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Effective Conductivity of 2D Ellipse – Elliptical Ring Composite Material

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For 2D composite material geometrically composed by an interior of ellipse E_1 (with semi-axes a_1, b_1) and an outer elliptical ring with an outer ellipse E_2 (with semi-axes a_2, b_2), where E_1 and E_2 are confocal ellipses; it is determined in an analytic form the x-component of the effective conductivity tensor as a sum of convergent power series with coefficients depending on conductivities of the components of the composite material and geometrical characteristics a_1, b_1 and a_2, b_2.

Для двовимірного композиційного матеріалу, що геометрично утворений внутрішністю еліпса E_1 з півосями a_1, b_1 та прилеглим еліптичним кільцем з зовнішнім еліпсом E_2 з півосями a_2, b_2, а E_1 та E_2 мають спільні фокуси, одержано аналітичний вираз x-компоненти тензору ефективної провідності у вигляді суми збіжного числового ряду, коефіцієнти якого залежать від провідностей компонентів композиту та геометричних характеристик a_1, b_1 та a_2, b_2.

Для плоского композиционного материала, геометрически образованного внутренностью эллипса E_1 с полуосями a_1, b_1 и прилежащим эллиптическим кольцом с внешним эллипсом E_2 с полуосями a_2, b_2, а E_1 и E_2 имеют общие фокусы, получено аналитическое выражение x-компоненты тензора эффективной проводимости в виде суммы сходящего числового ряда, коэффициенты которого зависят от проводимостей компонент композита и геометрических характеристик a_1, b_1 и a_2, b_2.

Keywords: 2D composites, ellipse − elliptical ring composite, heat conduction, $\mathbb{R}$-linear problem, effective conductivity

MSC: 74A40, 74G10, 30E25

1. Introduction.

Circular type 2D mechanical structures are subject of long investigations. In the pioneering work [8] (see also [9]) it had been found a complete solution of the problem of torsion and bending of elastic cylindric body reinforced by parallel cylindric beems S_k, $k = 1, \ldots, m$, of different materials. The author showed that this problem is equivalent to certain mathematical problem, which can be formulated in the following form.

Problem $\mathcal{M}$. *Let S_k, $k = 0, 1, \ldots, m$, are simple connected domains of the complex plane $\mathbb{C}$ with smooth boundaries C_0, $C_1, \ldots, C_m$, such that the contour C_0 embraces any contour C_k, $k = 1, \ldots, m$. Let $S := S_0 \setminus \cup_{k=1}^{m} S_k$. The problem is to find a function φ which harmonic in S, continuous in $\mathrm{cl}(S_0)$ and such that its normal derivative has a given jump on each contour C_k, $k = 0, 1, \ldots, m$.*

Here and anywhere $\mathrm{cl}(G)$ denote a closure of any domain $G \subset \mathbb{C}$.

In [8], it was proved that the problem is equivalent to a system of $m + 1$ Fredholm integral equations of the second kind which has solutions under some solvability conditions and the solution is unique up to an additive constant. The main aim of this article is to calculate some elastic characteristics of this elastic cylindric body reinforced by parallel cylindric beams. One of these is, so-called, elastic torsional rigidity, which, mathematically, is a real valued functional depending on the solution of the latter problem.

