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On differentiable and monogenic functions in a harmonic algebra

Dedicated to Prof. Yu. B. Zelinskii on the occasion of his 70^{th} *birthday*

For locally bounded and differentiable in the sense of Gâteaux functions Φ given in a three-dimensional commutative harmonic algebra with two-dimensional radical, we prove the following statement: if the function Φ domain is convex "in the radical direction" and the difference $\zeta_1-\zeta_2$ belongs to the radical, the difference $\Phi(\zeta_1)-\Phi(\zeta_2)$ belongs also to the radical. As a result, we prove that locally bounded and differentiable in the sense of Gâteaux functions are also differentiable in the sense of Lorch.

Для локально обмежених і диференційовних за Гато функцій Φ, визначених у тривимірній комутативній гармонічній алгебрі з двовимірним радикалом, ми доводимо наступне твердження: якщо область визначення функції Φ опукла "у напрямку радикала" і різниця $\zeta_1-\zeta_2$ належить радикалу, то різниця $\Phi(\zeta_1)-\Phi(\zeta_2)$ також належить радикалу. Як наслідок, доводиться, що локально обмежені і диференційовні за Гато функції є також диференційовними за Лорхом.

1. Introduction. In the algebra of complex numbers $\mathbb{C}$ a function $F\colon \mathbb{C} \longrightarrow \mathbb{C}$ is called monogenic at a point $\xi_0 \in \mathbb{C}$ if there exists the finite limit

$$\lim_{\xi\to\xi_0} \frac{F(\xi)-F(\xi_0)}{\xi-\xi_0} \tag{1}$$

and this limit, which is called the derivative of the function at the point ξ_0, is the same when ξ tends to ξ_0 by any way. A function, which is monogenic at all points of a domain $D \subset \mathbb{C}$, is called holomorphic in this domain. Class of holomorphic in D functions coincides with a class of analytic in D functions which are represented in a certain neighborhood of every point $\xi_0 \in D$ in the form of the sum of convergent power series (cf. [1]).

Every analytic function $F(\xi)$ of the complex variable $\xi = x+iy$ satisfies the two-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) F(\xi) \equiv F''(\xi)\ (1^2 + i^2) = 0$$

due to the equality $1^2 + i^2 = 0$ for the unit 1 and the imaginary unit i of the algebra of complex numbers.

An effectiveness of the analytic function methods in the complex plane for researching plane potential fields inspires mathematicians to develop analogous methods for spatial fields.

In the paper [2], analytic functions with values in a commutative algebra different from the algebra $\mathbb{C}$ are used for a construction of solutions of three-dimensional Laplace equation.

Let $\mathbb{A}$ be a commutative Banach algebra of a rank n, $3 \le n \le \infty$, over either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Let $\{e_1, e_2, e_3\}$ be a part of the basis of $\mathbb{A}$. P. W. Ketchum [2] has shown that if linearly independent elements $e_1, e_2, e_3 \in \mathbb{A}$ satisfy the condition

$$e_1^2 + e_2^2 + e_3^2 = 0\,, \tag{2}$$

then every analytic function $\Phi(\zeta)$ of the variable $\zeta = xe_1 + ye_2 + ze_3$ with real x, y, z satisfies the three-dimensional Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \Phi(\zeta) \equiv \Phi''(\zeta)\ (e_1^2 + e_2^2 + e_3^2) = 0\,, \tag{3}$$

where $\Phi''(\zeta)$ can be understood in a certain sense. An algebra $\mathbb{A}$ is called *harmonic* (cf. [2–4]) if in $\mathbb{A}$ there exists a triad of linearly independent vectors satisfying the equality (2).

It is clear that a characterization of functions satisfying the equalities (3) has relation to a question: in what sense the derivative is understood in the algebra $\mathbb{A}$.

It is well-known that there exist various definitions of differentiable functions given in algebras. Choosing concepts of a differentiable function and its derivative, it is natural to desire to combine the largest set of functions satisfying the equalities (3) with the preservation of the basic properties of analytic functions of a complex variable for functions of the mentioned set.

Some properties similar to properties of analytic functions of complex variable are established for functions differentiable in the sense of Lorch [5] in an arbitrary convex domain of commutative Banach algebra. In particular, the integral Cauchy theorem and the integral Cauchy formula, the Taylor expansion and the Morera theorem are proved in [5] in such a way as for analytic functions of complex variable. The convexity of domain in the mentioned results from [5] is withdrawn by E. K. Blum [6].

