

# Exact solvability of PDM systems with extended Lie symmetries

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Показано, що усі рівняння Шрьодінгера зі змінним параметром маси, які допускають алгебри інваріантності розмірності більше п'яти (повний список таких рівнянь наведено у роботі [*J. Math. Phys.* **58** (2017), 083508, 16 pp.], є точно розв'язними. Знайдено у явному вигляді відповідні розв'язки та показано їх суперсиметричну природу.

It is shown that all PDM Schrödinger equations admitting more than five-dimensional Lie symmetry algebras (whose completed list can be found in paper [*J. Math. Phys.* **58** (2017), 083508, 16 pp.] are exactly solvable. The corresponding exact solutions are presented. The supersymmetric aspects of the exactly solvable systems are discussed.

**1. Introduction.** Group classification of differential equations consists in the specification of non-equivalent classes of such equations which possess the same symmetry groups. It is a rather attractive research field which has both fundamental and application values.

A perfect example of group classification of fundamental equations of mathematical physics was presented by Boyer [3] who had specified all inequivalent Schrödinger equations with time independent potentials admitting symmetries with respect to Lie groups, see also [1, 7, 10], where particular important symmetries were discussed, and [14], where the Boyer results were corrected. These old results have a big impact since include a priori information about all symmetry groups which can be admitted by the fundamental equation of quantum mechanics. Let us mention also that the nonlinear Schrödinger equation as well as the generalized Ginsburg–Landau quasilinear equations have been classified also [11, 15] as well as symmetries of more general systems of reaction-diffusion equations [16, 17].

In contrary, the group classification of Schrödinger equations with position dependent mass (PDM) was waited for a very long time. There

were many papers devoted to PDM Schrödinger equations with particular symmetries, see, e.g., [5, 8, 20, 21]. But the complete group classification of these equations appears only recently in [18] and [13, 19] for the stationary and time dependent equations correspondingly. A systematic search for the higher order symmetries if the PDM systems started in [12]. So late making of such important job have to cause the blame for experts in group analysis of differential equations, taking into account the fundamental role played by such equations in modern theoretical physics!

Let us remind that the PDM Schrödinger equations are requested for the description of various condensed-matter systems such as semiconductors, quantum liquids, and metal clusters, quantum wells, wires and dots, super-lattice band structures, etc.

It happens that the number of PDM systems with different Lie symmetries is rather extended. Namely, in [13] seventy classes of such systems are specified. Twenty of them are defined up to arbitrary parameters, the remaining fifty systems include arbitrary functions.

The knowledge of all Lie groups which can be admitted by the PDM Schrödinger equations has both fundamental and application values. In particular, when construct the models with a priory requested symmetries we can use the complete lists of inequivalent PDM systems presented in [19] for  $d = 2$  and [13] for  $d = 3$ . Moreover, in many cases a sufficiently extended symmetry induces integrability or exact solvability of the system, and just this aspect will be discussed in the present paper.

It will be shown that all PDM systems admitting six parametric Lie groups of symmetries or more extended symmetries are exactly solvable. Moreover, the complete sets of solutions of the corresponding stationary PDM Schrödinger equations will be presented explicitly.

There exist a tight connection between the complete solvability and various types of higher symmetries and supersymmetries. We will see that extended Lie symmetries also can cause the exact solvability. Moreover, the systems admitting extended Lie symmetries in many cases are supersymmetric and superintegrable.

**2. PDM Schrödinger equations with extended Lie symmetries.** In [13] we present the group classification of PDM Schrödinger equations

$$L\psi \equiv \left( i \frac{\partial}{\partial t} - H \right) \psi = 0, \quad (1)$$

where  $H$  is the PDM Hamiltonian of the following generic form

$$H = \frac{1}{4}(m^\alpha p_a m^\beta p_a m^\gamma + m^\gamma p_a m^\beta p_a m^\alpha) + \hat{V}, \quad p_a = -i \frac{\partial}{\partial x_a}. \quad (2)$$

Here  $m = m(\mathbf{x})$  and  $\hat{V} = \hat{V}(\mathbf{x})$  are the mass and potential depending on spatial variables  $\mathbf{x} = (x_1, x_2, x_3)$ , and summation with respect to the repeating indices  $a$  is imposed over the values  $a = 1, 2, 3$ . In addition,  $\alpha$ ,  $\beta$  and  $\gamma$  are the ambiguity parameters satisfying the condition  $\alpha + \beta + \gamma = -1$ .

The choice of values of the ambiguity parameters can be motivated by physical reasons, see a short discussion of this point in [13].

Hamiltonian (2) can be rewritten in the following more compact form

$$H = \frac{1}{2} p_a f p_a + V, \quad (3)$$

where

$$V = \hat{V} + \frac{1}{4}(\alpha + \gamma) f_{aa} + \alpha \gamma \frac{f_a f_a}{2f} \quad (4)$$

with  $f = \frac{1}{m}$ ,  $f_a = \frac{\partial f}{\partial x_a}$  and  $f_{aa} = \Delta f = \frac{\partial^2 f}{\partial x_a^2}$ .

