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# Local behavior of mappings defined by their distributional derivatives

We study asymptotic behavior of spatial mappings at the origin. These mappings belong to the Orlic-Sobolev class and are defined by their Jacobi matrices. The theorem provides a result of a Schwarz Lemma type. The sharpness of conditions is illustrated by an example.

## 1. Introductory remarks.

**1.1.** Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $f = (f_1, \ldots, f_n)\colon D \longrightarrow \mathbb{R}^n$ be a $W^{1,1}_{\rm loc}$-mapping, and $f'(x) = (\partial f_i / \partial x_j)$ denote its Jacobi matrix at $x \in D$. Denote by $\mathbb{M}_n$ the space of $n \times n$ matrices.

In this paper, we study the asymptotic behavior of local solutions $y = f(x)$ of the matrix differential equation

$$y'(x) = M(x) \tag{1}$$

with given $M(x) \in \mathbb{M}_n$ (determined almost everywhere in $D$).

We shall use the both Hilbert-Schmidt and operator norms

$$\|M(x)\| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} m^2_{i,j}(x)}, \quad |M(x)| = \sup_{|h|=1} |M(x)\cdot h|$$

of matrices. Each matrix $M(x)$ determines the $p$-outer dilatation coefficient $(p \geq 1)$ of solution $f(x)$ by

$$K_p\left(M(x)\right) = \begin{cases} \frac{|M(x)|^p}{\det(M(x))}\,, & \text{if } \det M(x) \neq 0\,, \\ 1\,, & \text{if } M(x) = 0\,, \\ \infty\,, & \text{otherwise}\,. \end{cases} \tag{2}$$

We consider homeomorphisms $f \in W^{1,1}_{\rm loc}(D,\mathbb{R}^n)$ having *finite distortion*, i.e. such that $|M(x)|^n/\det M(x) < \infty$ almost everywhere in $D$. This means that $f'(x) = 0$ for almost all points of the set $Z = \{x \in D : \det f'(x) = 0\}$; see [8], [6].

**1.2.** Consider for a given convex increasing function $\varphi\colon [0,\infty) \longrightarrow [0,\infty)$, $\varphi(0) = 0$, the corresponding Orlic space $L^{\varphi}$ of functions $g\colon D \longrightarrow \mathbb{R}$, satisfying

$$\int\limits_D \varphi\left(\frac{|g(x)|}{\lambda}\right)\, dm(x) < \infty$$

for some $\lambda > 0$ (cf., e.g. [10]), where $m$ denotes the $n$-dimensional Lebesgue measure in $\mathbb{R}^n$.

The *Orlic-Sobolev class* $W^{1,\varphi}_{\rm loc}(D)$ is a collection of locally integrable functions $g$ in $D$ with first distributional derivatives, whose gradient $\nabla g$ belongs to the Orlic class locally in $D$. Note that $W^{1,\varphi}_{\rm loc} \subset W^{1,1}_{\rm loc}$ and $g \in W^{1,p}_{\rm loc}$ when $\varphi(t) = t^p$, $p \geq 1$.

For a locally integrable vector-function

$$f = (f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n))$$

with $f_i \in W^{1,1}_{\rm loc}$, we put

$$\int\limits_D \varphi\left(\|\nabla f(x)\|\right)\, dm(x) < \infty\,,$$

where $\|\nabla f(x)\| = \sqrt{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\left(\frac{\partial f_i}{\partial x_j}\right)^2}$, and $f \in W^{1,\varphi}_{\rm loc}$. We shall also use the notation $W^{1,\varphi}_{\rm loc}$ for more general functions $\varphi$, than for the Orlic classes, assuming the convexity of $\varphi$ with a normalization $\varphi(0) = 0$.