Although, the solution of the problem $\mathcal{M}$ is a solution of a system of Fredholm integral equations of the second kind, from the practical point of view we have evident difficulties to get its explicit expressions. Therefore, for the practice it is interesting any partial case of problem $\mathcal{M}$ which allows to find a method for its solving in an explicit form. For example, in case when $m = k = 1$ and S_0 and S are disks (or what is equivalent S_0' and S' are disks with common origin) the problem $\mathcal{M}$, is solved by the direct method based on expansions in Taylor's and Laurent series of a function $f(z)$, which is analytic in $S_1' \cup S'$, $S' = S_0' \setminus \mathrm{cl}\,(S_1')$. The function $f(z)$ is such that its real part $\mathrm{Re} f$ equals to an unknown required function φ in $S_1' \cup S'$. By using this solution the elastic torsional rigidity is calculated in an explicit form in [11]. The case when $m = k = 1$ and S_0 and S are confocal ellipses has been considered in [10]. The proposed method in this paper is based on the reduction of Problem $\mathcal{M}$ for confocal ellipses to similar boundary value problem but for two bounded rings, therefore, the method of solution of the latter problem based on conformal technique, which maps our ellipses, by using an analytic function of Zhukovskii type, to disks and after using Laurent expansions.

See also [5], where the method of functional equation (cf., [6]) is developed to derive analytical approximate formulae for the effective conductivity tensor of the two-dimensional composites with elliptical inclusions which number is greater then two.

The aim of present paper is to solve a problem similar to Problem $\mathcal{M}$ when $m = k = 1$ and S_0 and S are bounded by confocal ellipses $E_1 = C_1$ and $E_2 = C_0$, describing the 2D composite material with the elliptical inclusion, geometrically represented by a simply connected domain bounded by the ellipse E_1, and the matrix, geometrically represented by the elliptical annulus S bounded by two ellipses E_1 and E_2.

On the base of this solution, we calculate an analog of the elastic torsional rigidity, which is a x-component of, so-called, Effective Conductivity Tensor (see, e.g., [2, 4]). More precisely, we consider the problem of determination of the temperature distribution under perfect contact condition in the above described inhomogeneous media loaded by a simple heat flow. Note, that this problem is equivalent to the $\mathbb{R}$-linear conjugation problem on the complex plane (see [6, p. 45]). Note, that for a case of disk–ring composite material in [3] it was delivered for x-component of the effective conductivity tensor its exactly expression in terms of radius r of the internal disk and so-called contrast parameter ρ introduced by Bergmann [1] in the form of geometrical progression with respect to the powers of r^2.

2. Definitions and formulation of the problem. Let $\mathbb{C}$ be the field of complex numbers and the complex plane, $\mathbb{R}$ be the field of real numbers. For any ξ from $\mathbb{C}$ or $\mathbb{R}$ denote by $|\xi|$ its modulus. For any $z \in \mathbb{C}$ denote by $\operatorname{Re} z$ its real part, and by (η, θ), $0 < \eta < \infty$, $0 \leq \theta \leq 2\pi$ its polar coordinates, thus, $z = \eta \exp\{i\theta\}$.

Let Ω be an open or closed domain in $\mathbb{C}$ and let $\tau : \Omega \longrightarrow \mathbb{R}$ be a real valued function, then $\mathcal{C}^k(\Omega)$, $k \in 1, 2, \ldots$, denotes a class of functions having continuous partial derivatives up to k-th order. If $\tau \in \mathcal{C}^2(\Omega)$ then $\Delta\tau(x + iy) := \frac{\partial^2 \tau}{\partial x^2} + \frac{\partial^2 \tau}{\partial y^2}$.

Let G and G^o, $G \subsetneqq G^o$, are simply connected domains in $\mathbb{C}$. For any function $u : G^o \longrightarrow \mathbb{R}$ by u_1 and u_2 we mean its restrictions, respectively, to G and $\widetilde{G} := G^o \setminus \mathrm{cl}(G)$, i.e., $u_1 := u|_G$ и $u_1 := u|_{\widetilde{G}}$. Denote by E_1 and E_2, respectively, boundaries of G and G^o, i.e, $E_1 = \partial G$, $E_2 = \partial G^o$. Thus, $\partial\widetilde{G} = E_1 \cup E_2$.