In the following text representation (4) will be used.

In accordance with [13] there is a big variety of Hamiltonians (4) generating non-equivalent continuous point symmetries of equation (2). The corresponding potential and mass terms are defined up to arbitrary parameters or even up to arbitrary functions.

In the present paper we consider the PDM systems defined up to arbitrary parameters. Only such systems admit the most extended Lie symmetries. Using the classification results presented in [13, 18] we enumerate these systems in the following Table 1, where  $\varphi = \arctan \frac{x_2}{x_1}$  and the other Greek letters denote arbitrary constants parameters, which are supposed not to be zero simultaneously. Moreover,  $\lambda$  and  $\omega$  are either real or imaginary, the remaining parameters are real.

The symmetry operators presented in column 4 of the table are given by the following formulae

$$P_i = p_i = -i \frac{\partial}{\partial x_i}, \quad D = x_n p_n - \frac{3i}{2},$$

$$M_{ij} = x_i p_j - x_j p_i, \quad M_{0i} = \frac{1}{2}(K^i + P_i), \quad M_{4i} = \frac{1}{2}(K^i + P_i),$$

$$\begin{aligned}
B_1^1 &= \lambda \sin(\lambda t) M_{12} (\lambda^2 \varphi + \nu) \cos(\lambda t), & B_2^1 &= \frac{\partial}{\partial t} B_1^1, \\
B_1^2 &= \sin(\lambda t) D - \cos(\lambda t) (\lambda \ln(r) + \frac{\nu}{\lambda}), & B_2^2 &= \frac{\partial}{\partial t} B_1^2, \\
N_1^1 &= \omega \cos(\omega \sigma t) L_3 - \sin(\omega \sigma t) (i\partial_t - \omega^2 e^{-\sigma \Theta}), & N_2^1 &= \frac{\partial}{\partial t} N_1^1, \\
N_1^2 &= \omega \cos(\omega \sigma t) D + \sin(\omega \sigma t) (i\partial_t - \omega^2 r^{-\sigma}), & N_2^2 &= \frac{\partial}{\partial t} N_1^2, \quad (5)
\end{aligned}$$

where  $K_i = x_n x_n p_i - 2x_i D$  and indices  $i, j, k, n$  take the values 1, 2, 3.

Rather surprisingly, all systems (except ones given in items 4 and 5) presented in Table 1 are exactly solvable. In the following sections we present their exact solutions. To obtain these solutions we use some nice properties of the considered systems like superintegrability and supersymmetry with shape invariance. Let us remind that the quantum mechanical system is called superintegrable if it admits more integrals of motion than its number of degrees of freedom.

In accordance with Table 1 we can indicate 11 inequivalent PDM systems which are defined up to arbitrary parameters and admit Lie symmetry algebras of dimension five or higher. Notice that the systems fixed in items 4 and 5 admit five dimension symmetry algebras while the remaining systems admit more extended symmetries.

**3. Systems with fixed mass and potentials.** Firstly we consider those systems whose mass and potential terms are fixed, i.e., do not include arbitrary parameters. These systems are presented in items 1, 2 of Table 1 and others provided the mass does not depends on parameters and parameters of the potential are trivial.

**3.1. System invariant with respect to algebra  $\mathfrak{so}(4)$ .** Consider Hamiltonian (3) with functions  $f$  and  $V$  presented in item 1 of Table 1:

$$H = \frac{1}{2} p_a (1 + r^2)^2 p_a - 3r^2. \quad (6)$$

The eigenvalue problem for this Hamiltonian can be written in the following form

$$H\psi = 2E\psi, \quad (7)$$

where  $E$  are yet unknown numbers.

Equation (7) admits six integrals of motion  $M_{AB}$ ,  $A, B = 1, 2, 3, 4$ , presented in equation (5). Let us write them explicitly

$$M^{ab} = x^a p^b - x^b p^a, \quad M^{4a} = \frac{1}{2} (r^2 - 1) p^a - x^a x^b p^b + \frac{3i}{2} x^a. \quad (8)$$

Table 4. PDM systems with extended Lie symmetries.

