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Some inequalities for inner radii of partially overlapping domains

Dedicated to Prof. Yu. B. Zelinskii on the occasion of his 70^{th} *birthday*

In this paper we consider a problem on an extremal decomposition of the complex plane in the geometric function theory.

У даній роботі розглядається задача екстремального розбиття комплексної площини у геометричній теорії функцій.

1. Denotations and definitions. Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be its one-point compactification, and $\mathbb{R}^+ = (0,\infty)$. Let $r(D,a)$ be an inner radius of the domain $D \subset \overline{\mathbb{C}}$ with respect to the point $a \in D$ (cf., e.g., [1–4]). An inner radius is a generalization of a conformal radius for multiply connected domains. An inner radius of the domain D is associated with the generalized Green's function $g_D(z,a)$ of the domain D by the relations

$$g_D(z,a) = \ln \frac{1}{|z-a|} + \ln r(D,a) + o(1), \quad z \to a,$$

$$g_D(z,\infty) = \ln |z| + \ln r(D,\infty) + o(1), \quad z \to \infty.$$

For a system of points $A_n := \{a_k : a_0 = 0, |a_k| = 1, k = \overline{0,n}\}$ and for an open set D, $A_n \subset D$, we denote by $D(a_k)$ a connected component of D containing a_k, $k = \overline{0,n}$.

Denote by $P_k := \{w : \arg a_k < \arg w < \arg a_{k+1}\}$, $a_{n+1} := a_1$, $\alpha_k := \frac{1}{\pi}\arg\frac{a_{k+1}}{a_k}$, $\alpha_{n+1} := \alpha_1$, $k = \overline{1,n}$, $\sum_{k=1}^{n}\alpha_k = 2$.

We denote by

$$D_k(0) := D(0)\cap\overline{P}_k,\ D_k(a_k) := D(a_k)\cap\overline{P}_k,\ D_k(a_{k+1}) := D(a_{k+1})\cap\overline{P}_k,$$

for each $k = \overline{1,n}$, $a_{n+1} := a_1$.

The open set D, $A_n \subset D$, satisfies the non-overlapping condition with respect to the system of points A_n, if the equality $[D_k(0)\cap D_k(a_k)] \cup [D_k(0)\cap D_k(a_{k+1})] \cup [D_k(a_k)\cap D_k(a_{k+1})] = \emptyset$, $1 \le k \le n$, holds for all different points a_k which belong to $\overline{P}_k$.

The system of domains $\{D_k\}_{k=0}^{n}$ satisfies a *partially overlapping condition* with respect to the system of points A_n, if the open set $D = \cup_{k=0}^{n} D_k$ satisfies the non-overlapping condition with respect to the system A_n.

2. Formulation of the problem. The main goal of the work is to obtain a sharp upper bound for the functional

$$J_n(\gamma) = r^{\gamma}\left(D_0,0\right)\prod_{k=1}^{n} r\left(D_k,a_k\right)$$

where $\gamma \in \mathbb{R}^{+}$, $|a_k| = 1$, $a_0 = 0$, $\{D_k\}_{k=0}^{n}$ are the system of partially overlapping domains such that $a_k \in D_k \subset \overline{\mathbb{C}}$ for $k = \overline{0,n}$. The problem was formulated in the work [1].

3. Results and proofs. The following theorem strengthens the main result of the work [5].

Theorem 1. *Let $n \in \mathbb{N}$, $n \geqslant 4$, $\gamma \in (0,\gamma_n]$, $\gamma_4 = 4{,}17$, $\gamma_5 = 5{,}71$, $\gamma_6 = 7{,}5$, $\gamma_7 = 9{,}53$, $\gamma_8 = 11{,}81$, and $\gamma_n = 0{,}1215\,n^2$ for $n \geqslant 9$. Then for any different points of the unit circle $|a_k| = 1$ such that $0 < \alpha_k < 2/\sqrt{\gamma}$, $k = \overline{1,n}$, and for any system domains D_k, $a_k \in D_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, which satisfy the partially overlapping condition with respect to points of the unit circle, the following inequality holds*

$$J_n(\gamma) \le \left(\frac{4}{n}\right)^{n} \frac{\left(\frac{4\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1-\frac{\gamma}{n^2}\right)^{n+\frac{\gamma}{n}}} \left(\frac{1-\frac{\sqrt{\gamma}}{n}}{1+\frac{\sqrt{\gamma}}{n}}\right)^{2\sqrt{\gamma}}. \qquad (1)$$

The equality is attained if a_k and D_k, $k = \overline{0,n}$, are, respectively, poles *and circular domains of the quadratic differential*