Problem $\mathcal{A}$ for the system of domains $(G, \widetilde{G})$ is a problem on finding a pair of real-valued functions $u = (u_1, u_2)$, where $u_1 \in \mathcal{C}^2(G) \cap \mathcal{C}^1(\mathrm{cl}(G))$ and $u_2 \in \mathcal{C}^2(\widetilde{G}\,) \cap \mathcal{C}^1(\widetilde{G}\,)$, satisfying the following system of equations

$$\begin{cases} \Delta u(z) = 0 & \forall z \in G \cup \widetilde{G}, \\ u_2(t) = -\mathrm{Re}\, t & \forall t \in E_2, \\ u_1(t) = u_2(t) & \forall t \in E_1, \\ \lambda_1 \frac{\partial u_1}{\partial \mathbf{n}}(t) = \lambda_2 \frac{\partial u_2}{\partial \mathbf{n}}(t) & \forall t \in E_1, \end{cases} \tag{1}$$

where λ_1 and λ_2 are positive constants such, that the Bergmann's contrast parameter (see [1]) $\rho := (\lambda_1 + \lambda_2)^{-1}(\lambda_1 - \lambda_2)$ is different from zero (it is well known, that $-1 < \rho < 1$); $u_1(t)$ and $u_2(t)$ denote, the limiting on E_1 values of, respectively, $u_1(z)$, $z \in G$, and $u_2(z)$, $z \in \widetilde{G}$; $\frac{\partial u_1}{\partial \mathbf{n}}$ and $\frac{\partial u_2}{\partial \mathbf{n}}$ denote, respectively, the outward normal derivative of u_1 with respect to the boundary E_1 of the domain G and the inward normal derivative of u_2 with respect to the connected component E_1 of the boundary $\partial\widetilde{G}$ of $\widetilde{G}$.

Note, that Problem (1) (cf., e.g., [6, p. 45] with $\lambda_2 = 1$) can be reduced to the following $\mathbb{R}$-linear conjugation problem: to find two functions $\phi : G \cup \widetilde{G} \longrightarrow \mathbb{C}$, $\phi_0 : \mathbb{C} \cup \{\infty\} \longrightarrow \mathbb{C}$, $\phi_0(\infty) = 0$, such that ϕ is analytic in $G \cup \widetilde{G}$, ϕ_0 is analytic in $\mathrm{ext}\, G^o := \{z \in \mathbb{C} : z \overline{\in} \mathrm{cl}(G^o)\}$ and continuous in $\mathrm{cl}(\mathrm{ext}\, G^o)$, functions ϕ and ϕ_0 satisfy the following conjugation conditions

$$\phi^+(t) = \phi^-(t) - \rho\overline{\phi^-(t)} \quad \forall t \in E_1, \quad \phi^-(t) = \phi_0(t) - \overline{\phi_0(t)} - t \quad \forall t \in E_2, \tag{2}$$

where $\phi^+(t)$ and $\phi^+(t)$, $t \in E_1$, is boundary values of $\phi(z)$, accordingly, from G, $\widetilde{G}$, $\overline{u + iv} := u - iv$ for $u, v \in \mathbb{R}$.

In [7], it has been considered the problem of disturbance of a complex potential after insertion of a foreign inclusion in the form of a two-phase confocal elliptical annulus into a homogeneous medium. There were investigated the cases of an arbitrary distribution of singularities.

Our aim is to calculate an x-component of the effective conductivity tensor (see, e.g. [4]), i.e. a quantity λ^x_{eff}, satisfying the following equality

$$F^x \lambda^x_{eff} = \lambda_1 \int_G \frac{\partial\, u_1}{\partial x}\, dx\, dy + \lambda_2 \int_{\widetilde{G}} \frac{\partial\, u_2}{\partial x}\, dx\, dy, \tag{3}$$

where $u(x, y) = (u_1(x, y), u_2(x, y)) \equiv (u_1(x + iy), u_2(x + iy))$ is a solution of Problem $\mathcal{A}$ for the system of domains $(G, \widetilde{G})$, F^x is a complete flux in x-direction.