no.	inverse mass $f$	potential $V$	symmetries
1	$(r^2 + 1)^2$	$-3r^2$	$M_{41}, M_{42}, M_{43},$ $M_{21}, M_{31}, M_{32}$
2	$(r^2 - 1)^2$	$-3r^2$	$M_{01}, M_{02}, M_{03},$ $M_{21}, M_{31}, M_{32}$
3	$x_3^2$	$\nu \ln(x_3)$	$P_1, P_2, M_{12}, D + \nu t$
4	$\tilde{r}^3$	$\kappa x_3 + \lambda \tilde{r}$	$P_3 + \kappa t, D + it\partial_t, M_{12}$
5	$x_1^3$	$\lambda x_1 + \kappa x_3$	$P_3 + \kappa t, P_2, D + it\partial_t$
6	$x_3^{\sigma+2}$	$\kappa x_3^\sigma$	$P_1, P_2, M_{12}, D + i\sigma t\partial_t,$ $\sigma \neq 0, 1, -2$
7	$\tilde{r}^{\sigma+2} e^{\lambda\varphi}$	$\kappa \tilde{r}^\sigma e^{\lambda\varphi}$	$M_{12} + i\lambda t\partial_t, P_3,$ $D + i\sigma t\partial_t, \sigma \neq 0$
8	$\tilde{r}^2$	$\frac{\lambda^2}{2}\varphi^2 + \mu\varphi + \nu \ln(\tilde{r})$	$B_1^1, B_2^1, D + \nu t, P_3$
9	$\tilde{r}^2 e^{\sigma\varphi}$	$\kappa e^{\sigma\varphi} + \frac{\omega^2}{2} e^{-\sigma\varphi}$	$N_1^1, N_2^1, P_3, D, K_3$
10	$r^2$	$\nu \ln(r) + \frac{\lambda^2}{2} \ln(r)^2$	$B_1^2, B_2^2, L_1, L_2, L_3$
11	$r^{2+\sigma}$	$\kappa r^\sigma + \frac{\omega^2}{2} r^{-\sigma}$	$N_1^2, N_2^2, L_1, L_2, L_3$

Operators (8) form a basis of algebra  $\mathfrak{so}(4)$ . Moreover, the first Casimir operator of this algebra is proportional to Hamiltonian (6) up to the constant shift

$$C_1 = \frac{1}{2} M_{AB} M_{AB} = \frac{1}{2} (H - 9),$$

while the second Casimir operator  $C_2 = \varepsilon_{ABCD} M_{AB} M_{CD}$  appears to be zero.

Thus like the Hydrogen atom system (7) admits six integrals of motion belonging to algebra  $\mathfrak{so}(4)$  and is maximally superintegrable.

Using our knowledge of unitary representations of algebra  $\mathfrak{so}(4)$  is possible to find eigenvalues  $E$  algebraically

$$E = 4n^2 + 5, \tag{9}$$

where  $n = 0, 1, 2, \dots$  are natural numbers.

To find the eigenvectors of Hamiltonian (6) corresponding to eigenvalues (9) we use the rotation invariance of (7) and separate variables. Introducing spherical variables and expanding solutions via spherical functions

$$\psi = \frac{1}{r} \sum_{l,m} \phi_{lm}(r) Y_m^l, \quad (10)$$

we come to the following equations for radial functions

$$\begin{aligned} & \left( -(r^2 + 1)^2 \left( \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} \right) - 4r(r^2 + 1) \frac{\partial}{\partial r} - 2r^2 \right) \varphi_{lm} \\ & = (4n^2 + 1) \varphi_{lm}, \end{aligned}$$

where  $l = 0, 1, 2, \dots$  are parameters numerating eigenvalues of the squared orbital momentum. The square integrable solutions of these equations are

$$\varphi_{lm} = C_{lm}^n (r^2 + 1)^{-n - \frac{1}{2}} r^{l+1} \mathcal{F}([A, B], [C] - r^2), \quad (11)$$

where

$$A = -n + l + 1, \quad B = -n + \frac{1}{2}, \quad C = l + \frac{3}{2}.$$

$\mathcal{F}(\dots)$  is the hypergeometric function and  $C_{lm}^n$  are integration constants. Solutions (11) tend to zero at infinity provided  $n$  is a natural number and  $l \leq n - 1$ .

Thus the system (7) is maximally superintegrable and exactly solvable.

**3.2. System invariant with respect to algebra  $\mathfrak{so}(1, 3)$ .** The next Hamiltonian we consider corresponds to functions  $f$  and  $V$  presented in item 2 of Table 1. The related eigenvalue problem includes the following equation

$$H\psi \equiv -\frac{1}{2}(\partial_a(1 - r^2)^2 \partial_a + 6r^2)\psi = E\psi. \quad (12)$$

Equation (12) admits six integrals of motion  $M_{\mu\nu}$ ,  $\mu, \nu = 0, 1, 2, 3$ , given by equation (5), which can be written explicitly in the following form

$$\begin{aligned} M_{ab} &= x^a p^b - x^b p^a, \\ M_{0a} &= \frac{1}{2}(r^2 + 1)p^a - x^a x^b p^b + \frac{3i}{2}x^a, \quad a, b = 1, 2, 3. \end{aligned} \quad (13)$$

These operators form a basis of algebra  $\mathfrak{so}(1,3)$ , i.e., the Lie algebra of Lorentz group.

As in the previous section, the corresponding first Casimir operator is expressed via the Hamiltonian, namely

$$C_1 = \frac{1}{2}M^{ab}M^{ab} - M^{0a}M^{0a} = \frac{1}{2}(H + 9), \quad (14)$$

while the second one appears to be zero.

