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On shape of Hilbertian embedding of universal Teichmüller space

We provide restricted negative answer to the question of Sullivan whether the universal Teichmüller space $\mathbf{T}$ is biholomorhically equivalent to bounded convex domain in a complex Banach space and establish that Hilbertian embedding of $\mathbf{T}$ cannot be convex.

1. Problem and result. As is well-known, the Teichmüller spaces admit canonical complex structure and are pseudo-convex. An open question is about biholomorhic equivalence of these spaces to convex domains in complex Banach spaces. It was posed for the universal Teichmüller space by Sullivan and relates to Tukia's result [1] which explicitly yields a real analytic homeomorphism of this space onto a convex domain in a real Banach space.

Our goal is to prove the following theorem giving restricted negative answer.

Theorem. *The universal Teichmüller space* $\mathbf{T}$ *cannot not be mapped biholomorphically onto a bounded convex domain in the Hilbert space.*

Its proof involves conformally rigid domains whose existence was established by Thurston [2] (see also [3]). Such approach was applied in [4] for solving the related problem on starlikeness of the space $\mathbf{T}$ in the Bers embedding posed in [5].

We preceede the proof of the theorem by some introductory remarks. Recall that the space $\mathbf{T}$ is the set of quasisymmetric homeomorphisms of the unit circle $S^1 = \partial\Delta$ factorized by Möbius maps. Let

$$\Delta = \{z : \ |z| < 1\}, \quad \Delta^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : \ |z| > 1\}.$$

The canonical complex Banach structure on $\mathbf{T}$ is defined by factorization of the ball of Beltrami coefficients (conformal structures on Δ)

$$\mathrm{Belt}(\Delta)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\Delta^* = 0, \ \|\mu\| < 1\},$$

letting $\mu_1, \mu_2 \in \mathrm{Belt}(\Delta)_1$ be equivalent if the corresponding homeomorhisms of the Beltrami equation $\overline{\partial} w = \mu \partial w$ with $\mu = \mu_1, \ \mu_2$ coincide on S^1 (hence on $\overline{\Delta^*}$) and passing to Schwarzian derivatives

$$S_w(z) = \Big(\frac{w''(z)}{w'(z)}\Big)' - \frac{1}{2}\Big(\frac{w''(z)}{w'(z)}\Big)^2 \quad (z \in \Delta^*).$$

Let us normalize these solutions (quasiconformal maps) $w = f^\mu(z)$ by

$$f^\mu(z) = z + b_0 + b_1 z^{-1} + \dots \quad \text{in } \Delta^*, \ \ f^\mu(1) = 1.$$

Denote the collection of such univalent functions in Δ^* admitting quasiconformal extensions to $\mathbf{C}$ by Σ^0. Their Schwarzians S_{f^μ} run over a bounded domain in the Banach space $\mathbf{B}$ of hyperbolically bounded holomorphic functions on Δ^* with norm

$$\|\varphi\| = \sup_{\Delta^*} (|z|^2 - 1)^2 |\varphi(z)|.$$

This domain models the space $\mathbf{T}$. Note that $\varphi(z) = O(|z|^{-4})$ as $z \to \infty$ and that $\mathbf{B}$ is dual to the subspace $A_1(\Delta)$ formed in $L_1(\Delta)$ by integrable holomorphic functions in the disk.

The space $\mathbf{T}$ coincides with the union of inner points of the set

$$\mathbf{U} = \{\varphi = S_f \in \mathbf{B} : \ f \text{ univalent in } \Delta^*\};$$

on the other hand, by Thurston's theorem, $\mathbf{U} \setminus \overline{\mathbf{T}}$ has uncountable many isolated points $\varphi_0 = S_{f_0}$ which correspond to conformally rigid domains $f_0(\Delta^*)$.

2. Proof of Theorem. Assume, in the contrary, that there exists a biholomorphic homeomorphism χ of the universal Teichmüller space $\mathbf{T}$

onto a bounded convex domain D some Hilbert space X and take a function $f_* \in \Sigma^0$ whose domain $f_*(\Delta^*)$ is conformally rigid. Then the Schwarzians of the homotopy functions $f_r^*(z) = rf^*(z/r)$ $(0 < r < 1)$ are $S_{f_r^*}(z) = = r^{-2} S_{f^*}(z/r)$ and lie in the space $\mathbf{T}$.