**1.3.** We say that a matrix function $M\colon D \longrightarrow \mathbb{M}_n$ $(n \geq 3)$ has the $\mathcal{F}_{\varphi}$*-property* if:

1) $M(x) = 0$ for almost all points of the set $Z = \{x \in D : \det M(x) = 0\}$;

2) the function $\varphi\colon [0,\infty) \longrightarrow [0,\infty)$ is monotone increasing and satisfies a Calderon type condition $\int\limits_1^{\infty}\left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty$ (cf. [1]);

3) $\int\limits_D \varphi(\|M(x)\|)\, dm(x) < \infty\,.$

In fact, the $\mathcal{F}_\varphi$-property can be extended to the planar case ($n = 2$). In this case it suffices to require from $\varphi$ only the monotonicity property (since all mappings from $W^{1,1}_{\rm loc}(\mathbb{R}^2)$ are differentiable almost everywhere).

## 2. Moduli of surface families and capacities of condensers.

**2.1.** Let $\mathcal{S}$ be a $k$-dimensional surface in $\mathbb{R}^n$, which means that $\mathcal{S}\colon D_s \longrightarrow \mathbb{R}^n$ is a continuous image of a open domain $D_s \subset \overline{\mathbb{R}^k} := \mathbb{R}^k \cup \{\infty\}$. We denote by

$$N(\mathcal{S}, y) = \operatorname{card} \mathcal{S}^{-1}(y) = \operatorname{card}\{x \in D_s : \mathcal{S}(x) = y\}$$

the multiplicity function of the surface $\mathcal{S}$ at the point $y \in \mathbb{R}^n$. It is known that the multiplicity function is semicontinuous from below and therefore it is measurable with respect to the Hausdorff measure $H^k$ (cf. [12]).

For a given Borel function $\rho\colon \mathbb{R}^n \longrightarrow [0,\infty]$, the integral of $\rho$ over $\mathcal{S}$ is defined by

$$\int\limits_{\mathcal{S}} \rho\, d\mathcal{A} = \int\limits_{\mathbb{R}^n} \rho(y)\, N(\mathcal{S}, y)\, dH^k y.$$

Let $\mathcal{S}_k$ be a family of $k$-dimensional surfaces $\mathcal{S}$ in $\mathbb{R}^n$, $1 \le k \le n-1$ (curves for $k = 1$). The *p-module* of $\mathcal{S}_k$ is defined as

$$\mathcal{M}_p(\mathcal{S}_k) = \inf \int\limits_{\mathbb{R}^n} \rho^p(x)\, dm(x), \quad p \ge k,$$

where the infimum is taken over all Borel measurable functions $\rho \ge 0$ and such that

$$\int\limits_{\mathcal{S}} \rho^k\, d\mathcal{A} \ge 1$$

for every $\mathcal{S} \in \mathcal{S}_k$. We call each such $\rho$ an *admissible function* for $\mathcal{S}_k$ ( $\rho \in \operatorname{adm} \mathcal{S}_k$ ).

Following [9], a metric $\rho$ is said to be *extensively admissible* for $\mathcal{S}_k$ ($\rho \in \operatorname{ext}_p \operatorname{adm} \mathcal{S}_k$) with respect to $p$-module if $\rho \in \operatorname{adm}(\mathcal{S}_k \backslash \widetilde{\mathcal{S}}_k)$ such that $\mathcal{M}_p(\widetilde{\mathcal{S}}_k) = 0$.

Accordingly, we say that a property $P$ holds for almost every $k$-dimensional surface, if $P$ holds for all surfaces except a family of zero $\alpha$-module.

**2.2.** We also use especially another tool which is important in Potential Theory and Mathematical Analysis.

Following in general [11], a pair $\mathcal{E} = (A, C)$, where $A \subset \mathbb{R}^n$ is an open set and $C \subset A$ is a nonempty compactum, is called the *condenser*. We say that the condenser $\mathcal{E}$ is the *ring condenser*, if $\mathcal{R} = A \setminus C$ is a ring domain, i.e. its complement consists of two components. The condenser $\mathcal{E}$ is bounded, if $A$ is bounded. We also say that a condenser $\mathcal{E} = (A, C)$ lies in a domain $G$ when $A \subset G$. Obviously, for an open and continuous mapping $f\colon G \longrightarrow \mathbb{R}^n$ and for any condenser $\mathcal{E} = (A, C) \subset G$, the pair $(f(A), f(C))$ is a condenser in $f(G)$. In this case we shall use the notation $f(\mathcal{E}) = (f(A), f(C))$.