For any $\epsilon > 0$ by D_ϵ denote a disk $\{\zeta \in \mathbb{C} : |\zeta| < \epsilon\}$, γ_ϵ means a circle $\{\zeta \in \mathbb{C} : |\zeta| = \epsilon\}$. For a pair of positive real numbers r_1, r_2, $r_1 < r_2$, denote by A_{r_1,r_2} the set $\{\zeta \in \mathbb{C} : r_1 < |\zeta| < r_2\}$.

In [3], a quantity (3) is calculated in an exactly form for the system of domains $(\mathrm{D}_r, \mathrm{A}_{r,1})$, $0 < r < 1$, as a functional depending of r, analytically expressed as a sum of geometrical progression with respect to powers of r^2.

Assume further, that a boundary ∂G^o of a domain G^o is an ellipse E_2 with semi-axes a_2, b_2, $a_2 > b_2$, ∂G is an ellipse E_1 with semi-axes a_1, b_1, $a_1 > b_1$, moreover, ellipses E_1 and E_2 have common focuses $F_1 = -c$ and $F_2 = c$, where $c = \sqrt{a_2^2 - b_2^2} = \sqrt{a_1^2 - b_1^2}$. Then $a_2 > a_1$, $b_2 > b_1$, $2c$ is the focus distance, $\widetilde{G}$ is an elliptical ring bounded by ellipses E_1 and E_2.

3. Auxiliary $\mathcal{A}$–problem. Denote by

$$n = a_1 + b_1,\ m = \frac{c}{a_1 + b_1} < 1,\ R = \frac{a_2 + b_2}{a_1 + b_1} > 1. \tag{4}$$

Consider a mapping $\omega\colon \mathbb{C} \setminus \{\zeta \in \mathbb{C} : |\zeta| \le m\} \longrightarrow \mathbb{C} \setminus [F_1, F_2]$, $[F_1, F_2] := \{x : -c \le x \le c\}$ defined by

$$z = x + iy =: \omega(\zeta) \equiv \frac{n}{2}\left(\zeta + \frac{m^2}{\zeta}\right). \tag{5}$$

Obviously, that the mapping (5) is an analytic function, thus, it is one-to-one correspondence and the following formulae hold

$$x = x(\eta, \theta) \equiv \frac{n}{2}\left(\eta + \frac{m^2}{\eta}\right)\cos\theta,\ \ y = y(\eta, \theta) \equiv \frac{n}{2}\left(\eta - \frac{m^2}{\eta}\right)\sin\theta, \tag{6}$$

where $\zeta = \eta \exp\{i\theta\}$.

If a point $\zeta = \eta \exp\{i\theta\}$ runs through the circle γ_m, then $\omega(\zeta)$ twice runs through the segment $[F_1, F_2]$ and the following equality holds

$$\omega\left(m \exp\{i\theta\}\right) = \omega\left(m \exp\{-i\theta\}\right), 0 \le \theta \le 2\pi. \tag{7}$$

Valid the following equalities

$$\omega\left(\mathrm{A}_{m,1}\right) = G \setminus [F_1F_2],\ \omega(\gamma_1) = E_1,\ \omega(\mathrm{A}_{1,R}) = \widetilde{G},\ \omega(\gamma_R) = E_2. \tag{8}$$

Let $f(z)$ be analytic in $G \cup \widetilde{G}$ function such, that $\mathrm{Re} f(z) = u(z)$ is a solution of the system (1). Then the function $g(\zeta) := f(\omega(\zeta))$ is analytic in $\mathrm{A}_{m,1} \cup \mathrm{A}_{1,R}$ and a system (1) for a function $u(\zeta) \equiv v(\zeta) := \mathrm{Re}\, g(\zeta)$, taking in account equalities (6) and (8), transforms to the form