I. P. Mel'nichenko [7] suggested to consider doubly differentiable in the sense of Gâteaux functions in the equalities (3). Let us note that a priori the differentiability of the function Φ in the sense of Gâteaux is a restriction being weaker than the differentiability of this function in the sense of Lorch.

To prove analogues of principal theorems of the analytic function theory in the complex plane, in the papers [8–11] we considered monogenic functions (i.e., continuous differentiable in the sense of Gâteaux functions) in some harmonic algebras. We developed the following research scheme: at first, it is useful to obtain a constructive description of monogenic functions by means of analytic functions of complex variables; hereupon, to show that monogenic functions have the continuous Gâteaux derivatives of all orders and are differentiable in the sense of Lorch as well; and then to prove integral theorems and to obtain the Taylor and Laurent expansions. In the papers [12–14] such a scheme is extended to the case of monogenic functions in an arbitrary finite-dimensional commutative associative algebra.

The initial point of the mentioned research scheme is the following statement: for a monogenic function Φ, the difference $\Phi(\zeta_1)-\Phi(\zeta_2)$ belongs to a maximal ideal of a commutative finite-dimensional algebra $\mathbb{A}$ if the function Φ domain is convex "in the direction" of this ideal and the difference $\zeta_1 - \zeta_2$ belongs to the same ideal. For the first time, such a statement was proved in the papers [8, 9] for monogenic functions in a three-dimensional harmonic algebra $\mathbb{A}_3$ with two-dimensional radical.

In this paper, we prove similar statement for given in $\mathbb{A}_3$ functions Φ which are differentiable in the sense of Gâteaux and locally bounded, i.e. the assumption from [8, 9] on continuity of Φ is weakened. Obviously, the proved statement opens a way to similar generalizations of other results from [8–14].

2. A harmonic algebra $\mathbb{A}_3$. Let $\mathbb{A}_3$ be a three-dimensional commutative associative Banach algebra with the unit 1 over the field of complex numbers $\mathbb{C}$. Let $\{1, \rho_1, \rho_2\}$ be a basis of the algebra $\mathbb{A}_3$ with the multiplication table

$$\rho_1\rho_2 = \rho_2^2 = 0, \quad \rho_1^2 = \rho_2.$$

The algebra $\mathbb{A}_3$ is harmonic. A basis $\{e_1, e_2, e_3\}$ satisfying the equality (2) is called *harmonic*. All harmonic bases in $\mathbb{A}_3$ are described in Theorem 1.6 [4]. In particular, the basis $\{e_1, e_2, e_3\}$ is harmonic if decompositions of its elements with respect to the basis $\{1, \rho_1, \rho_2\}$ are of the form

$$\begin{aligned} &e_1 = 1, \\ &e_2 = n_1 + n_2\rho_1 + n_3\rho_2, \\ &e_3 = m_1 + m_2\rho_1 + m_3\rho_2, \end{aligned} \tag{4}$$

where n_k and m_k for $k = 1, 2, 3$ are complex numbers satisfying the system of equations

$$\begin{aligned} &1 + n_1^2 + m_1^2 &&= 0, \\ &n_1n_2 + m_1m_2 &&= 0, \\ &xn_2^2 + m_2^2 + 2(n_1n_3 + m_1m_3) &&= 0 \end{aligned} \tag{5}$$

and the inequality $n_2m_3 - n_3m_2 \neq 0$, and moreover, at least one of numbers in each of the pairs (n_1, n_2) and (m_1, m_2) is not equal to zero.

The algebra $\mathbb{A}_3$ have the unique maximal ideal $\mathcal{I} := \{\lambda_1\rho_1 + \lambda_2\rho_2 : \lambda_1, \lambda_2 \in \mathbb{C}\}$ which is also the radical of $\mathbb{A}_3$.

Consider the linear functional $f : \mathbb{A}_3 \to \mathbb{C}$ such that the maximal ideal $\mathcal{I}$ is its kernel and $f(1) = 1$. It is well known [15, p. 135] that f is also a multiplicative functional, i.e. the equality $f(ab) = f(a)f(b)$ is fulfilled for all $a, b \in \mathbb{A}_3$.

