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Lie–Bäcklund symmetry reduction of nonlinear and non-evolution equations

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Досліджено застосування операторів симетрії Лі–Беклунда, які допускаються звичайним диференціальним рівнянням, для редукції диференціальних рівнянь з частинними похідними. Анзаци для залежної змінної побудовано інтегруванням звичайних диференціальних рівнянь. Показано, що метод можна застосовувати для рівнянь еволюційного і нееволюційного типу. У рамках цього підходу знайдено ров'язок, що залежить від довільної функції одного аргументу.

The application of Lie–Bäcklund symmetry operators admitted by ordinary differential equations for reducing partial differential equations are studied. The ansatze for dependent variable are constructed by integrating ordinary differential equations. We show that the method is applicable for nonlinear evolution and non-evolution types equations. In the framework of the approach we construct the solution depending on arbitrary smooth function on one variable.

1. Introduction. It is a known fact that the symmetry groups of nonlinear PDEs are being used for finding special solutions invariant with respect to a certain subgroup of the complete symmetry group of the equation. Invariant solutions are constructed by solving a reduced equation with smaller number of independent variables than the initial equation, an ODE in particular. Conditional symmetry is a generalization of a classical Lie symmetry of differential equations and substantially extends the possibilities of construction of solutions of nonlinear PDEs. It must be noted, that the conditional symmetry method can be effectively used both for integrable (in some sense) and non-integrable equations. In [1, 5] concept of conditional Lie–Bäcklund symmetry of evolution equation, which is a generalization of point conditional symmetry, is proposed. In the framework of this approach we obtain reduced system of ODEs. The relationship of generalized conditional symmetry of evolution equations to compatibility of systems of differential equations is studied in [2]. In [3] Svirshchevskii used Lie–Bäcklund symmetry of linear homogeneous ODEs for reducing evolution equations to a system of ODEs. To apply this method we have to solve the inverse symmetry problem, namely to find linear homogeneous ODEs which admit given Lie–Bäcklund symmetry operator. We study the reduction of nonlinear generalization of the heat equation and modified Korteweg–de Vries equation by using Lie–Bäcklund symmetry property of linear and nonlinear ODEs [4]. It allows us to construct solutions for equations of evolution and non-evolution types.

2. Application of the symmetry reduction method. In this section we discuss the relationship between the Lie–Bäcklund symmetry of ordinary differential and reduction of generalized version of Korteweg–de Vries equation and nonlinear heat equation.

Example 1. We show how to apply the Lie–Bäcklund symmetry reduction using mKdV equation as an example. First step is finding an ODE or ODEs invariant under the operator $K[u] = K\big(x, u, \frac{\partial u}{\partial x}, \dots, \frac{\partial^p u}{\partial x^p}\big)$, in which case $K[u]$ is the right-hand side of the mKdV equation. Let p be a positive integer. Consider the ODE to be of the form $u_{xx}+g(u,u_x)=0$, where g is a differentiable function of u and u_x. Invariance condition for such ODE reads as

$$X^{\infty}\big(u_{xx}+g(u,u_x)\big)\big|_{u_{xx}+g(u,u_x)=0}=0, \tag{1}$$

where X^{∞} is a prolongation of the vector field $X=(u_{xxx}+u^p u_x)\frac{\partial}{\partial u}$ on the jet space. After necessary substitutions equation (1) becomes

$$\begin{aligned} &pu^{p-1}u_x(g_{u_x}u_x-3g)-u_x^3 g_{uuu}\\ &\quad+3u_x(u_x g_u g_{uu_x}+u_x g g_{uuu_x}+g g_{uu})\\ &\quad-3g(u_x g_{u_x} g_{uu_x}+u_x g_u g_{u_x u_x}+u_x g g_{uu_x u_x}+g g_{uu_x})\\ &\quad+g^2(3g_{u_x}g_{u_x u_x}+g g_{u_x u_x u_x})+p(p-1)u^{p-2}u_x^3=0. \end{aligned} \tag{2}$$