Using our knowledge of irreducible unitary representations of Lorentz group we find eigenvalues of  $C_1$  and  $C_2$  in the form [2, 9]:

$$c_1 = 1 - j_0^2 - j_1^2, \quad c_2 = 2ij_0j_1,$$

where  $j_0$  and  $j_1$  are quantum numbers labeling irreducible representations. Since the second Casimir operator  $C_2$  is trivial, we have  $c_1 = j_0 = 0$ . So there are two possibilities [9]: either  $j_1$  is an arbitrary imaginary number, and the corresponding representation belongs to the principal series, or  $j_1$  is a real number satisfying  $|j_1| \leq 1$ , and we come to the subsidiary series of IRs. So

$$j_1 = i\lambda, \quad c_1 = 1 - j_1^2 = \lambda^2 + 1, \quad (15)$$

where  $\lambda$  is an arbitrary real number, or, alternatively,

$$0 \leq j_1 \leq 1, \quad c_1 = 1 - j_1^2. \quad (16)$$

In accordance with (14) the related eigenvalues  $E$  in (12) are

$$E = -5 - j_1^2. \quad (17)$$

In view of the rotational invariance of equation (12) it is convenient to represent solutions in form (10). As a result we obtain the following radial equations

$$\begin{aligned} & \left( -(r^2 - 1)^2 \left( \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} \right) - 4r(r^2 - 1) \frac{\partial}{\partial r} - 2r^2 \right) \varphi_{lm} \\ & = (\tilde{E} + 4) \varphi_{lm}. \end{aligned} \quad (18)$$

The general solution of (18) is

$$\varphi_{lm} = (1 - r^2)^{-\frac{1}{2}-k} (C_{lm}^k r^{l+1} \mathcal{F}([A, B], [C], r^2))$$

$$+ \tilde{C}_{lm}^k r^{-l} \mathcal{F}([\tilde{A}, \tilde{B}], [\tilde{C}], r^2)), \quad (19)$$

where

$$\begin{aligned} A &= -k + l + 1, & B &= -k + \frac{1}{2}, & C &= l + \frac{3}{2}, \\ \tilde{A} &= -k - l, & \tilde{B} &= -k + \frac{1}{2}, & \tilde{C} &= \frac{1}{2} - l, & k &= \frac{1}{2} \sqrt{-\tilde{E} - 5} \end{aligned}$$

and is singular at  $r = 1$ . However, for  $\tilde{C}_{lm}^k = 0$  and  $k = j_1$  the solutions are normalizable in some specific metric [18].

Thus the system presented in item 7 of Table 1 is exactly solvable too. The corresponding eigenvalues and eigenvectors are given by equations (15), (16), (17) and (19), respectively.

**3.3. Scale invariant systems.** Consider one more PDM system which is presented in item 3 of the table and includes the following Hamiltonian: Let us note that the free fall effective potential appears also one more system specified in Table 1. Thus, considering the inverse mass and potential specified in item 3 we come to the following Hamiltonian

$$\begin{aligned} H &= -\frac{1}{2} \left( x_3 \frac{\partial}{\partial x_3} x_3 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_3} + x_3^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \right) \\ &+ \nu \ln(x_3). \end{aligned} \quad (20)$$

Equation (12) with Hamiltonian given in (20) can be easily solved by separation of variables in Cartesian coordinates. Expanding the wave function  $\psi$  via eigenfunctions of integrals of motion  $P_1$  and  $P_2$ :

$$\psi = \exp(-i(k_1 x_1 + k_2 x_2)) \Phi(k_1, k_2, x_3) \quad (21)$$

and introducing new variable  $y = \ln(x_3)$  we come to the following equation for  $\Phi = \Phi(k_1, k_2, x_3)$ :

$$-\frac{\partial^2 \Phi}{\partial y^2} + ((k_1^2 + k_2^2) \exp(2y) + 2\nu y) \Phi = \tilde{E} \Phi, \quad (22)$$

where  $\tilde{E} = 2E - \frac{1}{4}$ .

Here we consider the simplest version of equation (22) when parameter  $\nu$  is trivial

$$-\frac{\partial^2 \Phi}{\partial y^2} + (k_1^2 + k_2^2) \exp(2y) \Phi = \tilde{E} \Phi. \quad (23)$$



This equation is scale invariant and can be easily solved. Its square integrable solutions are given by Bessel functions

$$\Psi = C_{k_1 k_2}^E K_{i\sqrt{\tilde{E}}} \left( \sqrt{k_1^2 + k_2^2} \ln(x_3) \right),$$

where  $C_{k_1 k_2}^E$  are integration constants and  $\tilde{E}$  are arbitrary real parameters.

It is interesting to note that there are rather non-trivial relations between the results given in the present and previous sections. Equation (23) admits six integrals of motion which are nothing but the following operators

$$P_1, P_2, K_1, K_2, M_{12}, D, \quad (24)$$

which are presented in equations (5).