$$\begin{cases} \Delta v(\zeta) = 0 & \forall \zeta \in \mathrm{A}_{m,1} \cup \mathrm{A}_{1,R}, \\ v_2(\zeta) = -\frac{n}{2}\left(R + \frac{m^2}{R}\right)\cos\theta & \forall \zeta = R\exp\{i\theta\} \in \gamma_R, \\ v_1(\zeta) = v_2(\zeta) & \forall \zeta \in \gamma_1, \\ \lambda_1 \frac{\partial v_1(\eta \exp\{i\theta\})}{\partial \eta}\Big|_{\eta=1} = \lambda_2 \frac{\partial v_2(\eta \exp\{i\theta\})}{\partial \eta}\Big|_{\eta=1} & \forall \theta \in [0, 2\pi]. \end{cases} \tag{9}$$

A problem on finding a function $v = (v_1, v_2)$, where $v_1 \in C^2(\mathrm{A}_{m,1}) \cap C^1(\mathrm{cl}(\mathrm{A}_{m,1}))$ and $v_2 \in C^2(\mathrm{A}_{1,R}) \cap C^1(\mathrm{cl}(\mathrm{A}_{1,R}))$, which satisfy the system of equations (9) call an auxiliary $\mathcal{A}$–problem.

Using analyticity of $g_1 := g|_{A_{m,1}}$ and $g_2 := g|_{A_{1,R}}$, respectively, in $A_{m,1}$ and $A_{1,R}$, we have the following expansions in the Laurents's series

$$g_1(\zeta) = a'_0 + ib'_0 + \sum_{k=-\infty,\, k\neq 0}^{\infty} (a'_k + ib'_k)\zeta^k \quad \forall \zeta \in \mathrm{A}_{m,1}, \tag{10}$$

$$g_2(\zeta) = a''_0 + ib''_0 + \sum_{k=-\infty,\, k\neq 0}^{\infty} (a''_k + ib''_k)\zeta^k \quad \forall \zeta \in \mathrm{A}_{1,R}, \tag{11}$$

where a'_k, a''_k, b'_k, b''_k, $k = \pm 1, \pm 2, \ldots$, denote unknown real numbers. Going to real parts in (10) and (11), obtain formulas

$$v_1(\zeta) = a'_0 + \sum_{k=1}^{\infty} \left(a'_k \eta^k + a'_{-k}\eta^{-k}\right)\cos(k\theta) -$$

$$- \sum_{k=1}^{\infty} \left(b'_k \eta^k - b'_{-k}\eta^{-k}\right)\sin(k\theta) \quad \forall \zeta = \eta\exp\{i\theta\} \in \mathrm{A}_{m,1}, \tag{12}$$

$$v_2(\zeta) = a''_0 + \sum_{k=1}^{\infty} \left(a''_k \eta^k + a''_{-k}\eta^{-k}\right)\cos(k\theta) -$$

$$- \sum_{k=1}^{\infty} \left(b''_k \eta^k - b''_{-k}\eta^{-k}\right)\sin(k\theta) \quad \forall \zeta = \eta\exp\{i\theta\} \in \mathrm{A}_{1,R}. \tag{13}$$

Since ω maps exterior $\operatorname{ext} \mathrm{D}_m$ of the disk D_m in the complex plane of the variable ζ to the complex plane of the variable z with a cut along the segment $[F_1, F_2]$, then the problem (1) is equivalent to auxiliary $\mathcal{A}$–problem if and only if when the function (10) has coinciding limiting values on different sides of the cut γ_m, which, by virtue of the formula (7), is equivalent to the validity of the following equality $g_1\left(m \exp\{i\theta\}\right) = g_1\left(m \exp\{-i\theta\}\right), 0 \leq \theta \leq 2\pi$, which in turn leads equalities

$$a'_{-k} = m^{2k} a'_k, \quad b'_{-k} = m^{2k} b'_k, \quad k = 1, 2, \ldots. \tag{14}$$