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n+\gamma}{w^2(w^n-1)^2}\,dw^2.$$

Proof. Let domains D_k, $k=\overline{0,n}$, satisfy conditions of Theorem, and we have the open set $D=\cup_{k=0}^{n}D_k$. Then the inequality

$$r(D_k,a_k)\le r(D,a_k),$$

holds, and, obviously, we obtain

$$r^\gamma(D_0,0)\prod_{k=1}^{n} r(D_k,a_k)\le r^\gamma(D,0)\prod_{k=1}^{n} r(D,a_k).$$

Further, consider the system of functions:

$$\zeta=\pi_k(w)=-i\left(e^{-i\theta_k}w\right)^{\frac{1}{\alpha_k}},\quad k=\overline{1,n}.$$

The family of the functions $\{\pi_k(w)\}_{k=1}^{n}$ is called *admissible for the separating transformation of the open set D, with respect to the angles* $\{P_k\}_{k=1}^{n}$. Let $M_k^{(1)}$, $k=\overline{1,n}$, denote the domain of the plane ζ, obtained as a result of the union of the connected component of the set $\pi_k(D\bigcap\overline{P}_k)$ containing the point $\pi_k(a_k)$ with the own symmetric reflection with respect to the imaginary axis. In turn, by $M_k^{(2)}$, $k=\overline{1,n}$, one denotes the domain of the plain $\mathbb{C}_\zeta$, which are obtained as a result of the union of the connected component of the set $\pi_k(D\bigcap\overline{P}_k)$ containing the point $\pi_k(a_{k+1})$ with the own symmetric reflection with respect to the imaginary axis, $\pi_n(a_{n+1}) := \pi_n(a_1)$. Moreover, we denote $M_k^{(0)}$ as the domain of the plane $\mathbb{C}_\zeta$, obtained as a result of the union of the connected component of the set $\pi_k(D\bigcap\overline{P}_k)$ containing the point $\zeta=0$ with the own symmetric reflection with respect to the imaginary axis. Denote by $\pi_k(a_k) := m_k^{(1)}$, $\pi_k(a_{k+1}) := m_k^{(2)}$, $k=\overline{1,n}$, $\pi_n(a_{n+1}) := m_n^{(2)}$. From the definition of the function π_k, it follows that

$$|\pi_k(w)-m_k^{(1)}|\sim\frac{1}{\alpha_k}\cdot|w-a_k|,\quad w\to a_k,\quad w\in\overline{P_k},$$

$$|\pi_k(w)-m_k^{(2)}|\sim\frac{1}{\alpha_k}\cdot|w-a_{k+1}|,\quad w\to a_{k+1},\quad w\in\overline{P_k},$$

$$|\pi_k(w)| \sim |w|^{\frac{1}{\alpha_k}}, \quad w \to 0, \quad w \in \overline{P_k}.$$

Further, using the result of the papers [1,2], we obtain the inequality

$$r\left(D, a_k\right) \leq \left[\alpha_k \alpha_{k-1} r\left(M_k^{(1)}, m_k^{(1)}\right) r\left(M_k^{(2)}, m_k^{(2)}\right)\right]^{\frac{1}{2}}, \quad k = \overline{1,n}, \tag{2}$$

$$r\left(D, 0\right) \leq \left[\prod_{k=1}^{n} r^{\alpha_k^2}\left(M_k^{(0)}, 0\right)\right]^{\frac{1}{2}}. \tag{3}$$

From inequalities ((2)), ((3)), we obtain the inequality

$$J_n(\gamma) \leq \prod_{k=1}^{n} \alpha_k \left[\prod_{k=1}^{n} r^{\gamma\alpha_k^2}\left(M_k^{(0)}, 0\right) r\left(M_k^{(1)}, m_k^{(1)}\right) r\left(M_k^{(2)}, m_k^{(2)}\right)\right]^{\frac{1}{2}}.$$

Using the technique developed in [4, p. 269–274], we obtain the estimate

$$J_n(\gamma) \leqslant \left(\prod_{k=1}^{n} \alpha_k\right) \left[\prod_{k=1}^{n} r^{\alpha_k^2\gamma}\left(G_k^{(0)}, 0\right) r\left(G_k^{(1)}, -i\right) r\left(G_k^{(2)}, i\right)\right]^{\frac{1}{2}}, \tag{4}$$

where $G_k^{(0)}$, $G_k^{(1)}$, $G_k^{(2)}$ are circular domains of the quadratic differential

$$Q(w)dw^2 = \frac{(4 - \alpha_k^2\gamma)w^2 - \alpha_k^2\gamma}{w^2(w^2+1)^2}\, dw^2,$$

such that $0 \in G_k^{(0)}$, $-i \in G_k^{(1)}$, $i \in G_k^{(2)}$. Let

$$S(x) = 2^{x^2+6} \cdot x^{x^2} \cdot (2-x)^{-\frac{1}{2}(2-x)^2} \cdot (2+x)^{-\frac{1}{2}(2+x)^2}, \quad x \in [0,2].$$

