If a function $\Phi(\zeta)$ is N-times differentiable in the sense of Gateaux in Ω, then $\frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_k}\Phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_k^{\alpha_k}} = e_1^{\alpha_1} e_2^{\alpha_2} \ldots e_k^{\alpha_k} \Phi^{(\alpha_1+\alpha_2+\ldots+\alpha_k)}(\zeta) = = e_2^{\alpha_2} e_3^{\alpha_3} \ldots e_k^{\alpha_k} \Phi^{(N)}(\zeta)$. Therefore, due to the equality

$$\mathcal{L}_N \Phi(\zeta) = \Phi^{(N)}(\zeta) \sum_{\alpha_1+\alpha_2+\ldots+\alpha_k=N} C_{\alpha_1,\alpha_2,\ldots,\alpha_k} e_2^{\alpha_2} e_3^{\alpha_3} \ldots e_k^{\alpha_k}, \tag{35}$$

every N-times differentiable in the sense of Gateaux in Ω function Φ satisfies the equation $\mathcal{L}_N \Phi(\zeta) = 0$ in Ω if and only if

$$\sum_{\alpha_1+\alpha_2+\ldots+\alpha_k=N} C_{\alpha_1,\alpha_2,\ldots,\alpha_k} e_2^{\alpha_2} e_3^{\alpha_3} \ldots e_k^{\alpha_k} = 0. \tag{36}$$

Really, it follows from (36) that real-valued components $\mathrm{Re}\, U_k(x_1, x_2, \ldots, x_k)$ and $\mathrm{Im}\, U_k(x_1, x_2, \ldots, x_k)$ of the decomposition (8) are solutions of the equation (34).

In the case where $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$, it follows from Theorem 3 that the equality (35) holds for every monogenic function $\Phi \colon \Omega \longrightarrow \mathbb{A}_n^m$.

Thus, to construct solutions of the equation (34) in the form of components of monogenic functions, we must to find k linearly independent over the field $\mathbb{R}$ vectors (6) satisfying the characteristic equation (36) and to verify the condition: $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Then, the formula (23) gives a constructive description of all mentioned monogenic functions.

In the next theorem, we assign a special class of equations (34) for which $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. Let us introduce the polynomial

$$P(b_2, b_3, \ldots, b_k) := \sum_{\alpha_1+\alpha_2+\ldots+\alpha_k=N} C_{\alpha_1,\alpha_2,\ldots,\alpha_k} b_2^{\alpha_2} b_3^{\alpha_3} \ldots b_k^{\alpha_k}. \tag{37}$$

Theorem 4. *Suppose that there exist linearly independent over the field $\mathbb{R}$ vectors $e_1 = 1, e_2, \ldots, e_k$ in $\mathbb{A}_n^m$ of the form (6) that satisfy* *the equality* (36)*. If* $P(b_2, b_3, \ldots, b_k) \neq 0$ *for all real* $b_2, b_3, \ldots, b_k$*, then* $f_u(E_k) = \mathbb{C}$ *for all* $u = 1, 2, \ldots, m$*.*

Proof. Using the multiplication table of $\mathbb{A}_n^m$ we obtain the equalities $e_2^{\alpha_2} = \sum\limits_{u=1}^{m} a_{2u}^{\alpha_2} I_u + \Psi_{\mathcal{R}}$, $\ldots$, $e_k^{\alpha_k} = \sum\limits_{u=1}^{m} a_{ku}^{\alpha_k} I_u + \Theta_{\mathcal{R}}$, where $\Psi_{\mathcal{R}}, \ldots, \Theta_{\mathcal{R}} \in \mathcal{R}$. Now the equality (36) gets the form

$$\sum_{\alpha_1+\alpha_2+\ldots+\alpha_k=N} C_{\alpha_1,\alpha_2,\ldots,\alpha_k} \left(\sum_{u=1}^{m} a_{2u}^{\alpha_2} \ldots a_{ku}^{\alpha_k} I_u + \widetilde{\Psi}_{\mathcal{R}} \right) = 0, \qquad (38)$$

where $\widetilde{\Psi}_{\mathcal{R}} \in \mathcal{R}$. Moreover, due to the assumption that the vectors $e_1, e_2, \ldots, e_k$ of the form (6) satisfy the equality (36), exist complex coefficients a_{jr} for $j = 1, 2, \ldots, k$, $r = 1, 2, \ldots, n$ that satisfy the equality (38).

It follows from the equality (38) that

$$\sum_{\alpha_1+\alpha_2+\ldots+\alpha_k=N} C_{\alpha_1,\alpha_2,\ldots,\alpha_k}\, a_{2u}^{\alpha_2} \ldots a_{ku}^{\alpha_k} = 0, \qquad u = 1, 2, \ldots, m. \qquad (39)$$

Since $P(b_2, b_3, \ldots, b_k) \neq 0$ for all $\{b_2, b_3, \ldots, b_k\} \subset \mathbb{R}$, the equalities (39) can be satisfied only if for each $u = 1, 2, \ldots, m$ at least one of the numbers a_{2u}, $a_{3u}, \ldots, a_{ku}$ belongs to $\mathbb{C} \setminus \mathbb{R}$ that implies the relation $f_u(E_k) = \mathbb{C}$ for all $u = 1, 2, \ldots, m$. The theorem is proved.

We note that if $P(b_2, b_3, \ldots, b_k) \neq 0$ for all $\{b_2, b_3, \ldots, b_k\} \subset \mathbb{R}^k$, then $C_{N,0,0,\ldots,0} \neq 0$, because otherwise $P(b_2, b_3, \ldots, b_k) = 0$ for $b_2 = b_3 = \ldots = b_k = 0$.

Since the function $P(b_2, b_3, \ldots, b_k)$ is continuous on $\mathbb{R}^k$, the condition $P(b_2, b_3, \ldots, b_k) \neq 0$ means either $P(b_2, b_3, \ldots, b_k) > 0$ or $P(b_2, b_3, \ldots, b_k) < 0$ for all real $b_2, b_3, \ldots, b_k$. Therefore, it is obvious that for any equation (34) of the elliptic type, the condition $P(b_2, b_3, \ldots, b_k) \neq 0$ is always satisfied for all $\{b_2, b_3, \ldots, b_k\} \subset \mathbb{R}^k$. At the same time, exist equations (34) for which $P(b_2, b_3, \ldots, b_k) > 0$ for all $\{b_2, b_3, \ldots, b_k\} \subset \mathbb{R}$, but which are not elliptic. For example, such is the following equation in $\mathbb{R}^4$: $\frac{\partial^3 u}{\partial x_1^3} + \frac{\partial^3 u}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u}{\partial x_1 \partial x_3^2} + \frac{\partial^3 u}{\partial x_1 \partial x_4^2} = 0$.

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