Like operators (13) integrals of motion (24) form a basis of the Lie algebra of Lorentz group, and we again can find the eigenvalues of Hamiltonian (23) algebraically by direct analogy with the above. We will not present this routine procedure since there exist strong equivalence relations between Hamiltonians (23) with zero  $\nu$  and (6). To find them we note that basis (24) is equivalent to the following linear combinations of the basis elements

$$M_{01}, M_{02}, M_{04}, M_{41}, M_{42}, M_{12}, \quad (25)$$

whose expressions via operators (24) are given by equation (5). To reduce (25) to the set (13) it is sufficient to change subindices 4 to 3, i.e., to make the rotation in the plane 43. The infinitesimal operator for such rotation is given by the following operator

$$M_{43} = \frac{1}{2}(K_3 + P_3) = \frac{1}{2}(r^2 - 1)p_3 - x_3 x_b p_b + \frac{3i}{2}x_3,$$

which belongs to the equivalence group of equations. Solving the corresponding Lie equations and choosing the group parameter be equal  $\frac{\pi}{2}$  we easily find the requested equivalence transformations.

One more scale invariant system is presented in item 8 where all parameters of potential are zero. The relation Hamiltonian looks as follows

$$H = -\tilde{r} \frac{\partial}{\partial x_\alpha} \tilde{r} \frac{\partial}{\partial x_\alpha} - x_\alpha \frac{\partial}{\partial x_\alpha} - \tilde{r}^2 \frac{\partial^2}{\partial x_3^2}, \quad \alpha = 1, 2. \quad (26)$$

Considering the eigenvalue problem for (26) it is convenient to use the cylindrical variables

$$\tilde{r} = \sqrt{x_1^2 + x_2^2}, \quad \varphi = \arctan \frac{x_2}{x_1}, \quad x_3 = z \quad (27)$$

and expand solutions via eigenfunctions of  $M^{12}$  and  $P_3 = -i\frac{\partial}{\partial z}$ :

$$\Psi = \exp[i(\kappa\varphi + \omega z)]\Phi_{\kappa\omega}(\tilde{r}), \quad \kappa = 0, \pm 1, \pm 2, \dots, \quad -\infty < \omega < \infty.$$

As a result we come to the following equations for radial functions  $\Phi = \Phi_{\kappa\omega}(\tilde{r})$ :

$$-\left(\tilde{r}\frac{\partial}{\partial\tilde{r}}\tilde{r}\frac{\partial}{\partial\tilde{r}} + \tilde{r}\frac{\partial}{\partial\tilde{r}} + \omega^2\right)\Phi = (\tilde{E} - \kappa^2)\Phi.$$

Square integrable (with the weight  $\tilde{r}$ ) solutions of this equation are

$$\Phi_{\kappa\omega} = \frac{1}{\tilde{r}}J_\alpha(\omega\tilde{r}), \quad \alpha = \kappa^2 + 1 - \tilde{E}, \quad (28)$$

where  $J_\alpha(\omega\tilde{r})$  is Bessel function of the first kind. Functions (28) are normalizable and disappear at  $\tilde{r} = 0$  provided  $\alpha \leq 0$ . The rescaled energies  $\tilde{E}$  continuously take the values  $\kappa^2 \leq \tilde{E} < \infty$ .

The last scale invariant system which we have to consider is fixed in item 10 where  $\nu = \lambda = 0$ . We will do it later in the end of the following section.

**4. Systems defined up to arbitrary parameters.** In previous section we present exact solutions for systems with fixed potential and mass terms. In the following we deal with the systems defined up to arbitrary parameters.

**4.1. The system with oscillator effective potential.** Let us consider equation (1) with  $f$  and  $V$  are functions fixed in item 10 of Table 1, i.e.,

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\frac{\partial}{\partial x_a}r^2\frac{\partial}{\partial x_a} + \nu\ln(r) + \frac{\lambda^2}{2}\ln(r)^2\right)\psi.$$

These equations admit extended Lie symmetries (whose generators are indicated in the table) being invariant with respect to six-parametrical Lie group. Let us show that they also admit hidden supersymmetries.

In view of the rotational invariance and symmetry of the considered equations with respect to shifts of time variable, it is reasonable to search for their solutions in spherical variables, i.e., in the following form

$$\Psi = e^{-iEt} R_{lm}(r) Y_{lm}(\varphi, \theta), \quad (29)$$

where  $\varphi$  and  $\theta$  are angular variables and  $Y_{lm}(\varphi, \vartheta)$  are spherical functions, i.e., eigenvectors of  $L^2 = L_1^2 + L_2^2 + M_{12}^2$  and  $M_{12}$ . As a result we come to the following radial equations

$$\left( -r \frac{\partial R_{lm}}{\partial r} r \frac{\partial R_{lm}}{\partial r} - r \frac{\partial R_{lm}}{\partial r} + l(l+1) + \nu \ln(r) + \frac{\lambda^2}{2} \ln(r)^2 \right) R_{lm} = 2ER_{lm}. \quad (30)$$

Introducing new variable  $y = \sqrt{2} \ln(r)$  we can rewrite equation (30) in the following form

$$\left( -\frac{\partial^2}{\partial y^2} + l(l+1) + \nu y + \frac{\lambda^2}{2} y^2 \right) R_{lm}(y) = \tilde{E} R_{lm}(y), \quad (31)$$

where  $\tilde{E} = E - \frac{1}{4}$ .