Then, from inequality (4) according to [1, 2], we obtain the estimate

$$J_n(\gamma) \leqslant \gamma^{-n/2} \left(\prod_{k=1}^{n} \alpha_k\right) \left[\prod_{k=1}^{n} S(x)\right]^{\frac{1}{2}} \leqslant \gamma^{-n/2} \left[\prod_{k=1}^{n} L(x)\right]^{\frac{1}{2}},$$

where

$$L(x) = 2^{x^2+6} \cdot x^{x^2+2} \cdot (2-x)^{-\frac{1}{2}(2-x)^2} \cdot (2+x)^{-\frac{1}{2}(2+x)^2}, \quad x \in [0,2].$$

Consider the extremal problem

$$\prod_{k=1}^{n} L(x_k) \longrightarrow \max; \quad \sum_{k=1}^{n} x_k = 2\sqrt{\gamma},$$

$$x_k = \alpha_k \sqrt{\gamma}, \quad 0 < x_k \leqslant 2.$$

Let $F(x) = \ln(L(x))$ and $X^{(0)} = \left\{ x_k^{(0)} \right\}_{k=1}^{n}$ is any set of extremal points of the problem which is considered above.

Repeating the arguments of [5] we obtain the statement:
if $0 < x_k^{(0)} < x_j^{(0)} < 2$, $k \neq j$, *then the following equalities hold:*

$$F'(x_k^{(0)}) = F'(x_j^{(0)}),\ k,j = \overline{1,n},\ k \neq j,$$

$$F'(x) = 2x \ln 2x + (2-x)\ln(2-x) -$$

$$-(2+x)\ln(2+x) + \frac{2}{x} \text{ (see Fig. 1).}$$

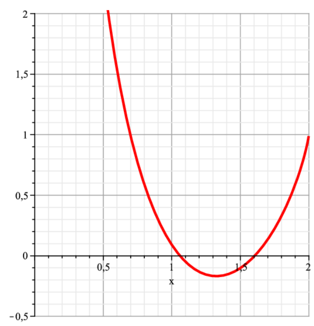

Fig. 1: A graph of the function $F'(x)$

Let us verify that for the above-accepted relations the following condition is valid: $x_1^{(0)} = x_2^{(0)} = \ldots = x_n^{(0)}$. Let $F'(x) = h$, $y_0 \leqslant h \leqslant 1$, $y_0 \approx -0{,}17$. Consider values h:

$$h_1 = 1, h_2 = 0{,}95, h_3 = 0{,}9, h_4 = 0{,}85, \cdots, h_{23} = -0{,}15, h_{24} = -0{,}17.$$

We need to find a solution of the equation:

$$F'(x) = h_k, k = \overline{1{,}24}. \tag{5}$$

For every $h_k \in [y_0, 1]$ the equation has two solutions:

$$x_1(h_k) \in (0, x_0],\ x_2(h_k) \in (x_0, 2],\ x_0 \approx 1{,}324683.$$

Results of direct calculations are given in the following table.

k	h_k	$x_1(h_k)$	$x_2(h_k)$	$3x_1(h_k) + x_2(h_{k+1})$	$4x_1(h_k) + x_2(h_{k+1})$
1	1,00	0,697331	2,000000		
2	0,95	0,708144	1,992640	4,084633	4,781964
3	0,90	0,719344	1,983233	4,107666	4,815810
4	0,85	0,730957	1,972549	4,130581	4,849925
5	0,80	0,743014	1,960786	4,153657	4,884614
6	0,75	0,755550	1,948028	4,177071	4,920085
7	0,70	0,768602	1,934315	4,200964	4,956513
8	0,65	0,782217	1,919654	4,225462	4,994064
9	0,60	0,796446	1,904035	4,250687	5,032904
10	0,55	0,811347	1,887429	4,276766	5,073211
11	0,50	0,826991	1,869791	4,303831	5,115178
12	0,45	0,843462	1,851059	4,332032	5,159023
13	0,40	0,860858	1,831149	4,361534	5,204996
14	0,35	0,879304	1,809955	4,392531	5,253389
15	0,30	0,898950	1,787338	4,425249	5,304553
16	0,25	0,919989	1,763115	4,459964	5,358914
17	0,20	0,942675	1,737044	4,497012	5,417001
18	0,15	0,967348	1,708794	4,536819	5,479494
19	0,10	0,994487	1,677892	4,579935	5,547283
20	0,00	1,059462	1,604865	4,588325	5,582811
21	-0,05	1,100561	1,559491	4,737878	5,797340
22	-0,10	1,152868	1,502748	4,804430	5,904991
23	-0,15	1,234855	1,416172	4,874775	6,027642
24	-0,17	1,324683	1,324683	5,029248	6,264103