We use the euclidian norm $\|a\| := \sqrt{|a_1|^2 + |a_2|^2 + |a_2|^2}$ in the algebra $\mathbb{A}_3$, where $a = a_1e_1 + a_2e_2 + a_3e_3$ and $a_1, a_2, a_3 \in \mathbb{C}$.

3. Differentiability in the sense of Lorch and in the sense of Gâteaux. Monogenic functions. In what follows, $\{e_1, e_2, e_3\}$ is a harmonic basis of the form (4), $E_3 := \{\zeta := xe_1 + ye_2 + ze_3 : x, y, z \in \mathbb{R}\}$ is the linear span generated by the vectors e_1, e_2, e_3 and $\zeta = xe_1 + ye_2 + ze_3$, where $x, y, z \in \mathbb{R}$.

Let Ω be a domain in $\mathbb{R}^3$. Associate with Ω the congruent domain $\Omega_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in \Omega\}$ in E_3. Associate similarly with any set $Q \subset \mathbb{R}^3$ the set $Q_\zeta \subset E_3$.

Consider a function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ and properties of differentiability of such a function.

The concepts of Fréchet derivative and Gâteaux derivative are used for mappings of linear normalized spaces. These derivatives are defined as linear operators. In the considered case, they are linear operators from E_3 into $\mathbb{A}_3$.

For a mapping given in a domain of a commutative Banach algebra, E.R. Lorch [5] introduced a derivative, which is understood as a function given in the same domain.

A function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ is called *differentiable in the sense of Lorch* (cf. [5]) in a domain $\Omega_\zeta \subset E_3$ if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'_L(\zeta) \in \mathbb{A}_3$ such that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $h \in E_3$ with $\|h\| < \delta$ the following inequality fulfilled:

$$\|\Phi(\zeta + h) - \Phi(\zeta) - h\Phi'_L(\zeta)\| \le \|h\|\,\varepsilon\,. \tag{6}$$

Obviously, in the inequality (6) the *Lorch derivative* $\Phi'_L(\zeta)$ is a function of the variable ζ, i.e., $\Phi'_L\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$.

At the same time, the mapping $B_\zeta\colon E_3 \longrightarrow \mathbb{A}_3$, which is defined by the equality $B_\zeta h := h\Phi'_L(\zeta)$, is a bounded linear operator. Therefore, a function Φ, which is differentiable in the sense of Lorch in a domain Ω_ζ, have the Fréchet derivative B_ζ in every point $\zeta \in \Omega_\zeta$ (cf. [15, p. 115]). The converse is not true, see an example in [15, p. 116].

Using the Gâteaux differential, I. P. Mel'nichenko [7] suggested to consider the Gâteaux derivative as a function $\Phi'_G\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ too.

We say that a function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ is called *differentiable in the sense of Gâteaux* in a domain $\Omega_\zeta \subset E_3$ if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'_G(\zeta) \in \mathbb{A}_3$ such that

$$\lim_{\delta \to 0+0} \left(\Phi(\zeta + \delta h) - \Phi(\zeta)\right)\delta^{-1} = h\Phi'_G(\zeta) \quad \forall h \in E_3. \tag{7}$$

Obviously, the *Gâteaux derivative* $\Phi'_G(\zeta)$ is a function of the variable ζ and is a generalization of the classical directional derivative.

The left-hand side of the equality (7) is called the *Gâteaux differential* of function Φ. It is well-known, in a general case, the Gâteaux differential may fail to be linear with respect to h. But, it is clear, if the Gâteaux derivative $\Phi'_G(\zeta)$ exists, the Gâteaux differential (7) is a bounded linear operator with respect to h. At the same time, the converse is not true as the same example in [15, p. 116] shows.

It is evident, the definition (6) of the Lorch derivative and the definition (7) of the Gâteaux derivative take into account the existence of noninvertible elements h in the algebra $\mathbb{A}_3$ because the division by elements of algebra is not used in them in contrast to the classical definition (1) of complex derivative.