The subscripts u and u_x denote differentiation with respect to u and u_x. We assume that $g(u,u_x)=\sum\limits_{i=j}^{k}\lambda_i(u)u_x^i$ for some integers j and k. In that case the left-hand side of the equation (2) becomes a power series of u_x. For every $k \geq 3$ and $j \leq -1$ the coefficients for the highest and lowest powers of u_x are $2k(k-1)(2k-1)\lambda_k^4$ and $2j(j-1)(2j-1)\lambda_j^4$ respectively, which implies $\lambda_i = 0$ for $i \notin \{0,1,2\}$, meaning $g = \lambda_2(u)u_x^2 + \lambda_1(u)u_x + \lambda_0(u)$. The six remaining coefficients in the now power series (2) become the determining equations. They are

$$12\lambda_2^4 - 30\lambda_2'\lambda_2^2 + 6\lambda_2'^2 + 9\lambda_2''\lambda_2 - \lambda_2''' = 0, \tag{3}$$

$$30\lambda_1\lambda_2^3 - 48\lambda_2'\lambda_1\lambda_2 - 15\lambda_1'\lambda_2^2 + 9\lambda_2'\lambda_1' + 9\lambda_2''\lambda_1 + 6\lambda_1''\lambda_2 - \lambda_1''' = 0, \tag{4}$$

$$24\lambda_0\lambda_2^3 + 23\lambda_1^2\lambda_2^2 - 42\lambda_2'\lambda_0\lambda_2 - 18\lambda_2'\lambda_1^2 - 21\lambda_1'\lambda_1\lambda_2 - 6\lambda_0'\lambda_2^2 + 6\lambda_2'\lambda_0' + 3\lambda_1'^2 - 9\lambda_2''\lambda_0 + 6\lambda_1''\lambda_1 + 3\lambda_0''\lambda_2 - \lambda_0''' - pu^{p-1}\lambda_2 + p(p-1)u^{p-2} = 0, \tag{5}$$

$$36\lambda_0\lambda_1\lambda_2^2 + 6\lambda_1^3\lambda_2 - 30\lambda_2'\lambda_0\lambda_1 - 18\lambda_1'\lambda_0\lambda_2 - 6\lambda_1'\lambda_1^2 - 6\lambda_0'\lambda_1\lambda_2 + 3\lambda_1'\lambda_0' + 6\lambda_1''\lambda_0 + 3\lambda_0''\lambda_1 - 2pu^{p-1}\lambda_1 = 0, \tag{6}$$

$$12\lambda_0^2\lambda_2^2 + 12\lambda_0\lambda_1^2\lambda_2 - 12\lambda_2'\lambda_0^2 - 9\lambda_1\lambda_1'\lambda_0 - 6\lambda_0'\lambda_0\lambda_2 + 3\lambda_0''\lambda_0 - 3pu^{p-1}\lambda_0 = 0, \tag{7}$$

$$6\lambda_0^2\lambda_1\lambda_2 - 3\lambda_0^2\lambda_1' = 0. \tag{8}$$

Based on equations (3)–(8) we will consider four cases:

Case (i): $\lambda_0 = 0$, $\lambda_2 = \frac{\omega}{u}$, $\omega \in \left\{-1, -\frac{1}{2}, 0\right\}$. Because we restricted p to be a nonzero natural number, any assumptions other than $\lambda_1 = \lambda_2 = 0$, $p = 1$ lead to contradictions, therefore a solution exists only for $p = 1$ and it is $\lambda_i = 0$, which means that the invariant equations is

$$u_{xx} = 0. \tag{9}$$

Case (ii): $\lambda_2 = 0$, $\lambda_1 = \kappa = \text{const}$. Here $\kappa = 0$, $\lambda_0 = \frac{1}{p+1}u^{p+1} + \alpha_1 u + \alpha_2$, $\alpha_i \in \mathbb{R}$, therefore the invariant equation is

$$u_{xx} + \tfrac{1}{p+1}u^{p+1} + \alpha_1 u + \alpha_2 = 0. \tag{10}$$

Case (iii): $\lambda_2 = -\frac{1}{2u}$, $\lambda_1 = \frac{\kappa}{u}$, $\kappa = \text{const}$. Here $\kappa = 0$, $\lambda_0 = \frac{1}{p+2}u^{p+1} + \beta_1 u + \frac{\beta_2}{u}$, $\beta_i \in \mathbb{R}$, therefore the invariant equation is

$$u_{xx} - \tfrac{u_x^2}{2u} + \tfrac{1}{p+2}u^{p+1} + \beta_1 u + \tfrac{\beta_2}{u} = 0. \tag{11}$$