Let $\mathcal{E} = (A, C)$ be a condenser. Denote by $\mathcal{C}_0(A)$ the set of all continuous functions $u\colon A \longrightarrow \mathbb{R}^1$ with compact support in $A$. Consider the set $\mathcal{W}_0(\mathcal{E}) = \mathcal{W}_0(A, C)$ of all nonnegative functions $u\colon A \longrightarrow \mathbb{R}^1$ such that 1) $u \in \mathcal{C}_0(A)$, 2) $u(x) \geq 1$ for $x \in C$ and 3) $u$ belongs ACL. Put

$$\operatorname{cap}_p \mathcal{E} = \operatorname{cap}_p (A, C) = \inf_{u \in \mathcal{W}_0(\mathcal{E})} \int_A |\nabla u|^p \, dx, \quad p \geq 1,$$

where, as usual

$$|\nabla u| = \left( \sum_{i=1}^{n} (\partial_i u)^2 \right)^{1/2}.$$

This quantity is called *$p$-capacity of condenser $\mathcal{E}$*.

It was proven in [7] that for $p > 1$

$$\operatorname{cap}_p \mathcal{E} = \mathcal{M}_p(\Delta(\partial A, \partial C; A \setminus C)), \tag{3}$$

where $\Delta(\partial A, \partial C; A \setminus C))$ denotes the set of all continuous curves which join the boundaries $\partial A$ and $\partial C$ in $A \setminus C$ (cf. [2,14]). For general properties of $p$-capacities and their relation to the mapping theory, we refer, for instance, to [5] and [13]. In particular, for $1 < p < n$,

$$\operatorname{cap}_p \mathcal{E} \geq n \Omega_n^{\frac{p}{n}} \left( \frac{n-p}{p-1} \right)^{p-1} [m(C)]^{\frac{n-p}{n}}, \tag{4}$$

where $\Omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$, and $mC$ is the $n$-dimensional Lebesgue measure of $C$.

## 3. Lower $Q$-homeomorphisms and auxiliary results.

We shall use the notations

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}, \ B_r = B(0,r), \ \mathbb{B}^n = B(0,1),$$
$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \ S_r = S(0,r).$$

Given a Lebesgue measurable function $Q : G \to [0,\infty]$, we define for the measurable sets $E \subset \mathbb{R}^n$ set

$$\fint_E Q(x)\, dm(x) = \frac{1}{m(E)} \int_E Q(x)\, dm(x)\,.$$

Suppose that $D$ and $D'$ are two domains in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$, and $Q\colon D \longrightarrow (0,\infty)$ is a Lebesgue measurable function. A homeomorphism $f\colon D \longrightarrow D'$ is called *lower $Q$-homeomorphism with respect to $p$-module* at $x_0$, if the following bound holds

$$\mathcal{M}_p\left(f(\Sigma_R)\right) \geq \inf_{\rho \in \mathrm{ext}_p \mathrm{adm}\, \Sigma_R} \int_R \frac{\rho^p(x)}{Q(x)}\, dm(x)$$

for each ring

$$R = R(x_0, \varepsilon_1, \varepsilon_2) = \{x \in \mathbb{R}^n : \varepsilon_1 < |x - x_0| < \varepsilon_2\}, 0 < \varepsilon_1 \leqslant \varepsilon_2 < d_0,$$

where $d_0 = \mathrm{dist}(x_0, \partial D)$, and $\Sigma_R$ denotes the family of spheres $S(x_0, r)$, $r \in (\varepsilon_1, \varepsilon_2)$.

The following statement is given in [3] and provides the necessary and sufficient condition for homeomorphisms to be lower $Q$-homeomorphisms.

**Lemma 1.** *Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$ and $Q\colon D \longrightarrow (0,\infty)$ be a measurable function. A homeomorphism $f\colon D \longrightarrow \mathbb{R}^n$ is lower $Q$-homeomorphism at $x_0$ with respect to $p$-module, $p > n-1$, if and only if the following inequality holds*

$$\mathcal{M}_p(f(\Sigma_R)) \geq \int_{r_1}^{r_2} \frac{dr}{\left(\int\limits_{S(x_0,r)} Q^{\frac{n-1}{p-n+1}}(x)\, d\mathcal{A}\right)^{\frac{p-n+1}{n-1}}}\,, \quad \forall\ 0 < \varepsilon_1 < \varepsilon_2 < d_0,$$

*where* $d_0 = \mathrm{dist}(x_0, \partial D)$ , $\Sigma_R$ *is the family of spheres* $S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}$ *with* $r \in (\varepsilon_1, \varepsilon_2)$.