Substituting (13) with $\eta = R$ to the second condition in (9), obtain

$$a''_0 = 0, \quad a''_{-1} = -\frac{n}{2}\left(R^2 + m^2\right) - R^2 a''_1, \tag{15}$$

$$a''_{-k} = -R^{2k} a''_k, k = 2, 3, \ldots, \quad b''_{-k} = R^{2k} b''_k, k = 1, 2, \ldots. \tag{16}$$

Substituting equalities (12) — (15) with $\eta = 1$ to the third condition in (9), obtain

$$a'_0 = 0, \quad a'_1 = \frac{1 - R^2}{1 + m^2} a''_1 - \frac{n}{2} \frac{R^2 + m^2}{1 + m^2}, \tag{17}$$

$$a'_k = \frac{1 - R^{2k}}{1 + m^{2k}} a''_k, k = 2, 3, \ldots, \quad b'_k = \frac{1 - R^{2k}}{1 - m^{2k}} b''_k, k = 1, 2, \ldots. \tag{18}$$

Now the equality (12) by using the relations (14), (17) and (18), turns to the form

$$v_1(\zeta) = \left(\frac{1 - R^2}{1 + m^2}\, a''_1 - \frac{n}{2} \frac{R^2 + m^2}{1 + m^2}\right)\left(\eta + \frac{m^2}{\eta}\right)\cos\theta +$$

$$+ \sum_{k=2}^{\infty} \frac{1 - R^{2k}}{1 + m^{2k}}\, a''_k \left(\eta^k + \frac{m^{2k}}{\eta^k}\right)\cos k\theta -$$

$$- \sum_{k=1}^{\infty} \frac{1 - R^{2k}}{1 - m^{2k}}\, b''_k \left(\eta^k - \frac{m^{2k}}{\eta^k}\right)\sin k\theta \quad \forall \zeta = \eta \exp\{i\theta\} \in \mathrm{A}_{m,1}. \tag{19}$$

Taking into account (15) and (16) we rewrite (13) in the form

$$v_2(\zeta) = \left(a''_1 \eta - \left(\frac{n}{2}(R^2 + m^2) + R^2 a''_1\right)\eta^{-1}\right)\cos\theta +$$

$$+\sum_{k=2}^{\infty} a''_k \left(\eta^k - \frac{R^{2k}}{\eta^k}\right)\cos k\theta -$$

$$-\sum_{k=1}^{\infty} b''_k \left(\eta^k - \frac{R^{2k}}{\eta^k}\right)\sin k\theta \quad \forall \zeta = \eta \exp\{i\theta\} \in \mathrm{A}_{1,R}. \tag{20}$$

Substituting delivered equalities (19) and (20) to the fourth condition in (9), obtain, after division on $\lambda_1 + \lambda_2$, the following relations

$$\left(\left(1 + m^2 R^2\right)\rho - m^2 - R^2\right) a''_1 = \frac{n}{2}(R^2 + m^2)(1 - m^2\rho), \tag{21}$$

$$\left(\left(1 + m^{2k} R^{2k}\right)\rho - m^{2k} - R^{2k}\right) a''_k = 0, \quad k = 2, 3, \ldots, \tag{22}$$

$$\left(\left(1 - m^{2k} R^{2k}\right)\rho + m^{2k} - R^{2k}\right) b''_k = 0, \quad k = 1, 2, \ldots. \tag{23}$$

Since

$$\rho < 1,\ (1 + m^{2k} R^{2k})^{-1}(m^{2k} + R^{2k}) > 1, k = 1, 2, \ldots, \tag{24}$$

and the modulus of the expression $(1 - m^{2k} R^{2k})^{-1}(R^{2k} - m^{2k})$ is greater then one for all natural k by virtue of inequalities $m < 1$ and $R > 1$, then relations (21) — (23) implies equalities

$$a''_1 = \frac{n}{2}\frac{\left(R^2 + m^2\right)\left(1 - m^2\rho\right)}{(1 + m^2 R^2)\rho - m^2 - R^2}, \quad a''_k = b''_{k-1} = 0, k = 2, 3, \ldots. \tag{25}$$