Case (iv): $\lambda_2 = -\frac{1}{u}$, $\lambda_1 = \frac{\kappa}{u^2}$, $\kappa = \text{const}$. Here $\kappa = 0$, $\lambda_0 = \frac{p}{(p+1)(p+2)}u^{p+1} + \gamma_1 + \frac{\gamma_2}{u}$, $\gamma_i \in \mathbb{R}$, therefore the invariant equation is

$$u_{xx} - \tfrac{u_x^2}{u} + \tfrac{p}{(p+1)(p+2)}u^{p+1} + \gamma_1 + \tfrac{\gamma_2}{u} = 0. \tag{12}$$

Second step is the variation of the parameters for the solution of the ODE for time dependence. Outcome of this step is an ansatz for the PDE, and in this case, the mKdV equation. Equation (9) is the only linear one and its solution is a trivial ansatz $u(x,t) = c_1(t)x + c_2(t)$. Equations (10)–(12) however, are nonlinear and generate implicit ansatzes ($\varepsilon = \pm 1$)

$$\varepsilon \int_0^{u(x,t)} \frac{\mathrm{d}a}{\sqrt{c_1(t) - \alpha_1 a^2 - 2\alpha_2 a - \frac{2}{(p+1)(p+2)} a^{p+2}}} = x + c_2(t),$$

$$\varepsilon \int_0^{u(x,t)} \frac{\mathrm{d}a}{\sqrt{c_1(t)a - 2\beta_1 a^2 + 2\beta_2 - \frac{2}{(p+1)(p+2)} a^{p+2}}} = x + c_2(t),$$

$$\varepsilon \int_0^{u(x,t)} \frac{\mathrm{d}a}{\sqrt{c_1(t)a^2 + 2\gamma_1 a + \gamma_2 - \frac{2}{(p+1)(p+2)} a^{p+2}}} = x + c_2(t),$$

respectively. For certain parameters the ansatzes can be written in an explicit form. For example when $\alpha_1 = 0$ and $p = 1$ the equation

$$u_{xx} + \tfrac{1}{2}u^2 + \alpha_2 = 0 \tag{13}$$

produces an explicit ansatz

$$u(x,t) = -12\wp\big(x + c_1(t), -\tfrac{1}{3}\alpha_2, c_2(t)\big),$$

where $\wp$ denotes the Weierstrass function $\wp(z, g_2, g_3)$.

Third step of the method would be substitution of the ansatz to the equation we wish to reduce. Let us consider equation (13) with $\alpha_2 = 0$, meaning

$$u_{xx} + \tfrac{1}{2}u^2 = 0. \tag{14}$$

It is not linearizable and it admits trivial symmetries $u_t\partial_u$, $(u_{xxx} + uu_x)\partial_u$. On the grounds of the aforementioned findings, the solution to (14) provides an ansatz and a reduction for the KdV equation $u_t = u_{xxx} + uu_x$. Before we proceed with the reduction, we can indulge in a side challenge of finding some other PDEs sharing the same ansatz. For this purpose we introduce independent variable z. It can identified with the time variable t or viewed as a second space variable. One can show the equation (14) is invariant under LBS operators

$$\big(u^2u_xu_z + 2u_{xz}u_x^2\big)\partial_u, \quad \big(u^2u_z^2 + 2u_{xz}u_xu_z\big)\partial_u.$$

Since a linear combination of symmetry operators is itself a symmetry operator, we can use the solution of equation (14) to reduce (1+2)-dimensional equations

$$u_t = u_{xxx} + uu_x + u^2 u_x u_z + 2u_{xz} u_x^2,$$
$$u_t = u_{xxx} + uu_x + u^2 u_z^2 + 2u_{xz} u_x u_z,$$

or a non-evolution equation in one of the forms

$$\varepsilon u_t + u^2 u_t u_x + 2u_{tx} u_x^2 + u_{xxx} + uu_x = 0, \tag{15}$$
$$\varepsilon u_t + u^2 u_t^2 + 2u_{tx} u_t u_x + u_{xxx} + uu_x = 0, \tag{16}$$

where ε is an arbitrary constant.