Obviously, if a function Φ is differentiable in the sense of Lorch in Ω_ζ, then it is also differentiable in the sense of Gâteaux, and $\Phi'_L(\zeta) = \Phi'_G(\zeta)$ for all $\zeta \in \Omega_\zeta$. The converse is clearly not true similarly to the fact that the existence of all directional derivatives at a point does not guarantee a strong differentiability (or even continuity) of function at that point.

Let us consider a concept of monogenic function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$.

We say that a function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ is *monogenic* in a domain $\Omega_\zeta \subset E_3$ if Φ is continuous and differentiable in the sense of Gâteaux at every point of Ω_ζ.

We use the notion of monogenic function in the sense of existence of derived numbers for this function (cf. [1,16]). In the scientific literature the denomination of monogenic function is used else for functions satisfying certain conditions similar to the classical Cauchy − Riemann conditions (cf. [17, 18]). Such functions are also called regular functions (cf. [19]) or hyperholomorphic functions (cf. [20, 21]).

In the paper [8] we obtained a constructive description of monogenic functions by means of analytic functions of complex variables (see also [9]). As a consequence of such a description, every monogenic in Ω_ζ function is differentiable in the sense of Lorch in Ω_ζ.

To explain it, without loss of generality, we assume that a function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ is monogenic in a convex domain Ω_ζ (if Ω_ζ is not convex, it is possible to consider a restriction of the function Φ to any ball lying in Ω_ζ). Denote $D := \{\xi = f(\zeta) : \zeta \in \Omega_\zeta\}$. Then for every monogenic function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ there exist complex-valued analytic functions F, F_1, F_2 in the domain D such that (see [8, 9])

$$\Phi(\zeta) = F(\xi)+\Big(F_1(\xi)+(n_2y+m_2z)F'(\xi)\Big)\rho_1+\Big(F_2(\xi)+(n_2y+m_2z)F_1'(\xi)+ + (n_3y + m_3z)F'(\xi) + \frac{(n_2y + m_2z)^2}{2}F''(\xi)\Big)\rho_2 \quad \forall\, \zeta \in \Omega_\zeta\,, \tag{8}$$

where $\xi = x+n_1y+m_1z$. The equality (8) can be rewritten in the following form (see [9])

$$\Phi(\zeta) = \frac{1}{2\pi i} \int\limits_{\Gamma_\zeta} \Big(F(t) + \rho_1 F_1(t) + \rho_2 F_2(t)\Big)(t-\zeta)^{-1} dt \qquad \forall \zeta \in \Omega_\zeta\,, \tag{9}$$

where Γ_ζ is an arbitrary closed Jordan rectifiable curve in D, which is homotopic to the point $f(\zeta)$ and embraces this point.

It follows from the equality (9) that the function Φ is differentiable in the sense of Lorch in Ω_ζ. Using the equality (9), we obtain the following expression for the Lorch n-th derivative, which coincides with the Gateaux n-th derivative:

$$\Phi^{(n)}(\zeta) = \frac{n!}{2\pi i} \int\limits_{\Gamma_\zeta} \Big(F(t) + \rho_1 F_1(t) + \rho_2 F_2(t)\Big)\Big((t-\zeta)^{-1}\Big)^{n+1} dt \qquad \forall \zeta \in \Omega_\zeta\,.$$

Thus, every monogenic function Φ satisfies the equalities (3).

Bellow, we show that similar statements are true for functions Φ which are differentiable in the sense of Gâteaux and locally bounded in Ω_ζ, i.e., the assumption from [8, 9] on continuity of Φ will be weakened.

4. Some special properties of locally bounded and differentiable in the sense of Gâteaux functions. Consider a function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ which is differentiable in the sense of Gâteaux in a domain Ω_ζ. It is follows from Theorem 1.3 in [4] that the function Φ satisfies the following conditions in Ω_ζ:

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x}\, e_2, \qquad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x}\, e_3. \tag{10}$$

All noninvertible elements in $\mathbb{A}_3$ belong to the radical $\mathcal{I}$, which is the kernel of functional f. Therefore, an element $\zeta = x + ye_2 + ze_3 \in E_3$ is noninvertible in $\mathbb{A}_3$ if and only if the point (x, y, z) belongs to the following straight line in $\mathbb{R}^3$ (see [8, 9]):

$$L: \quad \begin{cases} x + y\,\mathrm{Re}\, n_1 + z\,\mathrm{Re}\, m_1 = & 0, \\ \phantom{x +{}} y\,\mathrm{Im}\, n_1 + z\,\mathrm{Im}\, m_1 = & 0\,. \end{cases}$$

We say that a domain $\Omega \subset \mathbb{R}^3$ is *convex in the direction of the straight line* L if Ω contains every segment parallel to L and connecting two points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \Omega$. It may be said that the congruent domain Ω_ζ is convex "in the radical direction".