Substituting now (25) in (15) and (16), obtain equalities

$$a''_{-1} = -\frac{n}{2}\frac{(R^2 + m^2)(\rho - m^2)}{(1 + m^2 R^2)\rho - m^2 - R^2}, \quad a''_{-k} = 0, \quad k = 2, 3, \ldots. \tag{26}$$

Substituting expressions for a''_1 and b''_k from (25) to the (17) and (16), obtain

$$a'_1 = \frac{n}{2}\frac{(R^2 + m^2)(1 - \rho)}{(1 + m^2 R^2)\rho - m^2 - R^2}, \quad b''_{-k} = 0,\ k = 1, 2, \ldots. \tag{27}$$

Substituting expression for a'_1 from (27) to the (14) with $k = 1$, obtain

$$a'_{-1} = \frac{n}{2}\frac{m^2(R^2 + m^2)(1 - \rho)}{(1 + m^2 R^2)\rho - m^2 - R^2}. \tag{28}$$

Using (25) rewrite formulae (18) and (14) in the form $a'_k = a'_{-k} = 0, k = 2, 3, \ldots,\quad b'_k = b'_{-k} = 0, k = 1, 2, \ldots$.

Thus, substituting delivered relations to the formulas (12) and (13), obtain a solution of the auxiliary $\mathcal{A}$–problem with additional condition (14) in the form

$$v_1(\zeta) = (1-\rho)\psi(\rho)\frac{n}{2}\left(\eta + \frac{m^2}{\eta}\right)\cos\theta \quad \forall\zeta = \eta\exp\{i\theta\} \in \mathrm{A}_{m,1}, \tag{29}$$

$$v_2(\zeta) = \psi(\rho)\frac{n}{2}\left(\eta + \frac{m^2}{\eta} - \left(m^2\eta + \eta^{-1}\right)\rho\right)\cos\theta \quad \forall\zeta = \eta\exp\{i\theta\} \in \mathrm{A}_{1,R}, \tag{30}$$

where $\psi(\rho) := (R^2 + m^2)\big((1 + m^2R^2)\rho - m^2 - R^2\big)^{-1}$.

4. Solution of $\mathcal{A}$–problem for a system of domains bounded by confocal ellipses.

Now summarizing results of the latter section we can state the following theorem.

Theorem 1. *Let $\mathcal{A}$–Problem posed to the system of domains $(G, \widetilde{G})$ bounded by confocal ellipses E_1 (inner) and E_2 (outer), accordingly, with semi-axes a_1, b_1, $a_1 > b_1$, and a_2, b_2, $a_2 > b_2$, relations (4) hold.*

Then $\mathcal{A}$–Problem for the system of domains $(G, \widetilde{G})$ is equivalent to the auxiliary $\mathcal{A}$–problem if and only if when fulfilled a condition (14)*, where $v_1 = \operatorname{Re} g_1$ and g_1 is expressed by the formula* (10)*. Then these problems are uniquely solvable and the following formulas hold*

$$u(z) = v(\zeta), \quad z = x + iy \equiv \omega(\zeta). \tag{31}$$

A general solution $v = (v_1, v_2)$ of the auxiliary $\mathcal{A}$–problem is expressed by the formulae (29) *and* (30)*. Furthermore, the boundary values for any $\theta \in [0, 2\pi]$ have the form*

$$v_1(\exp\{i\theta\}) = v_2(\exp\{i\theta\}) \equiv v_{\gamma_1}(\theta) := \frac{n}{2}(1-\rho)\psi(\rho)\left(1 + m^2\right)\cos\theta, \tag{32}$$

$$v_2(R\exp\{i\theta\}) \equiv v_{\gamma_R}(\theta) := \frac{n}{2}\psi(\rho)\left(R + \frac{m^2}{R} - \left(m^2R + \frac{1}{R}\right)\rho\right)\cos\theta. \tag{33}$$