In the following lemma we establish the necessary condition characterizing the lower $Q$-homeomorphisms and obtain an upper bound for $\alpha$-module of the family of curves, $\alpha = p/(p-n+1)$.

**Lemma 2.** *Let* $D$ *be a domain in* $\mathbb{R}^n$, $n \geq 2$, $x_0 \in D$, *and* $Q\colon D \longrightarrow (0, \infty)$ *be measurable function. Suppose that* $f\colon D \longrightarrow \mathbb{R}^n$ *is a lower* $Q$*-homeomorphism at* $x_0$ *with respect to* $p$*-module with* $p > n-1$. *Then*

$$\mathcal{M}_{\frac{p}{p-n+1}}(\Gamma^*) \;\leq\; \left( \int\limits_{r_1}^{r_2} \frac{dr}{\left( \int\limits_{S(x_0,r)} Q^{\frac{n-1}{p-n+1}}(x) d\mathcal{A} \right)^{\frac{p-n+1}{n-1}}} \right)^{-\frac{n-1}{p-n+1}},$$

*where* $\Gamma^* = \Delta(f(S_1), f(S_2), f(D))$ *is the family of all curves connection* $S_j = S(x_0, r_j), j = 1, 2,$ *in* $f(D)$.

*Proof.* Pick arbitrary spheres $S_i = S(x_0, r_i)$, $i = 1, 2$, such that $0 < r_1 < r_2 < d(x_0, \partial D)$. Then due to the relations of moduli and capacities by Hesse [7] and Ziemer [15], we have

$$\mathcal{M}_{\frac{p}{p-n+1}}\left(f\left(\Delta(S_1, S_2, D)\right)\right) \leq \frac{1}{M_p^{\frac{n-1}{p-n+1}}\left(f\left(\Sigma_R\right)\right)}, \tag{5}$$

because $f(\Sigma_R) \subset \Sigma\left(f(S_1), f(S_2), f(D)\right)$. Here $\Sigma_R$ denotes a collection of all spheres centered at $x_0$, which lie between $S_1$ and $S_2$; $\Sigma\left(f(S_1), f(S_2), f(D)\right)$ is the family of all $(n-1)$-dimensional surfaces in $f(D)$, that separate $f(S_1)$ and $f(S_2)$. Now the assertion of the lemma follows from the inequality (5) and Lemma 1.

The following statement is crucial in the proof of the main result and follows from Theorem 5 of [4] obtained for open discrete mappings.

**Lemma 3.** *Let* $D$ *and* $D'$ *be domains in* $\mathbb{R}^n$, $n \geq 3$. *Assume that* $M\colon D \longrightarrow \mathbb{M}_n$ *possesses* $\mathcal{F}_\varphi$*-property and* $f\colon D \longrightarrow D'$ *is a homemorphic solution of the equation* (1). *Then* $f\colon D \longrightarrow D'$ *is lower* $Q$*-homeomorphism with respect to* $p$*-module with* $Q(x) = K_p\left(M(x)\right)$ *and* $p > n-1$.

## 4. Main result.

Our main result is the following theorem which implies an upper estimate for stretching of mappings reconstructed by the Jacobi matrix at the origin.