Let us prove the following statement for a function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ which is differentiable in the sense of Gâteaux and locally bounded in a domain Ω_ζ which is convex "in the radical direction".

Lemma 1. *Let a domain $\Omega \subset \mathbb{R}^3$ be convex in the direction of the straight line L and $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ be a locally bounded and differentiable in the sense of Gâteaux function in the domain Ω_ζ. If $\zeta_1, \zeta_2 \in \Omega_\zeta$ and $\zeta_2 - \zeta_1 \in L_\zeta$, then*

$$\Phi(\zeta_1) - \Phi(\zeta_2) \in \mathcal{I}. \tag{11}$$

Proof. Inasmuch as f is a linear continuous multiplicative functional, from the equalities (10) it follows that

$$f\left(\frac{\partial \Phi}{\partial y}\right) = f\left(\frac{\partial \Phi}{\partial x}\right) f\left(e_2\right), \qquad f\left(\frac{\partial \Phi}{\partial z}\right) = f\left(\frac{\partial \Phi}{\partial x}\right) f\left(e_3\right). \tag{12}$$

Consider the decomposition

$$\Phi(\zeta) = V_0(x,y,z) + V_1(x,y,z)\,\rho_1 + V_2(x,y,z)\,\rho_2\,, \tag{13}$$

of function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ with respect to the basis $\{1, \rho_1, \rho_2\}$.

Substituting the expressions (4), (13) into the equalities (12), we get the following equalities:

$$\frac{\partial V_0}{\partial y} = n_1 \frac{\partial V_0}{\partial x}\,, \qquad \frac{\partial V_0}{\partial z} = m_1 \frac{\partial V_0}{\partial x}\,. \tag{14}$$

Inasmuch as

$$\xi = f(\zeta) = (x + y\mathrm{Re}\,n_1 + z\mathrm{Re}\,m_1) + i(y\mathrm{Im}\,n_1 + z\mathrm{Im}\,m_1) =: \tau + i\eta\,, \tag{15}$$

from the equalities (14) we get

$$\frac{\partial V_0}{\partial \eta}\,\mathrm{Im}\,n_1 = i\,\frac{\partial V_0}{\partial \tau}\,\mathrm{Im}\,n_1\,, \qquad \frac{\partial V_0}{\partial \eta}\,\mathrm{Im}\,m_1 = i\,\frac{\partial V_0}{\partial \tau}\,\mathrm{Im}\,m_1\,. \tag{16}$$

It follows from the first equation of the system (5) that, at least one of the numbers $\mathrm{Im}\,n_1$, $\mathrm{Im}\,m_1$ is not equal to zero. Therefore, from (16) we get the equality

$$\frac{\partial V_0(x,y,z)}{\partial \eta} = i\,\frac{\partial V_0(x,y,z)}{\partial \tau} \tag{17}$$

which is fulfilled for all $(x, y, z) \in \Omega$.

Let us prove that $V_0(x_1, y_1, z_1) = V_0(x_2, y_2, z_2)$ for the points (x_1, y_1, z_1), $(x_2, y_2, z_2) \in \Omega$ such that the segment connecting these points is parallel to the straight line L.

Let us construct in Ω two surfaces Q and Σ satisfying the following conditions:

- Q and Σ have the same edge;
- the surface Q contains the point (x_1, y_1, z_1) and the surface Σ contains the point (x_2, y_2, z_2);
- restrictions of the functional f onto the sets Q_ζ and Σ_ζ are one-to-one mappings of these sets onto the same domain G of the complex plane.

As the surface Q, we can take an equilateral triangle having the center (x_1, y_1, z_1) and apexes A_1, A_2, A_3, and, in addition, the plane of this triangle is perpendicular to the straight line L.