Taking into consideration properties of the function $F'(x)$ and the condition of Theorem, we obtain the following inequality from the table, respectively, for $n = \overline{4{,}8}$, $h_k \leqslant h \leqslant h_{k+1}$, $k = \overline{1{,}23}$:

$$\sum_{k=1}^{n} x_k(h) > (n-1)x_1(h_k) + x_2(h_{k+1}) \geqslant 2\sqrt{\gamma_n},$$

Thus, the case $\left\{x_k^{(0)}\right\}_{k=1}^{n} \in (0, x_0]$, $x_0 \approx 1{,}324683$, $n = \overline{4{,}8}$, is possible only for the extremal set $X^{(0)}$, and, therefore, $x_1^{(0)} = x_2^{(0)} = \cdots = x_n^{(0)}$.

The inequality

$$(x_1(h_k) - 0{,}6973)\, n + (x_2(h_{k+1}) - x_1(h_k)) > 0, n \geqslant 9,$$

the proof of which is based on the technique developed in [5], is true for roots of the equation (5).

Then

$$nx_1(h_k) + (x_2(h_{k+1}) - x_1(h_k)) > 0{,}6973\,n.$$

Solving the inequality

$$0{,}6973n > 2\sqrt{\gamma_n},$$

conclude that $\gamma_n = 0{,}1215\,n^2$, for $n \geqslant 9$.

Therefore, in the case $n \geqslant 9$, the set of the points $\left\{x_k^{(0)}\right\}_{k=1}^{n}$ can not be the extremal, provided $x_n^{(0)} \in (x_0; 2]$. Then, the case may be only for the extremal set $\left\{x_k^{(0)}\right\}_{k=1}^{n}$, when $x_k^{(0)} \in (0, x_0]$, $k = \overline{1,n}$, and $x_1^{(0)} = x_2^{(0)} = = \ldots = x_n^{(0)}$. For all $\gamma < \gamma_n$, $n \geqslant 9$, all the previous reasoning holds. The Theorem is proved

Corollary 1. *Let $n \in \mathbb{N}$, $n \geqslant 4$, $\gamma \in (0, \gamma_n]$, $\gamma_4 = 4{,}17$, $\gamma_5 = 5{,}71$, $\gamma_6 = 7{,}5$, $\gamma_7 = 9{,}53$, $\gamma_8 = 11{,}81$, and $\gamma_n = 0{,}1215\,n^2$ for $n \geqslant 9$. Then for any different points of the unit circle $|a_k| = 1$ such that $0 < \alpha_k < 2/\sqrt{\gamma}$, $k = \overline{1,n}$, and for any system domains D_k, $a_k \in D_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, which satisfy the partially overlapping condition with respect to points of a unit circle, the following inequality holds*

$$r^{\gamma}\left(D_0,0\right) \prod_{k=1}^{n} r\left(D_k,a_k\right) \leq r^{\gamma}\left(\Lambda_0,0\right) \prod_{k=1}^{n} r\left(\Lambda_k,\lambda_k\right),$$

where λ_k and Λ_k, $k = \overline{0,n}$, are, respectively, poles and circular domains of the quadratic differential

$$Q(w)dw^2 = -\frac{(n^2-\gamma)w^n + \gamma}{w^2(w^n-1)^2}\,dw^2.$$

Corollary 2 ([5,6])**.** *Let $n \in \mathbb{N}$, $n \geqslant l \in \{5{,}4\}$ ($l = 5$ in [5], $l = 4$ in [6]), $\gamma \in (0, \gamma_n]$, $\gamma_n = n$. Then for different points of the unit circle $|a_k| = 1$ such that $0 < \alpha_k < 2/\sqrt{\gamma}$, $k = \overline{0,n}$, and for any non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, $a_0 = 0 \in B_0$, the inequality* (1) *holds. The equality is attained under the same condition as in Theorem 1.*

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