Pick a sequence $r_n \to 1$, then $S_{f_{r_n}^*}$ are convergent to S_{f^*} locally uniformly on Δ^* and weakly* in $\mathbf{B}$. The corresponding sequence $x_n = \chi(S_{f_{r_n}^*}) \in D$ is weakly compact in the space X, and one can assume that this sequence is convergent to some point x_0 in X. Then

$$\|x_0\|_X \le \lim_{n\to\infty} \|x_n\|_X. \tag{1}$$

Our goal now is to show that only the equality is possible here, i.e., $\|x_0\|_X = \lim\limits_{n\to\infty} \|x_n\|_X$. To this end, we consider the space X as a real space with the same norm (admitting multiplication of $x \in X$ only with $c \in \mathbf{R}$). Denote this real space by $\widetilde{X}$. The domain D is convex in $\widetilde{X}$, thus its Minkowski functional

$$\alpha_D(x) = \inf\{t > 0 : \ t^{-1}x \in D\} \quad (x \in X)$$

determines on this space a norm equivalent to initial norm $\|x\|_X$. Denote the space with new norm by $\widetilde{X}_\alpha$ and notice that the domain D is its unit ball.

The sequence x_n is weakly convergent also on $\widetilde{X}_\alpha$; thus, similar to (1),

$$\alpha_D(x_0) \le \lim_{n\to\infty} \alpha_D(x_n) \le 1.$$

This implies that the point x_0 belongs to the closure of domain D in the initial norm of X.

Were $\alpha_D(x_0) < \lim\limits_{n\to\infty} \alpha_D(x_n)$ or $\alpha_D(x_0) = \lim\limits_{n\to\infty} \alpha_D(x_n) < 1$, in both these cases the point x_0 must lie inside D. Then its inverse image $\chi^{-1}(x_0) \in \mathbf{T}$ and thus is the Schwarzian S_{f_0} of some function $f_0 \in \Sigma^0$. Since $\chi^{-1}(x_n) = S_{f_{r_n}^*}$ are convergent locally uniformly on Δ^* to S_{f^*}, it must be $f_0 = f^*$, which yields that S_{f^*} must lie in $\mathbf{T}$, in contradiction that is contradicts as an isolated point of $\mathbf{S}$.

It remains the case $\alpha_D(x_0) = \lim\limits_{n\to\infty} \alpha_D(x_n) = 1$ which is equivalent to

$$\lim_{n\to\infty} \|x_n\|_X = \|x_0\|_X \quad \text{and} \quad x_0 \in \partial D. \tag{2}$$

In view of the properties of X, the weak convergence $x_n \to x_0$ in X and the equality (2) together imply the strong convergence $\lim\limits_{n\to\infty} \|x_n - x_0\|_X = 0$.

Then, since χ is a biholomorphic homeomorphism, the inverse images $\chi^{-1}(x_n) = S_{f^*_{r_n}}$ must approach the boundary of **T** in **B** and therefore S_{f^*} must be a boundary point of **T**, contradicting that it is an isolated points of the set **U**. This completes the proof of the theorem.

Remarks.
1. The above arguments are extended straightforwardly to more general uniformly convex Banach spaces X (which means that for any $x_n,\ y_n$ satisfying $\|x_n\| \le 1,\ \|y_n\| \le 1,\ \|x_n + y_n\| \to 2$ must be $\|x_n - y_n\| \to 0$).

The uniformly convex spaces are reflexive and have another important property (essentially used in the proof): any bounded subset $E \subset X$ is weakly compact; moreover, if a sequence $\{x_n\} \subset X$ is weakly convergent to x_0 and $\|x_n\| \to \|x_0\|$, then $x_n \to x_0$ in the strong topology of the space X induced by its norm.
2. A Hilbert model of the universal Teichmüller space was defined and studied from differential geometric point of view in [6] using a collection of Hilbert space inner products on tangent spaces. This manifold has uncountable many components.

References

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