**Theorem.** *Let a matrix $M\colon D \longrightarrow \mathbb{M}_n$ be defined in the unit ball $\mathbb{B}^n$ and possess $\mathcal{F}_\varphi$-property. Assume that $f\colon \mathbb{B}^n \longrightarrow \mathbb{B}^n, n \geq 3$, is a homeomorphic solution of the equation (1) in $\mathbb{B}^n$ normalized by $f(0) = 0$. If for $p \in (n, +\infty)$*

$$k_p = \liminf_{\varepsilon\to 0} \left( ⨍_{B_\varepsilon} K_p^\alpha\left(M(x)\right)\, dm(x)\right)^{\frac{1}{\alpha}} < \infty\,, \quad \alpha = \frac{n-1}{p-n+1}, \tag{6}$$

*then*

$$\liminf_{x\to 0} \frac{|f(x)|}{|x|} \leq \nu_0 \cdot\ k_p^{\frac{1}{p-n}} < \infty\,, \tag{7}$$

*where $\nu_0$ is a positive constant depending only on $n$ and $p$.*

*Proof.* Consider a spherical ring $R = R(0, \varepsilon, 2\varepsilon)$ with $0 < \varepsilon < \frac{1}{2}$. Then $\mathcal{E} = \big(B_{2\varepsilon}, \overline{B}_\varepsilon\big)$ and $f(\mathcal{E}) = \big(f(B_{2\varepsilon}), f(\overline{B}_\varepsilon)\big)$ are the ring condensers in $\mathbb{B}^n$. Consider the curve family $\Gamma_\varepsilon^* = \Delta(f(S_\varepsilon), f(S_{2\varepsilon}), f(R))$. Then from (3),

$$\mathrm{cap}_q\ f(\mathcal{E}) = \mathcal{M}_q\left(\Gamma_\varepsilon^*\right)\,,$$

with $q = p/(p-n+1)$.

By Lemmas 2 and 3, one gets

$$\mathrm{cap}_q\ f(\mathcal{E}) \leq \left( \int\limits_\varepsilon^{2\varepsilon} \frac{dr}{\left(\int\limits_{S_r} K_p^\beta\left(M(x)\right)\, d\mathcal{A}\right)^{1/\beta}} \right)^{-\beta}, \tag{8}$$

where $\beta = (n-1)/(p-n+1)$.

Noting that

$$\varepsilon = \int\limits_\varepsilon^{2\varepsilon} \left( \int\limits_{S_r} K_p^\beta\left(M(x)\right) d\mathcal{A}\right)^{1/q} \frac{dr}{\left(\int\limits_{S_r} K_p^\beta\left(M(x)\right) d\mathcal{A}\right)^{1/q}}$$

and applying the Fubini theorem and the Hölder inequality with the exponents $q=\frac{p}{p-n+1}$, $q'=\frac{p}{n-1}$, one obtains

$$\left(\int\limits_{\varepsilon}^{2\varepsilon}\frac{dr}{\left(\int\limits_{S_r}K_p^{\beta}\left(M(x)\right)d\mathcal{A}\right)^{1/\beta}}\right)^{-\beta}\leq\frac{1}{\varepsilon^q}\int\limits_{R}K_p^{\beta}\left(M(x)\right)\,dm(x)\,. \tag{9}$$

Combining (9) and (8) yields

$$\operatorname{cap}_q\, f(\mathcal{E})\leq\frac{1}{\varepsilon^q}\int\limits_{R}K_p^{\beta}\left(M(x)\right)\,dm(x)\,, \tag{10}$$

where $q=\frac{p}{p-n+1}$.

On the other hand, due to the inequality (4), we have

$$\operatorname{cap}_q\, f(\mathcal{E})\geq c_1\left[m(f(B_\varepsilon))\right]^{\frac{n-q}{n}}\,, \tag{11}$$

with the same $q=\frac{p}{p-n+1}$ and a positive constant $c_1$ depending only on $n$ and $p$.

Comparing (10) and (11), one obtains the following upper estimate

$$\frac{m(f(B_\varepsilon))}{\Omega_n\varepsilon^n}\leq c_2\,\left(⨍_{B_{2\varepsilon}}K_p^{\beta}\left(M(x)\right)\,dm(x)\right)^{\frac{n}{n-q}}\,. \tag{12}$$

Here $c_2>0$ also depends only $n$ and $p$.