To construct the surface Σ, first, consider a triangle with the center (x_2, y_2, z_2) and apexes A'_1, A'_2, A'_3 such that the segments $A'_1A'_2$, $A'_2A'_3$, $A'_1A'_3$ are parallel to the segments A_1A_2, A_2A_3, A_1A_3, respectively, and, in addition, the length of $A'_1A'_2$ is less than the length of A_1A_2. Inasmuch as the domain Ω is convex in the direction of the straight line L, the prism with vertexes $A'_1, A'_2, A'_3, A''_1, A''_2, A''_3$ is completely contained in Ω, where the points A''_1, A''_2, A''_3 are located in the plane of triangle $A_1A_2A_3$ and the edges $A'_mA''_m$ are parallel to L for $m = \overline{1,3}$.

Further, set a triangle with apexes B_1, B_2, B_3 such that the point B_m is located on the segment $A'_mA''_m$ for $m = \overline{1,3}$ and the truncated pyramid with vertexes $A_1, A_2, A_3, B_1, B_2, B_3$ and lateral edges A_mB_m, $m = \overline{1,3}$, is completely contained in the domain Ω.

At last, in the plane of triangle $A'_1A'_2A'_3$ set a triangle T with apexes C_1, C_2, C_3 such that the segments C_1C_2, C_2C_3, C_1C_3 are parallel to the segments $A'_1A'_2$, $A'_2A'_3$, $A'_1A'_3$, respectively, and, in addition, the length of C_1C_2 is less than the length of $A'_1A'_2$. It is evident that the truncated pyramid with vertexes $B_1, B_2, B_3, C_1, C_2, C_3$ and lateral edges B_mC_m, $m = \overline{1,3}$, is completely contained in the domain Ω.

Now, as the surface Σ, denote the surface formed by the triangle T and the lateral surfaces of mentioned truncated pyramids $A_1A_2A_3B_1B_2B_3$ and $B_1B_2B_3C_1C_2C_3$.

For each $\xi \in G$ define two complex valued functions H_1 and H_2 so that

$$H_1(\xi) := f(\Phi(\zeta)) \equiv V_0(x, y, z) \ \text{ for } (x, y, z) \in Q,$$

$$H_2(\xi) := f(\Phi(\zeta)) \equiv V_0(x, y, z) \ \text{ for } (x, y, z) \in \Sigma,$$

where the correspondence between the points (x, y, z) and $\xi \in G$ is determined by the relation (15). The functions H_1, H_2 are analytic in the domain G due to the equality (17) and Theorem 6 [22].

Inasmuch as H_1, H_2 are continuous in the closure of domain G and $H_1(\xi) \equiv H_2(\xi)$ on the boundary of G, this identity is fulfilled everywhere in G. Therefore, $V_0(x_1, y_1, z_1) = V_0(x_2, y_2, z_2)$ and the equalities

$$f(\Phi(\zeta_2) - \Phi(\zeta_1)) = f(\Phi(\zeta_2)) - f(\Phi(\zeta_1)) = 0,$$

are fulfilled for $\zeta_1 := x_1e_1 + y_1e_2 + z_1e_3$ and $\zeta_2 := x_2e_1 + y_2e_2 + z_2e_3$. Thus, $\Phi(\zeta_2) - \Phi(\zeta_1)$ belongs to the kernel $\mathcal{I}$ of functional f. The lemma is proved.

Now, using Lemma, in such a way as in the papers [8,9], we obtain the expression (8) or, that is the same, the expression (9) for a locally bounded and differentiable in the sense of Gâteaux function $\Phi : \Omega_\zeta \to \mathbb{A}_3$ in the case where a domain Ω is convex in the direction of the straight line L.

As a result, we obtain the following statement.

Theorem 1. *For a function $\Phi\colon \Omega_\zeta \longrightarrow \mathbb{A}_3$ given in an arbitrary domain $\Omega_\zeta \subset E_3$ the following properties are equivalent:*

(I) Φ *is a locally bounded and differentiable in the sense of Gâteaux function in Ω_ζ;*

(II) Φ *is a monogenic function in Ω_ζ;*

(III) Φ *is a differentiable in the sense of Lorch function in Ω_ζ.*

Certainly, the property of function to be locally bounded and differentiable in the sense of Gâteaux in Ω_ζ is also equivalent to the various definitions of monogenic function, that are stated in Theorem 1.15 [9].

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