A general solution of $\mathcal{A}$–Problem for the system of domains $(G, \widetilde{G})$ in coordinates (x, y) has a form

$$u_1(z) = \psi(\rho)(1-\rho)x \quad \forall z = x + iy \in G, \tag{34}$$

$$u_2(z) = \psi(\rho)\,(x - x'\rho) \quad \forall z = x + iy \in \widetilde{G}, \tag{35}$$

where $x' = \operatorname{Re}\omega(\zeta^{-1})$, $z = \omega(\zeta)$.

5. Effective conductivity. In the considered case x-component F^x of the complete flux can be normalized as follows $F^x = -\pi a_2 b_2$ (see [6]).

Theorem 2. *Under assumptions of Theorem 1 x-component of the effective conductivity tensor is expressed by the formula*

$$\lambda_{eff}^x = \lambda_2\, P_1(\rho) \frac{R^2 + m^2}{m^2 + R^2 - \rho(1 + m^2 R^2)}, \tag{36}$$

where $P_1(\rho) := \frac{n^2}{4\, a_2 b_2} \left(R^2 - \frac{m^4}{R^2} + \left(1 - m^4 - m^2 R^2 + \frac{m^2}{R^2} \right) \rho \right)$.

Furthermore, the function (3) is the sum of the following convergent series

$$\lambda_{eff}^x = \lambda_2\, P_1(\rho) \left(1 + \sum_{k=1}^{\infty} \rho^k \left(\frac{1 + m^2 R^2}{m^2 + R^2} \right)^k \right). \tag{37}$$

Proof. By the Green's formula

$$F^x \lambda_{eff}^x = \lambda_1 \oint_{E_1} u_1(x,y)\, dy + \lambda_2 \left(\oint_{E_2} - \oint_{E_1} \right) u_2(x,y)\, dy =$$

$$= \oint_{E_1} \left(\lambda_1 u_1(x,y) - \lambda_2 u_2(x,y) \right) dy + \lambda_2 \oint_{E_2} u_2(x,y)\, dy,$$

where in contour integrals the orientation is assumed to be opposite to the clockwise direction. Doing a change of variables (6) with using of the relations (8), (32) and (33), obtain

$$F^x \lambda_{eff}^x = (\lambda_1 - \lambda_2) \int_0^{2\pi} v_{\gamma_1}(\theta)\, dy(1,\theta) + \lambda_2 \int_0^{2\pi} v_{\gamma_R}(\theta)\, dy(R,\theta) =$$

$$= \frac{n^2}{4} \psi(\rho) \int_0^{2\pi} (\cos\theta)^2\, d\theta \left((1-\rho)(\lambda_1 - \lambda_2)(1 - m^4) + \right.$$

$$\left. + \lambda_2 \left(R^2 - \frac{m^4}{R^2} \right) - \lambda_2 \rho \left(R - \frac{m^2}{R} \right) \left(m^2 R + \frac{1}{R} \right) \right).$$

Substituting equalities $(1-\rho)(\lambda_1-\lambda_2)=2\lambda_2\rho$, $\int_0^{2\pi}(\cos\theta)^2\,d\theta=\pi$ to the last formula, and doing elementary transformations with using of the expressions of $\psi(\rho)$ and F^x, obtain the required equality (36).

Then the equality (37) follows from the obtained formula after substitution to it the following series expansion

$$\frac{R^2+m^2}{m^2+R^2-\rho(1+m^2R^2)}=1+\sum_{k=1}^{\infty}\rho^k\left(\frac{1+m^2R^2}{m^2+R^2}\right)^k, \tag{38}$$

which is valid due to the inequalities (24) with $k=1$. The proof is completed.

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