The solution of the ODE (14) is the Weierstrass elliptic function $u(x) = -12\wp(x + c_1, 0, c_2)$, meaning the ansatz we substitute into the presented PDEs for reduction is

$$u(x,t) = -12\wp(x + c_1(t), 0, c_2(t)).$$

After such substitution, equation (15) for example, reduces to a system

$$\varepsilon c_2' = 0, \quad \varepsilon c_1' - 144c_2' = 0.$$

This means that for $\varepsilon = 0$ equation (15) has a class of solutions

$$u(x,t) = -12\wp(x + c_1(t), 0, c_2), \quad c_2 = \text{const}$$

and for $\varepsilon \neq 0$ there is only a stationary solution

$$u(x,t) = -12\wp(x + c_1, 0, c_2), \quad c_1, c_2 = \text{const}.$$

For equation (16) the reduced equations are

$$c_1'(144c_2' - \varepsilon) = 0, \quad c_2'(144c_2' - \varepsilon) = 0.$$

The solutions to this system are

$$c_1 = \text{const}, \quad c_2 = \text{const}$$

or

$$c_1(t) \text{ is arbitrary function}, \quad c_2 = \tfrac{\varepsilon}{144} t + c_0, \quad c_0 = \text{const}.$$

This means that the equation (16) has a class of solutions

$$u(x,t) = -12\wp\big(x + c_1(t), 0, \tfrac{\varepsilon}{144}t + c_0\big), \quad c_0 = \text{const}$$

as well as a stationary solution

$$u(x,t) = -12\wp(x + c_1, 0, c_2), \quad c_1, c_2 = \text{const}.$$

Example 2. Equations

$$u_x = \tfrac{1-u^2}{\sqrt{2}} \tag{17}$$

and

$$u_{xx} + u - u^3 = 0 \tag{18}$$

share the same kink solution

$$u = \tanh\big(\tfrac{x+c}{\sqrt{2}}\big), \quad c = \text{const}. \tag{19}$$

It can be easily shown that both $\big(u_{xx} + u - u^3\big)\partial_u$ and $u_t\partial_u$ are Lie–Bäcklund symmetry operators of equation (17). It follows that the substitution

$$u = \tanh\big(\tfrac{x+c(t)}{\sqrt{2}}\big) \tag{20}$$

reduces equation

$$u_t = u_{xx} + u\big(1 - u^2\big) \tag{21}$$

to a first-order differential equation $c'(t) = 0$, which means that equation (21) admits a stationary solution

$$u(x,t) = u(x) = \tanh\big(\tfrac{x+c}{\sqrt{2}}\big).$$

The second-order ODE (18) is a differential consequence of equation (17), therefore the right-hand side of the reduced equation vanishes on the ansatz solution. To obtain non-stationary solutions using this particular ansatz and this particular first-order ODE, we can add to the evolutionary equation first-order terms corresponding to the contact symmetries of (17). Contact symmetry of (17) in general form can be written as

$$f\big(\tfrac{u_x}{1-u^2}, \tfrac{1-u}{1+u}\mathrm{e}^{\sqrt{2}x}\big)u_x\partial_u,$$

where f is an arbitrary smooth function of two arguments. Substitution of (20) into

$$u_t = u_{xx} + u\big(1 - u^2\big) + f\big(\tfrac{u_x}{1-u^2}, \tfrac{1-u}{1+u}\mathrm{e}^{\sqrt{2}x}\big)u_x \tag{22}$$

reduces this equation to a simple first-order ODE

$$c'(t) = f\big(\tfrac{1}{\sqrt{2}}, \mathrm{e}^{-\sqrt{2}c(t)}\big).$$

Equation (22) will have a kink solution if $\frac{\partial f}{\partial c(t)} = 0$ and $f \neq 0$.

3. Conclusion. We show the application of the Lie–Bäcklund symmetry method for reducing the generalized version of Korteweg–de Vries equation of and nonlinear heat equation. We construct the class of ordinary differential equations which admit given Lie–Bäcklund symmetry operator and the corresponding ansatze reducing the equation under study to the system of two ordinary differential equations. The method enables us to find solutions which contain arbitrary functions on one variable for the equations (15), (16) and the solution generalizing the kink solution for nonlinear heat equation.

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