Denote $l_f(\varepsilon)=\min\limits_{|x|=\varepsilon}|f(x)|$. Since $f(0)=0$, $\Omega_n\, l_f^n(\varepsilon)\leq m(f(B_\varepsilon))$ or equivalently,

$$l_f(\varepsilon)\leq\left(\frac{m(f(B_\varepsilon))}{\Omega_n}\right)^{\frac{1}{n}}\,,$$

one gets

$$\liminf_{x\to 0}\frac{|f(x)|}{|x|}=\liminf_{\varepsilon\to 0}\frac{l_f(\varepsilon)}{\varepsilon}\leq\liminf_{\varepsilon\to 0}\left(\frac{m(f(B_\varepsilon))}{\Omega_n\varepsilon^n}\right)^{\frac{1}{n}}\,,$$

and, combining with (12),

$$\liminf_{x\to 0}\frac{|f(x)|}{|x|}\leq c_0\,\liminf_{\varepsilon\to 0}\left(⨍_{B_{2\varepsilon}}K_p^{\beta}\left(M(x)\right)\,dm(x)\right)^{\frac{1}{n-q}}=c_0\,k_p^{\frac{p-n+1}{(n-1)(p-n)}}$$

with $q = \frac{p}{p-n+1}$ and a positive constant $c_0$ which depends only on $n$ and $p$. The proof is completed.

Now we show that the sufficient condition (6) of the theorem can not be dropped. The following example shows that in the case of infinite lower limit in (6), the limit in (7) controlling the asymptotic behavior can be also infinite. Indeed, consider the homeomorphic automorphism of the unit ball $\mathbb{B}^n$ defined by

$$f(x) = \frac{x}{|x|}\left(1 + (p-n)\int\limits_{|x|}^{1} \frac{dt}{t^{p-n+1}\ln^{\frac{p-n+1}{n-1}}\left(\frac{e}{t}\right)}\right)^{-\frac{1}{p-n}}$$

for any $|x| < 1$, $x \neq 0$ and $f(0) = 0$ for any fixed $p$ in the interval $(n, \infty)$.

Because of the radial symmetry of the mapping, one can rewrite it via

$$f(x) = \frac{x}{|x|}\varphi(|x|), \quad x \neq 0, \quad \text{and} \quad f(0) = 0,$$

with

$$\varphi(|x|) = \left(1 + (p-n)\int\limits_{|x|}^{1} \frac{dt}{t^{p-n+1}\ln^{\frac{p-n+1}{n-1}}\left(\frac{e}{t}\right)}\right)^{-\frac{1}{p-n}}.$$

Note that $\varphi(|x|) \to 0$ as $x \to 0$, and $\varphi(|x|) \to 1$ as $|x| \to 1$. In addition, we can restrict ourselves to the case when $x = (r, 0, 0, \ldots, 0)$, $r \in (0, 1)$. Then the Jacobi matrix of $f$ is of the form

$$f'(x) = \begin{pmatrix} \varphi'(r) & 0 & 0 & \ldots & 0 \\ 0 & \frac{\varphi(r)}{r} & 0 & \ldots & 0 \\ 0 & 0 & \frac{\varphi(r)}{r} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{\varphi(r)}{r} \end{pmatrix},$$

at any $x \in \mathbb{B}^n$, $x \neq 0$. A direct computation yields

$$\left(\frac{\varphi(r)}{r}\right)^{p-n+1} = \varphi'(r)\ln^{\frac{p-n+1}{n-1}}\left(\frac{e}{r}\right),$$

and since for $0 < r < 1$, $\varphi(r)/r > \varphi'(r)$, the $p$-outer dilatation coefficient $K_p(M(x))$ defined by (2) assumes the form

$$K_p(M(x)) = \frac{|M(x)|^p}{\det M(x)} = \frac{\left(\frac{\varphi(r)}{r}\right)^p}{\left(\frac{\varphi(r)}{r}\right)^{n-1}\varphi'(r)} = \ln^{\frac{p-n+1}{n-1}}\left(\frac{e}{|x|}\right).$$

For this mapping,

$$\liminf_{\varepsilon\to 0} \fint_{B(0,\varepsilon)} (K_p(M(x)))^{\frac{n-1}{p-n+1}}\, dm(x) = \infty\,, \tag{13}$$

hence the condition (6) does not hold.

On the other hand, L'Hospital's rule yields $\frac{|f(x)|}{|x|} \to \infty$ as $x \to 0$.

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