Let  $\lambda \neq 0$  then equation (31) is reduced to the 1D harmonic oscillator up to the additional term  $l(l+1)$ . The admissible eigenvalues  $\tilde{E}$  are given by the following formula

$$\tilde{E} = n + l(l+1),$$

where  $n$  is a natural number. The corresponding eigenfunctions are well known and we will not present them here. The same is true for supersymmetric aspects of the considered system.

If parameter  $\lambda$  is equal to zero then (31) reduces to equation with free fall potential slightly modified by the term  $l(l+1)$ . The corresponding solutions can be found in textbooks devoted to quantum mechanics. If both parameters  $\nu$  and  $\lambda$  are zero, equation (31) is solved by trigonometric or hyperbolic functions. The corresponding PDM Schrödinger equation is scale invariant, i.e., belongs to the class considered in the previous section.

**4.2. The systems with potentials equivalent to 3d oscillator.** Consider now the system represented in item 11 of the table. The

corresponding equation (1) takes the following form

$$i\frac{\partial\psi}{\partial t} = \left(-\frac{1}{2}\partial_a r^{\sigma+2}\partial_a + \kappa r^{2\sigma} + \frac{\omega^2}{r^{2\sigma}}\right)\psi. \quad (32)$$

Like in previous section we represent the wave function in the form given in (29) and came to the following radial equation

$$\begin{aligned} -r^{2\sigma+2}\frac{\partial^2 R_{lm}}{\partial r^2} - (2\sigma+4)r^{2\sigma+1}\frac{\partial R_{lm}}{\partial r} \\ + (r^{2\sigma}(l(l+1)+\kappa) + \omega^2 r^{-2\sigma})R_{lm} = 2ER_{lm}. \end{aligned} \quad (33)$$

Using the Liouville transform

$$r \rightarrow z = r^{-\sigma}, \quad R_{lm} \rightarrow \tilde{R}_{lm} = z^{\frac{\sigma+3}{2\sigma}} R_{lm},$$

we reduce (33) to the following form

$$-\sigma^2\frac{\partial^2 \tilde{R}_{lm}}{\partial z^2} + \left(\frac{l(l+1)+\delta}{z^2} + \omega^2 z^2\right)\tilde{R}_{lm} = 2E\tilde{R}_{lm}, \quad (34)$$

where  $\delta = \frac{3}{4}(\sigma+1)(\sigma+3) + 2\kappa$ .

Equation (34) describes a deformed 3d harmonic oscillator including two deformation parameters, namely,  $\sigma$  and  $\kappa$ .

Let

$$2\kappa = -\sigma^2 - 3\sigma - 2,$$

then equation (34) is reduced to the following form

$$H_l \tilde{R}_{lm} \equiv \left(-\sigma^2\frac{\partial^2}{\partial z^2} + \frac{(2l+1)^2 - \sigma^2}{4z^2} + \omega^2 z^2\right)\tilde{R}_{lm} = 2E\tilde{R}_{lm}. \quad (35)$$

Equation (35) is shape invariant. Hamiltonian  $H_r$  can be factorized

$$H_l = a_l^+ a_l - C_l, \quad (36)$$

where

$$\begin{aligned} a = -\sigma\frac{\partial}{\partial z} + W, \quad a^+ = \sigma\frac{\partial}{\partial z} + W, \\ W = \frac{2l+1+\sigma}{2z} + \omega z, \quad C_l = \omega(2l+2\sigma+1). \end{aligned}$$

The superpartner  $\hat{H}_l$  of Hamiltonian (36) has the following property

$$\hat{H}_l \equiv a_l a_l^+ + C_l = H_{l+\sigma} + C_l.$$

Thus our Hamiltonian is shape invariant.

Thus to solve equation (35) we can use the standard tools of SUSY quantum mechanics and find the admissible eigenvalues in the following form

$$E_n = \omega (2n\sigma + l + \sigma + \frac{1}{2}) = \omega (2n + l + \frac{3}{2}) + \delta\omega(2n + 1), \quad (37)$$

where  $\delta = \sigma - 1$ .

Equation (37) represents the spectrum of 3d isotropic harmonic oscillator deformed by the term proportional to  $\delta$ .

For equation (34) we obtain in the analogous way

$$E_n = \frac{\omega}{2} (\sigma(2n + 1) + \sqrt{(2l + 1)^2 + \tilde{\kappa}}), \quad (38)$$

where  $\tilde{\kappa} = 8(\kappa + 1) + \sigma(\sigma + 3)$ . The related eigenvectors are expressed via the confluent hypergeometric functions  $\mathcal{F}$ :

$$R_n = e^{-\frac{\omega r^\sigma}{2\sigma}} r^{\sigma n - \frac{E_n}{\omega}} \mathcal{F} \left( -n, \frac{E_n}{\sigma\omega} - n, \frac{\omega}{\sigma} r^{-\sigma} \right),$$

where  $n$  is integer and  $E_n$  is eigenvalue (38).

**4.3. System with angular oscillator potential.** The next system which we consider is specified by the inverse mass and potential presented in item 8 of the table. The corresponding Hamiltonian is

$$H = p_a r^2 p_a + \frac{\lambda^2}{2} \varphi^2 + \sigma\varphi + \nu \ln(\tilde{r}).$$

The corresponding eigenvalue equation is separable in cylindrical variables, thus it is reasonable to represent the wave function as follows

$$\psi = \Psi(\tilde{r})\Phi(\varphi) \exp(-ikx_3). \quad (39)$$

As a result we obtain the following equations for radial and angular variables

$$(-\tilde{r}\partial_{\tilde{r}}\tilde{r}\partial_{\tilde{r}} - \tilde{r}\partial_{\tilde{r}} + \nu \ln(\tilde{r}) + k^2\tilde{r}^2 - \mu) \Psi(\tilde{r}) = 0 \quad (40)$$

and

$$\left(-\frac{\partial^2}{\partial\varphi^2} + \frac{\lambda^2}{2}\varphi^2 + \sigma\varphi - \mu\right)\Phi(\varphi) = 0, \quad (41)$$

where  $\mu$  is a separation constant.

For  $\lambda$  nonzero equation (41) is equivalent to the Harmonic oscillator. The specificity of this system is that, in contrast with (31), it includes angular variable  $\varphi$  whose origin is

$$0 \leq \varphi \leq 2\pi. \quad (42)$$

For trivial  $\lambda$  our equation (41) is reduced to equation with free fall potential, but again for the angular variable satisfying (42).

The radial equation (40) is simple solvable too. In the case  $k = 0$  we again come to the free fall potential.

**4.4. Systems with Morse effective potential.** The next system we consider is specified by the inverse mass and potentials represented in item 9 of Table 1. The corresponding Hamiltonian is

$$H = -\frac{\partial}{\partial x_a} \tilde{r}^2 e^{\sigma\varphi} \frac{\partial}{\partial x_a} + \kappa e^{\sigma\varphi} + \frac{\omega^2}{2} e^{-\sigma\varphi}.$$

Introducing again the cylindric variables and representing the wave function in the form (39) we come to the following equations for the radial and angular variables

$$\left(-\left(\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial y}\right) + \mu + k^2 e^{2y}\right)\Psi(\tilde{r}) = \mu\Psi(\tilde{r})$$

and

$$\left(-e^{\sigma\varphi} \left(\frac{\partial^2}{\partial\varphi^2} + \kappa - \mu\right) + \frac{\omega^2}{2} e^{-\sigma\varphi}\right)\Phi(\varphi) = \tilde{E}\Phi(\varphi). \quad (43)$$

Dividing all terms in (43) by  $\exp(\sigma\varphi)$  we obtain the following equation

$$\left(-\left(\frac{\partial^2}{\partial\varphi^2} + \kappa - \mu\right) + \frac{\omega^2}{2} e^{-2\sigma\varphi}\right)\Phi(\varphi) = e^{-\sigma\varphi} \tilde{E}\Phi(\varphi).$$

or

$$\left(-\left(\frac{\partial^2}{\partial\varphi^2}\right) + \frac{\omega^2}{2} e^{-2\sigma\varphi} - \tilde{E}e^{-\sigma\varphi}\right)\Phi(\varphi) = \hat{E}\Phi(\varphi), \quad (44)$$

where we denote  $\hat{E} = \mu - \kappa$ .

Formula (44) represents the Schrödinger equation with Morse potential. This equation is shape invariant and also can be solved using tools of SUSY quantum mechanics. We demonstrate this procedure using another system.

Considering the mass and potential presented in item 6 of Table 1 we come to the following Hamiltonian

$$H = \frac{1}{2}p_a x_3^{\sigma+2} p_a + \kappa x_3^\sigma.$$

Equation (12) with Hamiltonian (20) can be solved by separation of variables in Cartesian coordinates. Expanding the wave function  $\psi$  via eigenfunctions of integrals of motion  $P_1$  and  $P_2$  in the form (21) and introducing new variable  $y = \ln(x_3)$  we reduce the problem to the following equation for  $\Phi(k_1, k_2, x_3)$ :

$$\left( -\frac{\partial}{\partial x_3} x_3^{\sigma+2} \frac{\partial}{\partial x_3} + x_3^{\sigma+2} k^2 + 2\kappa x_3^\sigma \right) \Phi = 2E\Phi, \quad (45)$$

where  $k^2 = k_1^2 + k_2^2$ .

Dividing all terms in (45) by  $x_3^\sigma$  we can rewrite it in the following form

$$\left( -\frac{\partial^2}{\partial y^2} - (\sigma + 1) \frac{\partial}{\partial y} - 2E \exp(-\sigma y) + k^2 \exp(2y) + 2\kappa \right) \Phi = 0.$$

In the particular case  $\sigma = 2$  we again come to the equation with Morse effective potential.

One more system which can be related to Morse potential is represented in item 7 and include the following Hamiltonian

$$H = \frac{1}{2}p_a \exp(\lambda\varphi) \tilde{r}^{\sigma+2} p_a + \nu \exp(\lambda\varphi) \tilde{r}^\sigma.$$

The corresponding equation (12) is separable in the cylindrical variables (27) provided  $\sigma \cdot \lambda = 0$  and again includes the Morse effective potential.

Let us return to equation (33) and solve it using approach analogous to the presented above. In other words, we will change the roles of eigenvalues and coupling constants.

First we divide all terms in (33) by  $r^{2\sigma}$  and obtain

$$-r^2 \frac{\partial^2 R_{lm}}{\partial r^2} - (2\sigma + 4)r \frac{\partial R_{lm}}{\partial r}$$

$$+ (\omega^2 r^{-4\sigma} + \mu r^{-2\sigma}) R_{lm} = \varepsilon R_{lm}, \quad (46)$$

where

$$\varepsilon = -l(l+1) - 2\kappa, \quad \mu = -2E. \quad (47)$$

Applying the Liouville transform

$$r \rightarrow \rho = \ln(r), \quad R_{lm} \rightarrow \tilde{R}_{lm} = e^{-\frac{\sigma+3}{2}\rho} R_{lm}$$

we reduce (46) to a more compact form

$$H_\nu \tilde{R}_{lm} \equiv \left( -\frac{\partial^2}{\partial \rho^2} + \omega^2 e^{-2\sigma\rho} + (2\omega\nu + \omega\sigma)e^{-\sigma\rho} \right) \tilde{R}_{lm} = \hat{\varepsilon} \tilde{R}_{lm}, \quad (48)$$

where

$$\hat{\varepsilon} = \varepsilon - \left( \frac{\sigma+3}{2} \right)^2, \quad \nu = \frac{\mu}{2\omega} - \frac{\sigma}{2}. \quad (49)$$

Like (44) equation (48) includes the familiar Morse potential and so is shape invariant. Indeed, denoting  $\mu = 2\omega(\nu + \frac{\sigma}{2})$  we can factorize Hamiltonian  $H_\nu$  like it was done in (36) where index  $l$  should be changed to  $\nu$  and

$$W = \nu - \omega e^{-a\rho}, \quad C_\nu = \nu^2$$

and the shape invariance is easily recognized.

To find the admissible eigenvalues  $\varepsilon$  and the corresponding eigenvectors we can directly use the results presented in [4], see item 4 of Table 4.1 there

$$\hat{\varepsilon} = \hat{\varepsilon}_n = -(\nu - n\sigma)^2, \quad (\tilde{R}_{lm})_n = y^{\frac{\nu}{\sigma} - n} e^{-\frac{y}{2}} L_n^{2(\frac{\nu}{\sigma} - n)}(y),$$

where  $y = \frac{2\omega}{\sigma} r^{-\sigma}$ .

Thus we find the admissible values of  $\hat{\varepsilon}_n$ . Using definitions (47) and (49) we can find the corresponding values of  $E$  which are in perfect accordance with (38).

**Discussion.** The results presented above in Section 2 include the complete list of continuous symmetries which can be admitted by PDM Schrödinger equations, provided these equations are defined up to arbitrary parameters. All such systems appear to be exactly solvable.



It is important to note that the list of symmetries presented in the fourth column of the table is valid only for the case of nonzero parameters defined the potential and mass terms. If some (or all) of these parameters are trivial, the corresponding PDM Schrödinger equation can have more extended set of symmetries. For example, it is the case for the potential and PDM presented in item 3 of the table, compare the list of symmetries presented in column 4 with (24). The completed list of non-equivalent symmetries can be found in [13] which generalizes the Boyer results [3] to the case of PDM Schrödinger equations. As other extensions of results of [3] we can mention the group classification of the nonlinear Schrödinger equations [15] and the analysis of its conditional symmetries [6].

Thanks to their extended symmetries the majority of the presented systems is exactly solvable. In Sections 3 and 4 we present the corresponding solutions explicitly and discuss supersymmetric aspects of some of them. However, two of the presented systems (whose mass and potential are presented in items 4 and 5 of Table 1) are not separable, if both arbitrary parameters  $\kappa$  and  $\lambda$  are nonzero. And just these systems have the most small symmetry. On the other hand, all systems admitting six- or higher-dimensional Lie symmetry algebras are separable and exactly solvable.

In addition to the symmetry under the six parameter Lie group, equation (32) (which we call deformed 3d isotropic harmonic oscillator) possesses a hidden dynamical symmetry with respect to group  $SO(1,2)$ . The effective radial Hamiltonian is shape invariant, and its eigenvalues can be found algebraically. In spite on the qualitative difference of its spectra (37) and (38) of the standard 3d oscillator, it keeps the main supersymmetric properties of the latter. We show that the shape invariance of PDM problems usually attends their extended symmetries.

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